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UNDULOIDS AND THEIR GEOMETRY

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Abstract. In this paper we consider non-compact cylinder-like surfaces called unduloids and study some aspects of their geometry. In particular, making use of a Kenmotsu-type representation of these surfaces, we derive explicit formulas for the lengths and areas of arbitrary segments, along with a formula for the volumes enclosed by them.

1. Introduction

The unduloids, which are members of the family of constant mean curvature surfaces, prove themselves ideal for modelling the interfaces that are used to explain the very elongated myelin shapes which look like cylinders or strings of beads [3], charged diblock copolymers [5] or pearling instabilities of fluid membrane tubes [1]. As mathematical objects, the unduloids were discovered and described in an analytical form a long time ago by Delaunay [2] as constant mean curvature surfaces of revolution in $\mathbb{R}^3$ that are generated by the trace of a focus of an ellipse which rolls without sliding on the axis of revolution. Many years after that, Kenmotsu [9] found and solved a complex non-linear differential equation which describes these surfaces (up to integration). It should be noted also that the integrals which appear in Delaunay's approach are not even well defined while the parameters entering in the Kenmotsu representation do not have a direct geometrical interpretation.

Elsewhere we have used an alternative approach for describing this class of surfaces and have found their explicit parametrizations [10]. Here we will clarify the geometrical meaning of the parameters entering in this parametrization and present various results about the lengths, surface areas and volumes of appropriate parts of these infinitely long surfaces.

2. Surface Geometry

A parametrized surface $\mathcal{S}\colon \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$ is determined by its first and second fundamental forms:

$$(1) \qquad I = E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\mathrm{d}v + G\,\mathrm{d}v^2\,, \quad II = L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\mathrm{d}v + N\,\mathrm{d}v^2$$

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where the respective coefficients are given by

$$E = \mathbf{x}_u \cdot \mathbf{x}_u\,, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v\,, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v \tag{2}$$
$$L = \mathbf{x}_{uu} \cdot \mathbf{n}\,, \quad M = \mathbf{x}_{uv} \cdot \mathbf{n}\,, \quad N = \mathbf{x}_{vv} \cdot \mathbf{n}\,.$$

Here $\mathbf{n}$ is the unit normal vector to $\mathcal{S}$,

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}\cdot \tag{3}$$

Intuitively, the *metric coefficients* E, F and G describe the stretching necessary to map a piece of the plane up to the surface under the parametrization. As can be seen from the definition, the coefficients L, M and N of II have more to do with acceleration and, hence, curvature. Indeed, there are classical formulas which describe two types of curvatures at every point of the surface. These are the *Gauss* and *mean* (meaning "average") curvatures, denoted by K and H respectively. The formulas are

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{EN + GL - 2FM}{2(EG - F^2)}\cdot \tag{4}$$

As usual, we will think of the axisymmetric surface $\mathcal{S}$ in Euclidean space by specifying its meridional section, a curve $u \mapsto (x(u), z(u))$ in the XOZ plane, where we take u to be the so-called *natural* arclength parameter. Such a surface can be presented in ordinary Euclidean space $\mathbb{R}^3$ with a fixed orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, by making use of the parameter u and the angle v specifying the rotation of the YOZ plane via the vector-valued function

$$\mathbf{x}(u,v) = x(u)\mathbf{e}_1(v) + z(u)\mathbf{e}_2(v)\,, \qquad 0 < u \le L\,, \quad 0 \le v < 2\pi\,. \tag{5}$$

