ARCHIVUM MATHEMATICUM (BRNO)
Tomus 43 (2007), 485 – 490

PROPORTIONALITY PRINCIPLE FOR CUSPED MANIFOLDS

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Abstract. We give a short proof of the proportionality principle for cusped hyperbolic manifolds.

1. Introduction

Let M be a hyperbolic manifold of finite volume $\mathrm{Vol}\,(M)$. The Gromov-Thurston proportionality principle is the following theorem, where $\|M, \partial M\|$ denotes the simplicial volume and V_n is the volume of a regular ideal simplex in hyperbolic n-space.

Theorem 1 ([6], Theorem 6.5.4.)**.** *Let M be a compact, orientable n-manifold such that* $\mathrm{int}\,(M) = M - \partial M$ *admits a hyperbolic metric of finite volume* $\mathrm{Vol}\,(M)$*. Then*

$$\|M, \partial M\| = \frac{1}{V_n}\,\mathrm{Vol}\,(M)\ .$$

The proportionality principle for closed manifolds was proved by Gromov ([3, Section 2.2]), for a more detailed elaboration of the proof see [1], Theorem C.4.2. (An alternative proof using measure homology was proposed by Thurston and completed in [4].)

For cusped manifolds (i.e., non-compact manifolds of finite volume), the proportionality principle was stated in [6], Theorem 6.5.4. The proof in [6] is very brief and uses another definition of the simplicial volume. Thus it remained an open question whether Gromov's proof has a direct generalization to cusped manifolds. Such a generalization is actually possible, all necessary arguments are contained in the recent paper [2].

The aim of our paper is to give yet another, somewhat shorter, proof of the proportionality principle for cusped manifolds, using bounded cohomology and measure chains to give a more direct generalization of Gromov's argument.

We give a short outline of the argument. Consider the map which pinches all boundary tori to points, and let $(M', \partial M')$ be the quotient, with $\partial M'$ a finite number of points. It follows from Gromov's theory of multicomplexes that this

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2000 *Mathematics Subject Classification.* 57N16.
Key words and phrases. hyperbolic volume, simplicial volume.

map induces an isometry of Gromov norms. There is an obvious isometry between $H_*(M', \partial M')$ and the (absolute) homology theory constructed from ideal simplices with all ideal vertices in the cusps of M. The main point in the proof of the wanted inequality is then to check that to the latter homology theory one can apply Gromov's smearing construction. A little care is needed because one can not apply the smearing construction to ideal simplices: one is not allowed to have ideal simplices with vertices not in cusps. In section 2.3, which is the main part of this paper, this technical point is surmounted.

2. Proof

2.1. Crushing.

Definition 1. For a manifold M with boundary ∂M we denote by M^{crush} resp. $\partial M^{\text{crush}}$ the quotient $M/\sim$ of M resp. $\partial M/\sim$ of ∂M under the following equivalence relation: $x \sim y$ if and only if x and y belong to the same connected component of ∂M (or if $x = y \notin \partial M$). We denote by crush: $(M, \partial M) \to \left(M^{\text{crush}}, \partial M^{\text{crush}}\right)$ the canonical projection.

Lemma 1. *Let M be an n-dimensional compact, orientable, connected manifold with boundary ∂M. If each connected component of ∂M has amenable fundamental group, then* $(\text{crush})_n : H_n(M, \partial M) \to H_n\left(M^{\text{crush}}, \partial M^{\text{crush}}\right)$ *is an isometry of Gromov norms.*

Proof. We have $H_n(M, \partial M) \simeq H_n\left(M^{\text{crush}}, \partial M^{\text{crush}}\right) \simeq \mathbb{R}$. This implies, by the argument in [3], section 1.1, that for proving equality of Gromov-norms it suffices to show that $(\text{crush})^*$ induces an isometric isomorphism in bounded cohomology.

