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ON LOCAL GEOMETRY OF FINITE MULTITYPE HYPERSURFACES

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Abstract. This paper studies local geometry of hypersurfaces of finite multitype. Catlin's definition of multitype is applied to a general smooth hypersurface in $\mathbb{C}^{n+1}$. We prove biholomorphic equivalence of models in dimension three and describe all biholomorphisms between such models. A finite constructive algorithm for computing multitype is described. Analogous results for decoupled hypersurfaces are given.

1. Introduction

Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth hypersurace and p be a Levi degenerate point on M. When $n = 1$, the basic local CR invariant of M at p is the type of the point, as defined by J. J. Kohn in [10]. It measures the maximal order of contact of M with complex curves passing through p. On the next level, important local invariants are extracted from the invariantly defined model hypersurface at p ([11]).

In higher dimensions, local geometry of Levi degenerate hypersurfaces is much more complicated. In order to obtain invariants relevant for analysis of the $\bar\partial$ equation on the domain bounded by M, one has to consider orders of contact with singular complex varieties. If d_k denotes the maximal order of contact of M with complex varieties of dimension k, the n-tuple $(d_n, \dots, d_1)$ is called the D'Angelo multitype of M at p.

For pseudoconvex hypersurfaces David Catlin ([4]) introduced a different, more algebraic notion of multitype. One of its important advantages is that it provides a well defined weighted-homogeneous model hypersurface, an essential tool for local analysis.

The two multitypes coincide on an important class of hypersurfaces called h-extendible ([17]), or semiregular ([7]). This class contains for example all decoupled and all convexifiable hypersurfaces. Such hypersurfaces have been studied in [7], [3], [17].

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In this paper we use Catlin's definition of multitype for a general smooth hypersurface in $\mathbb{C}^{n+1}$, not necessarily pseudoconvex. As an important consequence, we obtain a well defined model, with all the convenient properties familiar from dimension two.

Since the definition itself is nonconstructive, and the models are not uniquely defined, it is not a priori clear what is the relation between various models. In many situations, when low order boundary invariants are needed, it is enough to choose an arbitrary model. However, in order to study higher order CR invariants it is essential to understand the non uniqueness in the definition of models. In particular, it is not a priori obvious whether all models are necessarily biholomorphically equivalent.

In the case of h-extendible pseudoconvex hypersurfaces , biholomorphic equivalence of models was proved by N. Nikolov in [16]. The main tool in the proof is Pinchuk's scaling technique. Pseudoconvexity is an essential assumption for using this technique.

In the first part of this paper we prove biholomorphic equivalence of models for general hypersurfaces of finite multitype in $\mathbb{C}^3$. A finite algorithm for calculating multitype is also given. Then we determine explicitly the non-uniqueness of models, describing biholomorpisms between different models. Our main tool is the technique developed in the work of S. S. Chern and J. K. Moser for analysis of biholomorpisms in weighted coordinates. For Levi degenerate hypersurfaces this technique was used in [11], [12].

In the second part we obtain analogous results for decoupled hypersurfaces. Biholomorphic equivalence of models is proved, where the biholomorphisms are given by an explicitly described polynomial transformation.

2. Hypersurfaces of finite multitype

Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth hypersurface and $p \in M$ be a point of finite type in the sense of Kohn and Bloom-Graham.

Consider holomorphic coordinates (z, w), where $z = (z_1, z_2, \dots, z_n)$ and $z_j = x_j + y_j$, $w = u + iv$, centered at p. The hyperplane $\{v = 0\}$ is assumed to be tangent to M at p. M is described near p as the graph of a uniquely determined real valued function

$$v = F(z, \bar{z}, u)\,.$$

We will apply Catlin's definition of multitype to M at p. In the following, α, β, will denote multiindices, and we will use the standard multiindex notation.

Definition 2.1. A weight is an n-tuple of nonnegative rational numbers $\Lambda = (\lambda_1, \dots, \lambda_n)$, where $0 \le \lambda_j \le \frac{1}{2}$, and $\lambda_j \ge \lambda_{j+1}$, such that there exist integers $k_1, \dots, k_n$ satisfying

$$\sum_{j=1}^{n} k_j \lambda_j = 1\,.$$

The weighted degree of a monomial $c_{\alpha\beta} z^\alpha \bar{z}^\beta$ is

$$\text{wt. } (c_{\alpha\beta} z^\alpha \bar{z}^\beta) = \sum_{i=1}^{n} (\alpha_i + \beta_i)\lambda_i \, .$$

A real valued polynomial $P(z, \bar{z})$ is Λ-homogeneous of weighted degree γ if it is a sum of monomials of weight γ.