Here the vector $\mathbf{e}_2(v)$ is the new position of $\mathbf{j}$ after a rotation at some angle v

$$\mathbf{e}_2(v) = \cos v\mathbf{j} + \sin v\mathbf{k}\,. \tag{6}$$

Since the rotation is about the first axis $\mathbf{i}$, the vector representing it in (5) is a constant, $\mathbf{e}_1(v) = \text{const} = \mathbf{i}$. The pair $\{\mathbf{e}_1, \mathbf{e}_2\}$ can be completed to the orthonormal basis set $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in $\mathbb{R}^3$ by letting the third vector $\mathbf{e}_3(v)$ be the cross product of the vectors $\mathbf{e}_1(v)$ and $\mathbf{e}_2(v)$:

$$\mathbf{e}_3(v) = \mathbf{e}_1(v) \times \mathbf{e}_2(v) = \mathbf{i} \times \mathbf{e}_2(v) = -\sin v\mathbf{j} + \cos v\mathbf{k}\,.$$

We will deal only with surfaces of revolution which have parametrizations of the general form (5) (up to permutation of coordinates). It is easy to compute that, for such surfaces, we always have $F = 0 = M$ (see, for instance [12]), so the formulas for Gauss and mean curvature reduce accordingly. More detailed specification of the surface requires us to find some other important characteristics of the generating curve. This relies mostly on the derivatives of $\mathbf{x}(u,v)$. For example, the tangent vector at each point of the generating curve is given by the first derivative with respect to u,

$$\mathbf{t}(u,v) = \mathbf{x}_u(u,v) = x'(u)\mathbf{i} + z'(u)\mathbf{e}_2(v)\,. \tag{7}$$

In equation (7), and elsewhere in this paper, the prime denotes a derivative with respect to the meridional arclength parameter u. Let us also introduce $\theta(u)$, which measures the angle between the tangent vector $\mathbf{t}$ and $\mathbf{i}$. Then, the coordinates $x(u)$ and $z(u)$ depend on $\theta(u)$ via the equations

$$x'(u) = \cos\theta(u) \tag{8}$$

$$z'(u) = \sin\theta(u). \tag{9}$$

Using these equations, we can express the tangent vector as

$$\mathbf{t}(u,v) = \cos\theta(u)\mathbf{i} + \sin\theta(u)\mathbf{e}_2(v)\,. \tag{10}$$

By differentiating the last relation with respect to the parameter u, we get

$$\mathbf{x}_{uu} = -(\sin\theta(u)\mathbf{i} - \cos\theta(u)\mathbf{e}_2(v))\,\theta'(u)\,. \tag{11}$$

Next, we compute the first and second order derivatives of $\mathbf{x}(u,v)$ with respect to v:

$$\mathbf{x}_v = z(u)(\mathbf{e}_2(v))_v = z(u)\mathbf{e}_3(v) \tag{12}$$

$$\mathbf{x}_{vv} = z(u)(\mathbf{e}_3(v))_v = -z(u)\mathbf{e}_2(v) \tag{13}$$

and finally, the mixed derivative

$$\mathbf{x}_{uv} = \mathbf{x}_{vu} = \cos\theta(u)\mathbf{e}_3(v)\,. \tag{14}$$

Another important object that we will need to know is the unit normal vector, which is easily found to be

$$\mathbf{n}(u,v) = \sin\theta(u)\mathbf{i} - \cos\theta(u)\mathbf{e}_2(v)\,. \tag{15}$$

The last couple of relations are sufficient to obtain the coefficients of the first fundamental form of $\mathcal{S}$,

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = 1, \qquad F = \mathbf{x}_u \cdot \mathbf{x}_v = 0, \qquad G = \mathbf{x}_v \cdot \mathbf{x}_v = z^2(u)\,. \tag{16}$$

We also find the second fundamental form of $\mathcal{S}$ to be

$$\begin{gathered} L = \mathbf{n}\cdot\mathbf{x}_{uu} = -\theta'(s)\,, \qquad M = \mathbf{n}\cdot\mathbf{x}_{uv} = 0 \\ N = \mathbf{n}\cdot\mathbf{x}_{vv} = z(u)\cos\theta(u)\,. \end{gathered} \tag{17}$$