Because each connected component of ∂M has amenable fundamental group, the Relative Mapping Theorem ([3], section 4.1) states that $(\text{crush})^*$ is indeed an isometric isomorphism in bounded cohomology. □

For a topological space X, the measure chain complex $\mathcal{C}_*(X)$ is the vector space of signed measures on $\operatorname{map}(\Delta^*, X)$ with compact determination set and finite variation. The total variation gives a norm on $\mathcal{C}_*(X)$ which descends to a seminorm on the homology of this complex (for the usual boundary operator), the so-called measure homology $\mathcal{H}_*(X)$. (For a detailed account on measure homology see [4], section 2). For a pair of spaces (X, Y), $\mathcal{C}_*(X, Y)$ is the vector space of signed measures on $\operatorname{map}(\Delta^*, X) / \operatorname{map}(\Delta^*, Y)$ with compact determination set and finite variation.

Lemma 2 ([4])**.** *If N is a compact manifold with π_1-injective boundary ∂N, then the inclusion $i_*\colon C_*(N, \partial N) \to \mathcal{C}_*(N, \partial N)$ induces an isometric isomorphism in homology.*

Proof. This is [4], Theorem 1.1, in the case $\partial N = \emptyset$. We explain how the proof can be adapted to the relative case.
Let $p\colon \widetilde{N} \to N$ be the universal covering. The main step in [4] is the construction of a Borel map $s_k\colon \operatorname{map}\left(\Delta^k, N\right) \to \operatorname{map}\left(\Delta^k, \widetilde{N}\right)$ with $p_k s_k = \text{id}$. The construction of s_k consists in the construction of a countable covering $\operatorname{map}\left(\Delta^k, \widetilde{N}\right) = \cup_{j=1}^{\infty} V_j$

such that $p\mid_{V_j}$ is a homeomorphism, and to succesively define $s_k = p^{-1}$ on $p(V_n) - \cup_{j=1}^{n-1} p(V_j)$.

In the proof of [4, Theorem 4.1], s_k is used to define a map $v_k\colon \operatorname{map}\left(\widetilde{N}^{k+1}, \mathbb{R}\right)^{\pi_1 N} \to \operatorname{map}\left(\operatorname{map}(\Delta^k, N), \mathbb{R}\right)$ by $v_k(f) := f\left(s_k(\sigma)(e_0), \ldots, s_k(\sigma)(e_k)\right)$.

Let $W_j = V_j \cap \operatorname{map}\left(\Delta^k, p^{-1}(\partial N)\right)$. Since $p\mid_{V_j}$ is a homeomorphism, also its restriction $p\mid_{W_j}\colon W_j \to p(W_j) \subset \operatorname{map}(\Delta^k, \partial N)$ is a homeomorphism. Therefore s_k maps $\operatorname{map}(\Delta^k, \partial N)$ to $\operatorname{map}\left(\Delta^k, p^{-1}(\partial N)\right)$.

This means that v_k maps $\operatorname{map}\left(p^{-1}(\partial N)^{k+1}, \mathbb{R}\right)$ to $\operatorname{map}\left(\operatorname{map}(\Delta^k, \partial N), \mathbb{R}\right)$. Hence we get a relative cochain map of norm ≤ 1. Since its composition with i^* is an isometric isomorphism (which follows from independence of bounded cohomology from the choice of strong relatively injective resolutions), i^* must actually be an isometry. By duality, the claim follows. $\square$

2.2. **Ideal simplices.**

Definition 2. We denote $C_*^{\mathrm{str,ideal}}(\mathbb{H}^n)$ the chain complex of straight, possibly ideal, simplices in hyperbolic space $\mathbb{H}^n$. For a hyperbolic manifold $N = \Gamma\backslash\mathbb{H}^n$ we denote

$$C_*^{\mathrm{str,ideal}}(N) = \Gamma\backslash C_*^{\mathrm{str,ideal}}(\mathbb{H}^n)\,.$$

Assume that M is an orientable manifold with boundary such that its interior $N = M - \partial M$ admits a complete hyperbolic metric, i.e. $N = \Gamma\backslash\mathbb{H}^n$ for some discrete, torsionfree subgroup $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$. Assume that this hyperbolic metric has finite volume. It is then well-known that points of $\partial M^{\mathrm{crush}}$ (the so-called cusps) correspond to Γ-orbits of points of the ideal boundary $\partial_\infty \mathbb{H}^n$ which are fixed points for some parabolic element of Γ. Therefore each n+1-tuple of vertices in M^{crush} corresponds to a Γ-orbit of straight simplices with vertices either in $\mathbb{H}^n$ or in parabolic fixed points of Γ. In particular, we get a straight ideal simplex $\mathrm{str}(\sigma)$ with the same vertices and homotopic rel. vertices to a given simplex σ in M^{crush}.