The variables w and u are given weight one. Hence the weighted degree of a monomial $c_{\alpha,\beta,m} z^\alpha \bar{z}^\beta u^m$ is

$$\text{wt. } (c_{\alpha,\beta,m} z^\alpha \bar{z}^\beta u^m) = m + \sum_{i=1}^{n} (\alpha_i + \beta_i)\lambda_i \, .$$

and the weighted degree of $c_{\alpha,m} z^\alpha w^m$ is equal to $m + \sum_{i=1}^{n} \alpha_i \lambda_i$.

A weight Λ will be called admissible if there exist coordinates (z, w) in which the defining equation has form

$$v = P(z, \bar{z}) + o_{\text{wt.}}(1) \, , \tag{2.1}$$

where $P(z, \bar{z})$ is a Λ-homogeneous polynomial of weighted degree one without harmonic terms, and $o_{\text{wt.}}(1)$ denotes terms in the Taylor expansion of weight greater then one.

Clearly, for any real $\delta > 0$ there are only finitely many rational n-tuples for which $\lambda_n > \delta$ and such that $(\lambda_1, \dots, \lambda_n)$ is a weight. We denote by $\Lambda_0 = (\mu_1, \dots, \mu_n)$ the lexicographically smallest admissible weight.

The multitype of M at p is defined to be the n-tuple $(m_1, m_2, \dots, m_n)$, where $m_j = \frac{1}{\mu_j}$ if $\mu_j \neq 0$ and $m_j = \infty$ if $\mu_j = 0$. If none of the m_j is infinity, we say that M is of finite multitype at p.

Coordinates corresponding to an admissible weight Λ, in which the local description of M has form (2.1), with P being Λ-homogeneous, will be called Λ-adapted. Λ_0 will be called the multitype weight.

In the following, when using this terminology and the weight is not explicitly specified, the multitype weight is always understood, e.g. adapted coordinates mean Λ_0-adapted.

If (2.1) is the defining equation in some adapted coordinates, we define a model hypersurface to M at p to be

$$M_H = \left\{ (z, w) \in \mathbb{C}^{n+1} \mid v = P(z, \bar{z}) \right\} . \tag{2.2}$$

Models are useful for many geometric and analytic results (see e.g. [3], [17]).

In order to deal with biholomorphisms between models, we introduce the following terminology.

Definition 2.2. Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be an admissible weight. A transformation

$$w^* = w + g(z_1, \dots, z_n, w) \, , \quad z_i^* = z_i + f_i(z_1, \dots, z_n, w)$$

is called

- Λ-homogeneous if $g = 0$ and f_i is a Λ-homogeneous polynomial of weighted degree λ_i

– Λ-subhomogeneous if f_i is a polynomial consisting of monomials of weighted degree less or equal to λ_i and g consists of monomials of weighted degree less or equal to one.
– Λ-superhomogeneous if the Taylor expansion of f_i consists of terms of weighted degree greater or equal to λ_i and g consists of monomials of weighted degree greater than one.

3. Biholomorphic equivalence of models

In this section we will consider $M \subseteq \mathbb{C}^3$, hence $z = (z_1, z_2)$ and $\Lambda_0 = (\mu_1, \mu_2)$. We will consider biholomorphic transformations of the form

$$\begin{aligned} z_1^* &= z_1 + f_1(z_1, z_2, w) \\ z_2^* &= z_2 + f_2(z_1, z_2, w) \\ w^* &= w + g(z_1, z_2, w)\,, \end{aligned} \tag{3.1}$$

and write $f = (f_1, f_2)$. Let $v^* = F^*(z^*, \bar z^*, u^*)$ be the defining equation in the new coordinates. Substituting (3.1) into $v^* = F^*(z^*, \bar z^*, u^*)$ we get the transformation formula

$$\begin{aligned} &F^*\big(z + f(z, u + iF),\ \overline{z + f(z, u + iF)},\ u + Re\, g(u + iF)\big) \\ &\quad = F(z, \bar z, u) + Im\ g(z, u + iF)\,, \end{aligned} \tag{3.2}$$

where the argument of F is $(z, \bar z, u)$.