Once we have these, we can easily find the *mean curvature* H of $\mathcal{S}$. We now make use of the standard formula for H (see (4)) which appears in the textbooks on classical differential geometry (see, for example, [12]),

$$H = \frac{1}{2}(k_\mu + k_\pi)$$

where

$$k_\mu = L/E = -\theta'(u) \qquad \text{and} \qquad k_\pi = N/G = \frac{\cos\theta(u)}{z(u)} \tag{18}$$

are the *principal curvatures* along the respective meridional and parallel directions, to express the mean curvature in the form

$$H = -\frac{1}{2}\left(\theta'(u) - \frac{\cos\theta(u)}{z(u)}\right)\,. \tag{19}$$

3. Unduloid Geometry

Elsewhere (see [10]) the following result was shown.

Theorem 3.1. *The profile curve of an unduloid has a parametrization*

$$
\begin{aligned}
x(u) &= aF\left(\frac{\mu u}{2} - \frac{\pi}{4}, k\right) + cE\left(\frac{\mu u}{2} - \frac{\pi}{4}, k\right) \\
z(u) &= \sqrt{m \sin \mu u + n}
\end{aligned}
\tag{20}
$$

where

$$
\mu = \frac{2}{a+c}\,, \quad k^2 = (c^2 - a^2)/c^2\,, \quad m = (c^2 - a^2)/2\,, \quad n = (c^2 + a^2)/2 \tag{21}
$$

and the elliptic integrals of the first kind $F(\varphi, k)$ and second kind $E(\varphi, k)$ are functions of the two real parameters a and c.

From the representation specified in (5), any profile curve $(x(u), z(u))$ produces a surface of revolution of the form

$$
\mathbf{x}(u, v) = \big(x(u),\, z(u)\cos v,\, z(u)\sin v\big)\,, \quad u \in \mathbb{R}\,, \quad 0 \le v < 2\pi\,. \tag{22}
$$

If we insert the explicit expressions from (20) into this representation and vary appropriately the free parameters a and c, we can produce four of the six families of surfaces in Delaunay list: namely, unduloids, nodoids, spheres and cylinders (more details can be found in [4] and [9]).

Having the explicit form of the parametrization generated by the meridional section of the surface, we can easily find corresponding geometrical characteristics. To do this, we need only the first and second fundamental forms of the surface

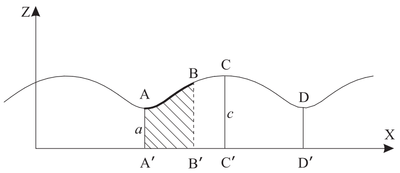

Figure 1: A half of the meridional section of a finite segment of the unduloid.

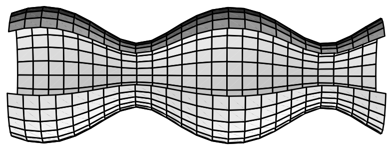

Figure 2: An open part of the unduloid generated by a partial revolution of the profile curve shown on the left.

under consideration. In our case, these forms are

$$I = \mathrm{d}u^2 + \frac{1}{2}\Big(a^2 + c^2 + (c^2 - a^2)\sin\Big(\frac{2u}{a+c}\Big)\Big)\,\mathrm{d}v^2 \tag{23}$$

$$II = \frac{(c-a)(c-a+(a+c)\sin(\frac{2u}{a+c}))}{(a+c)(a^2+c^2+(c^2-a^2)\sin(\frac{2u}{a+c}))}\,\mathrm{d}u^2 + \frac{1}{2}\Big(a+c+(c-a)\sin\Big(\frac{2u}{a+c}\Big)\Big)\,\mathrm{d}v^2 \tag{24}$$

and we are led to

$$k_\mu = \frac{(c-a)\;(c-a+(a+c)\;\sin(\frac{2u}{a+c})\,)}{(a+c)\;(a^2+c^2+(c^2-a^2)\;\sin(\frac{2u}{a+c})\,)} \tag{25}$$

$$k_\pi = \frac{a+c+(c-a)\sin(\frac{2u}{a+c})}{a^2+c^2+(c^2-a^2)\sin(\frac{2u}{a+c})} \tag{26}$$

and we have

Theorem 3.2. *The respective mean and Gauss curvatures of the unduloid having profile curve as in Theorem 3.1 are*

$$H = \frac{1}{a+c} \tag{27}$$

and

$$K = \frac{1 - \left(\dfrac{a\,c}{z^2(u)}\right)^2}{(a+c)^2} \tag{28}$$

where $z(u)$ is as in Theorem 3.1.