Lemma 3.

$$\mathrm{str}\colon C_*\left(M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right) \to C_*^{\mathrm{str,ideal}}(N)$$

is a chain map of norm 1.

Proof. Obvious.

Corollary 1. $\|M, \partial M\| \geq \frac{1}{V_n}\mathrm{Vol}(M)$.

Proof. Let $\sum a_i\sigma_i$ be a representative of $\left[M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right]$. We consider the volume form of the hyperbolic metric on $N = M - \partial M = M^{\mathrm{crush}} - \partial M^{\mathrm{crush}}$. It defines a measure ν on N which extends to a measure on M^{crush} by putting $\nu\left(\partial M^{\mathrm{crush}}\right) = 0$. With respect to this measure we have $\mathrm{Vol}(N) = \sum a_i \mathrm{Vol}(\mathrm{str}(\sigma_i))$ and $|\mathrm{Vol}(\mathrm{str}(\sigma_i))| \leq V_n$, from which we get

$$\frac{1}{V_n}\mathrm{Vol}(M) \leq \|\,\mathrm{str}\left[M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\right]\| \leq \|M^{\mathrm{crush}}, \partial M^{\mathrm{crush}}\| = \|M, \partial M\|\,.$$

(The first inequality is true because $\sum a_i \operatorname{str}(\sigma_i)$ represents $\operatorname{str}_*\left[M^{\text{crush}}, \partial M^{\text{crush}}\right]$, the second by Lemma 3, the last equality is Lemma 1.) $\square$

2.3. **Smearing construction.** The purpose of this section is to construct representatives of $[M, \partial M] \in \mathcal{H}_n(M, \partial M)$ of norm close to $\frac{1}{V_n}\operatorname{Vol}(M)$. We assume that $N = M - \partial M$ is a complete hyperbolic manifold of finite volume. For $\epsilon > 0$ let $N_{[\epsilon,\infty]} = \{x \in N : \operatorname{inj}(x) \geq \epsilon\}$ be the ϵ-thick part and $N_{[0,\epsilon]} = \{x \in N : \operatorname{inj}(x) \leq \epsilon\}$ the ϵ-thin part.

For some small $\delta > 0$, let $\sigma\colon \Delta^n \to N$ be some fixed straight regular simplex with $\operatorname{Vol}(\sigma) = V_n - \delta$. Abbreviate $G = \operatorname{Isom}^+(\mathbb{H}^n)$. We define $\sigma^*\colon \Gamma\backslash G \to \operatorname{map}(\Delta^n, N)$ by $\sigma^*[g] = g\sigma$. We denote by Haar the Haar measure on $\Gamma\backslash G$, normalised such that $\operatorname{Haar}(\Gamma\backslash G) = \operatorname{Vol}(N)$.

As in [5], we define $\operatorname{smr}(\sigma) \in \mathcal{C}_*(N)$ to be the measure chain given by $\operatorname{smr}(\sigma)(B) := \operatorname{Haar}\left((\sigma^*)^{-1}B\right)$ for $B \subset \operatorname{map}(\Delta^n, N)$, and $\operatorname{Avg}(\sigma) := \frac{1}{2}\operatorname{smr}(\sigma) - \frac{1}{2}\operatorname{smr}(r\sigma)$ for some fixed orientation-reversing isometry $r \in \operatorname{Isom}(\mathbb{H}^n)$. By the same proof as in [5], we have $\partial \operatorname{Avg}(\sigma) = 0$.