Now we consider Λ-adapted coordinates for a fixed weight Λ and transformations of the form (3.1) which preserve form (2.1). We will show that this is the case if and only if the transformation is Λ-superhomogeneous. Hence we assume that F^* has the same form as F,

$$v^* = P^*(z^*, \bar z^*) + o_{\mathrm{wt.}}(1)\,, \tag{3.3}$$

where P^* is a Λ-homogeneous polynomial of weighted degree one without harmonic terms.

Lemma 3.1. A transformation of the form (3.1) transforms Λ-adapted coordinates into Λ-adapted coordinates if and only if it is Λ-superhomogeneous.

Proof. Decoupled linear transformations

$$z_1^* = \delta_1 z_1\,, \quad z_2^* = \delta_2 z_2\,, \quad w^* = d_3 w\,,$$

where $\delta_1, \delta_2 \in \mathbb{C}$, $d_3 \in \mathbb{R}$ act on P in a trivial way. Hence, without any loss of generality we may assume that this linear part of the transformation is normalized, i.e.,

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} = \frac{\partial g}{\partial w} = 0 \qquad \text{at} \qquad z = w = 0\,.$$

If $\lambda_1 = \lambda_2$, then for any subhomogeneous transformation the component f is linear. Comparing terms of weight less or equal to one in (3.2) we check that g cannot contain terms of weight less or equal to one, since in that case F^* would contain a harmonic term of weight less or equal to one. Let us now consider the case $\lambda_1 > \lambda_2$. Consider a transformation which is not superhomogeneous. By the same argument as above, if g contains terms of weight less or equal to one, then the new coordinates are not Λ-adapted. Separating the subhomogeneous part in f_1 we write

$$z_1^* = z_1 + \sum_{i=1}^{l} c_i z_2^i + O_{\text{wt.}}(\lambda_1)\,,$$
$$z_2^* = z_2 + \sum_{i=1}^{l} d_i z_2^i + O_{\text{wt.}}(\lambda_1)$$

where $l = \left[\frac{\lambda_1}{\lambda_2}\right]$ and $c_i, d_i \in \mathbb{C}$. Let γ be the lowest index such that $c_\gamma \neq 0$. We have $z_1 = z_1^* - \sum_{i=1}^{l} e_i (z_2^*)^i + O_{\text{wt.}}(\mu_1)$, where $e_\gamma = c_\gamma$ and $e_i = 0$ for $i < \gamma$. Let us write P as

$$P(z, \bar z) = \sum_{|\alpha|\mu_1 + |\beta|\mu_2 = 1} C_{\alpha,\beta} z_1^{\alpha_1} z_2^{\alpha_2} \bar z_1^{\beta_1} \bar z_2^{\beta_2}\,. \tag{3.4}$$

By assumption, the restriction of P to the z_1-axis is a nonzero homogeneous polynomial of degree m_1, of the form

$$P_1(z_1, \bar z_1) = \sum_{j=1}^{m_1-1} a_j z^j \bar z^{m_1-j}\,,$$

where $a_j = \bar a_{m_1-j}$. Let a_l be the first nonzero coefficient in this formula. Substituting into (3.4) we obtain in F^* the term (with stars dropped)

$$(m_1 - l) a_l e_\gamma z_1^l \bar z_1^{m_1-l-1} (\bar z_2)^\gamma\,,$$

since by (3.2) no other entry in F has influence on this term. It has weight less than one, hence (z^*, w^*) are not Λ-adapted coordinates. □

The previous lemma will be especially useful when Λ is the multitype weight. It allows to describe explicitly biholomorphisms between different models.

Lemma 3.2. Let M_H and $\tilde M_H$ be two models for M at p. Then there is a homogeneous transformation which maps M_H to $\tilde M_H$.

Proof. By the previous lemma, the coordinates in which M_H is the model are related to those in which $\tilde M_H$ is the model by a superhomogeneous transformation. By the transformation formula, terms of weight greater than μ_i in f_i influence only terms of weight greater then one in F^*. Hence $\tilde M_H$ is obtained by the homogeneous part of this transformation. □

Analyzing homogeneous transformations is straightforward. It follows immediately from Definition 2.2 that if $c = \frac{\mu_1}{\mu_2}$ is not an integer, homogeneous transformation are just the decoupled linear transformations $z_1^* = \delta_1 z_1,\ z_2^* = \delta_2 z_2$. If c is an integer, homogeneous transformations are of the form

$$z_1^* = \delta_1 z_1 + \beta z_2^c, \qquad z_2^* = \delta_2 z_2,$$

where $\delta_1, \delta_2 \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$.

As a corollary we obtain

Corollary 3.3. Any two models are biholomorphically equivalent by a polynomial transformation.