Proof. The formula for H follows easily by adding the principal curvatures and dividing by 2. The formula for K will be proved using Maple in §5. □

The formulas above for the principle curvatures show that, if we take $a = 0$, $c = 0$, and $c = a$, we indeed get spheres in the first two cases and a cylinder in the third — in full agreement with the classification presented in [9]. Notice also that combining (18), (20) and (26) allows us to write down the last relation in the form

$$\cos\theta(u) = \frac{z(u)}{a+c} + \frac{ac}{(a+c)z(u)} \tag{29}$$

which, according to Eells [4], should be recognized as the *Gauss map* or general equation describing the Delaunay surfaces. As we shall see very soon, all formulas above are indispensable in problems such as finding the length, surface area or volume of the unduloid.

Before we find these quantities, let's give a geometrical interpretation of the numerical parameters a and c which appear in the formulas above. Note first that, as $\sin\mu u$ oscillates between -1 and $+1$, the function $z(u)$ in (20) takes values between a and c. Since $\sin\mu u = -1$ corresponds to the minimum of $z(u)$

and $\sin \mu u = 1$ to its maximum, it follows that $c \geq a$. However, for the non-degenerate unduloids in which we are interested, we have the strict inequality $c > a$. This condition will be assumed from now on.

Because we are interested in finding the corresponding coordinates of the points depicted in Fig. 1, we choose A as a starting point, for which we have

$$\frac{2\mathring{u}}{a+c} = -\frac{\pi}{2} \quad \Longrightarrow \quad \mathring{u} = -\frac{\pi(a+c)}{4} \cdot \tag{30}$$

The arclength of the curve from A to any other point B with coordinate u is given by the integral

$$\mathcal{L}(A, B) = \int_{\mathring{u}}^{u} \mathrm{d}\tilde{u} = u - \mathring{u} = u + \frac{\pi(a+c)}{4} \cdot \tag{31}$$

In particular, we have the

Proposition 3.3. *The distance along the meridian from the point A to the point C, for which $u = \frac{\pi(a+c)}{4}$, is given by the formula*

$$\mathcal{L}(A, C) = \frac{\pi(a+c)}{2} \cdot$$

Therefore, the path from one minimum of the profile curve to the next one along the surface of the unduloid (or of one full period) is of length $\pi\,(a+c)$.

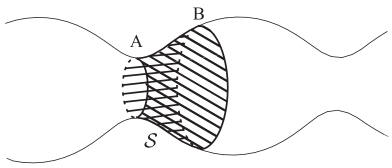

Figure 3: The area of the striped part of the unduloid surface $\mathcal{S}$ is given by the formula (37).

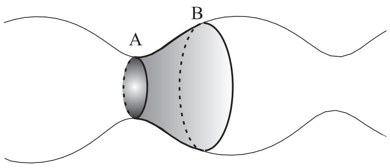

Figure 4: The dotted domain presents the volume enclosed by the unduloid surface and the two disks through the points A and B.

Next we turn to the problem of finding the area $\mathcal{A}(A, B)$ of the unduloid surface confined between the two respective circles passing through the points A and B (i.e. the striped region in Fig. 3). We have

Theorem 3.4. *The surface area $\mathcal{A}$ of one complete period of the unduloid is*

$$\mathcal{A} = 4\pi\,(a+c)\,c\,E(k)$$

where $E(k)$ denotes the complete elliptic integral of the second kind, $E(\pi/2, k)$.