Let ϵ_0 be smaller than the Margulis constant for G. It follows from the Margulis lemma (see [1, D.3.3]) that, for each $\epsilon_1 \leq \epsilon_0$, $N_{[0,\epsilon_1]}$ is a product and one thus has a homotopy-equivalence $h_{\epsilon_1} : \left(N, N_{[0,\epsilon_1]}\right) \to \left(N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right)$.

Further we fix some positive constants $\epsilon < \epsilon_1 < \epsilon_0$ such that

$$d\left(\partial N_{[\epsilon_0,\infty]}, \partial N_{[\epsilon_1,\infty]}\right) > l(\delta),$$

$$d\left(\partial N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon,\infty]}\right) > l(\delta),$$

where $l(\delta)$ denotes the edge-length of the regular simplex of volume $V_n - \delta$.

Definition 3. For a hyperbolic manifold N and σ, ϵ as above we define

$$C_\epsilon(N) = \left\{\tau \in \operatorname{map}(\Delta^n, N) : \tau(e_0), \ldots, \tau(e_n) \in N_{[\epsilon,\infty]}\right\}$$

and a signed measure $\operatorname{Avg}_\epsilon(\sigma) \in \mathcal{C}_*(N)$ by

$$\operatorname{Avg}_\epsilon(\sigma)(B) = \operatorname{Avg}(\sigma)\left(B \cap C_\epsilon(N)\right).$$

Lemma 4. *$\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on* $\operatorname{map}\left(\Delta^n, N_{[0,\epsilon_1]}\right)$.

Proof. We know that $\partial \operatorname{Avg}(\sigma) = 0$. Therefore $\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on simplices which are, at the same time, face of some n-simplex in $C_\epsilon(N)$ and some n-simplex not in $C_\epsilon(N)$, both simplices being regular simplices with volume $V_n - \delta$ and edgelength $l(\delta)$. Since $d\left(\partial N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon,\infty]}\right) > l(\delta)$, this implies that $\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on simplices with vertices in $N_{[0,\epsilon_1]}$. It is well-known that $N_{[0,\epsilon_1]}$ is convex, therefore $\partial \operatorname{Avg}_\epsilon(\sigma)$ is determined on simplices in $N_{[0,\epsilon_1]}$. $\square$

Therefore $\operatorname{Avg}_\epsilon(\sigma) \in \mathcal{C}_n\left(N, N_{[0,\epsilon_1]}\right)$ is a relative cycle. Applying h_{ϵ_1}, we get a relative cycle

$$\left(h_{\epsilon_1}\right)_* \operatorname{Avg}_\epsilon(\sigma) \in \mathcal{C}_n\left(N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right) \simeq \mathcal{C}_n(M, \partial M).$$

Lemma 5. *Let M be a compact, orientable n-manifold such that $N = \operatorname{int}(M)$ admits a hyperbolic metric of finite volume. Then*

(i) $\|\left(h_{\epsilon_1}\right)_* \operatorname{Avg}_\epsilon(\sigma)\| \leq \operatorname{Vol}\left(N_{[\epsilon,\infty]}\right)$,

(ii) $(h_{\epsilon_1})_* \operatorname{Avg}_\epsilon(\sigma)$ *represents* $\frac{K}{\operatorname{Vol}(N)}\left[N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right]$ *with* $K \geq (V_n - \delta)\left(\operatorname{Vol}(N) - (n+1)\operatorname{Vol}(N_{[0,\epsilon_0]})\right)$.

Proof. (i) We have $\| (h_{\epsilon_1})_* \operatorname{Avg}_\epsilon(\sigma) \| \leq \| \operatorname{Avg}_\epsilon(\sigma) \|$. Let $\operatorname{smr}_\epsilon(\sigma)$ be the positive part of $\operatorname{Avg}_\epsilon(\sigma)$. Then $\| \operatorname{Avg}_\epsilon(\sigma) \| = \frac{1}{2}\| \operatorname{smr}_\epsilon(\sigma) \| + \frac{1}{2}\| \operatorname{smr}_\epsilon(r\sigma) \| = \| \operatorname{smr}_\epsilon(\sigma) \|$, such that it suffices to compute the latter. This is

$$\begin{aligned}\| \operatorname{smr}_\epsilon(\sigma)\| &= \operatorname{smr}_\epsilon(\sigma)\big(\operatorname{map}(\Delta^n, N)\big) = \operatorname{smr}(\sigma)\big(C_\epsilon(N)\big) \\ &= \operatorname{Haar}\left((\sigma^*)^{-1} C_\epsilon(N)\right) \leq \operatorname{Vol}\left(N_{[\epsilon,\infty]}\right).\end{aligned}$$