Now we desribe explicitely the process of finding multitype.

Lemma 3.4. Let (z, w) be local holomorphic coordinates in which M is described by (2.1) for some admissible weight Λ. Then Λ is not the multitype weight if and only if there is a Λ-homogeneous transformation such that in the new coordinates P^* is independent of z_2.

Proof. Clearly, if there is such a transformation, then Λ is not the multitype weight. Conversely, assume Λ is not the multitype weight. By definition, there exist a biholomorphic transformation which takes the Λ-adapted coordinates (z, w) into Λ_0-adapted. Note that any Λ_0-adapted coordinates are also Λ-adapted (we just truncate P at weight one with respect to Λ). Hence we may use Lemma 3.1. It follows that the transformation has to be Λ-superhomogeneous. Denote $\tilde{P}$ the leading polynomial with respect to the weight Λ in the new coordinates. It is obtained by the homogeneous part (with respect to Λ of this transformation. However, a polynomial which is in the same time Λ-homogeneous and Λ_0-homogeneous has to be independent of z_2. $\square$

4. Decoupled hypersurfaces

In this section we consider decoupled hypersurfaces and obtain results analogous to those of the previous section. Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth hypersurface and p be a point on M.

Definition 4.1. M is called decoupled at p if there exist local holomorphic coordinates around p such that the defining equation has form

$$v = \sum_{j=1}^{n} f_j(z_j)\,.$$

We will assume that M is a decoupled hypesurface at p, of finite multitype. Since removing low order harmonic terms in f_j is obtained by transformations preserving this decoupled form, we may assume that

$$f_j(z_j) = P_j(z_j, \bar{z}_j) + o(|z|^{m_j})\,,$$

where P_j is a nonzero homogeneous polynomial of degree m_j, without harmonic terms, and we fix coordinates (z, w) with this property. Again, we denote $\Lambda_0 = (\mu_1, \mu_2, \ldots, \mu_n)$ the multitype weight, where, as before, $\mu_i = \frac{1}{m_i}$.

We will consider biholomorphic transformations of the form

$$\begin{aligned} z_j^* &= z_j + f_j(z_1, \ldots, z_n, w) \\ w^* &= w + g(z_1, \ldots, z_n, w)\,, \end{aligned} \tag{4.1}$$

where we denote $f = (f_1, \ldots, f_n)$.

Lemma 4.2. If M is decoupled at $p \in M$, then any transformation which maps the coordinates (z, w) into adapted coordinates is superhomogeneous.

Proof. With no loss of generality, we may again assume that the linear part of the transformation is partly normalized and satisfies

$$\frac{\partial f_j}{\partial z_j} = \frac{\partial g}{\partial w} = 0 \qquad \text{at} \qquad z = w = 0\,,$$

for all $j = 1, \dots, n$. By the same reasoning as above, g cannot contain terms of weight less or equal to one. Let us assume that the transformation is not superhomogeneous, and take a variable z_j in which a strictly subhomogeneous term appears. We obtain

$$z_j = z_j^* + \alpha \prod_{i>j} (z_i^*)^{l_i} + O_{\mathrm{wt.}}(\beta)\,,$$

where $\beta = \sum l_i \mu_i < \mu_j$ and $\alpha \neq 0$. $P(z, \bar{z})$ when restricted to the coordinate axis z_j gives a subharmonic but not harmonic real valued homogeneous polynomial of degree m_j

$$P_j(z_j, \bar{z}_j) = \sum_{k=1}^{m_j - 1} a_k z_j^k \bar{z}_j^{m_j - k}\,, \tag{4.2}$$

where $a_k = \bar{a}_{m_j - k}$. Let a_l be the first nonzero coefficient in this formula. Then F^* contains a term (with stars ommited)

$$a_l (m_j - l) \alpha z_j^l \bar{z}_j^{m_j - l - 1} \prod_{i>j} \bar{z}_i^{l_i}\,,$$

which again, by (3.2), cannot come from any other term in F, and the coefficient is different from zero. This term has weight less than one, hence the transformation does not preserve form (2.1), i.e. the coordinates (z^*, w^*) are not adapted. □

By the same argument as in Lemma 3.2., we obtain

Lemma 4.3. Let M_H and $\tilde{M}_H$ be two models for M at p. Then there is a homogeneous transformation which maps M_H to $\tilde{M}_H$. In particular, M_H and $\tilde{M}_H$ are biholomorphic by a polynomial transformation.

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