(ii) To determine the homology class of a relative n-cycle it suffices to evaluate it against the volume form dvol of $N_{[\epsilon_1,\infty]}$. We compute

$$\begin{aligned}\left\langle \operatorname{dvol}, (h_{\epsilon_1})_* \operatorname{smr}_\epsilon(\sigma)\right\rangle &= \left\langle (h_{\epsilon_1})^* \operatorname{dvol}, \operatorname{smr}_\epsilon(\sigma)\right\rangle \\ &= \int_{\operatorname{map}(\Delta^n,N)} \Big(\int_\tau (h_{\epsilon_1})^* \operatorname{dvol}\Big) \operatorname{dsmr}_\epsilon(\sigma)(\tau) \\ &= \int_{C_\epsilon(N)} \Big(\int_\tau (h_{\epsilon_1})^* \operatorname{dvol}\Big) \operatorname{dsmr}(\sigma)(\tau) \\ &= \int_{C_\epsilon(N)} \operatorname{vol}(h_{\epsilon_1}\tau) \operatorname{dsmr}(\sigma)(\tau) = \int_{\Gamma\backslash G} \operatorname{vol}(h_{\epsilon_1} g\sigma) \operatorname{dHaar}(\Gamma g).\end{aligned}$$

Let $v_0 = \sigma(e_0), \ldots, v_n = \sigma(e_n)$ be the vertices of σ. Let $g \in G$. If all $gv_i \in N_{[0,\epsilon_1]}$, then $g\sigma \subset N_{[0,\epsilon_1]}$ because $d\left(\partial N_{[0,\epsilon]}, \partial N_{[0,\epsilon_1]}\right) > l(\delta)$. Thus $\operatorname{vol}(h_{\epsilon_1} g\sigma) = \operatorname{vol}(g\sigma) = V_n - \delta$ except possibly if $gv_i \notin N_{[0,\epsilon_1]}$ for some v_i. But the set

$$\left\{g \in \Gamma\backslash G : gv_i \notin N_{[0,\epsilon_1]} \text{ for some } i\right\}$$

has Haar measure smaller than $(n+1)\operatorname{Vol}\left(N_{[0,\epsilon_0]}\right)$. This implies the claimed inequality

$$\left\langle \operatorname{dvol}, (h_{\epsilon_1})_* \operatorname{smr}_\epsilon(\sigma)\right\rangle \geq (V_n - \delta)\big(\operatorname{Vol}(N) - (n+1)\operatorname{Vol}(N_{[0,\epsilon_0]})\big).$$

□

Theorem 2. *Let M be a compact, orientable n-manifold such that* $\operatorname{int}(M)$ *admits a hyperbolic metric of finite volume.*

Then $\|M\| = \frac{1}{V_n}\operatorname{Vol}(M)$.

Proof. The lower bound follows from Corollary 1. To get the upper bound we observe that, using $(M, \partial M) \simeq \left(N_{[\epsilon_1,\infty]}, \partial N_{[\epsilon_1,\infty]}\right)$, Lemma 5 implies

$$\|M, \partial M\| \leq \frac{\operatorname{Vol}(M)}{V_n - \delta} \frac{\operatorname{Vol}(N_{[\epsilon,\infty]})}{\big(\operatorname{Vol}(N) - (n+1)\operatorname{Vol}(N_{[0,\epsilon_0]})\big)}.$$

This implies the wanted inequality if we let $\delta, \epsilon_0, \epsilon \to 0$. □

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