Proof. We calculate surface area in the standard way from the metric of the surface.

$$\mathcal{A}(\mathcal{S}) = \mathcal{A}(A,B) = \iint\limits_{\mathcal{S}} \mathrm{d}\mathcal{A}(\mathcal{S}) = \int_0^{2\pi} \int_{\mathring{u}}^{u} \sqrt{EG - F^2}\, \mathrm{d}\tilde{u}\, \mathrm{d}\tilde{v}\,. \tag{32}$$

In our case the explicit form of the last integral on the right is (in the notation of (21))

$$\int_0^{2\pi} \int_{\mathring{u}}^{u} \sqrt{m \sin \mu\tilde{u} + n}\, \mathrm{d}\tilde{u}\, \mathrm{d}\tilde{v}\,. \tag{33}$$

After performing the trivial integration with respect to the angular variable v, we obtain a factor of 2π and end up with the problem of evaluating the integral

$$\int_{\mathring{u}}^{u} \sqrt{m \sin \mu\tilde{u} + n}\, \mathrm{d}\tilde{u}\,. \tag{34}$$

The integral can be converted into a standard elliptic integral of the second kind by making the substitutions

$$\sin \mu\tilde{u} = 1 - 2\,\mathrm{sn}^2(\tilde{t},k)\,, \qquad \mathrm{d}\tilde{u} = -\,(a+c)\mathrm{dn}(\tilde{t},k)\mathrm{d}\tilde{t}\,, \tag{35}$$

where $\mathrm{sn}(\tilde{t},k)$ and $\mathrm{dn}(\tilde{t},k)$ are Jacobian elliptic functions with argument $\tilde{t}$ and *elliptic modulus* k. (Details about elliptic functions, their integrals, and properties can be found in [8]). Choosing $k^2 = 2m/(m+n) = (c^2 - a^2)/c^2$, we end up with the canonical form of the elliptic integral of the second kind $E(\varphi, k)$. We then have

$$\begin{aligned} \mathcal{A}(A,B) &= -2\pi(a+c)c \int_{\mathring{t}}^{t} \mathrm{dn}^2(\tilde{t},k)\mathrm{d}\tilde{t} \\ &= 2\pi(a+c)c\left(E(\mathrm{am}(\mathring{t},k),k) - E(\mathrm{am}(t,k),k)\right)\,. \end{aligned} \tag{36}$$

A little bit more work shows that the Jacobian amplitude function, $\mathrm{am}(t,k)$, which appears above can be replaced by $\pi/4 - \mu u/2$, so that

$$\begin{aligned} \mathcal{A}(A,B) &= 2\pi(a+c)c\,(E(\pi/4 - \mu\mathring{u}/2,k) - E(\pi/4 - \mu u/2,k)) \\ &= 2\pi(a+c)c\,(E(k) - E(\pi/4 - \mu u/2,k)) \\ &= 2\pi(a+c)c\,(E(k) + E(\mu u/2 - \pi/4,k)) \end{aligned} \tag{37}$$

where $E(k)$ denotes the complete elliptic integral of the second kind $E(\pi/2,k)$. Again, the formula for the area of half of the period of the unduloid surface simplifies considerably. We obtain

$$\mathcal{A}(A,C) = 2\pi(a+c)cE(k)\,, \tag{38}$$

which means that the area $\mathcal{A}$ of the complete period is $4\pi(a+c)cE(k)$. □

Finally, let's find the volume of the unduloid segment enclosed between the two disks passing through the points A and B (i.e. the dotted region in Fig. 4). We have

$$
\begin{aligned}
\mathcal{V}(A,B) &= \pi \int_{\mathring{u}}^{u} z^2(\tilde{u})\,\mathrm{d}x(\tilde{u}) \\
&= \frac{\pi}{a+c}\int_{\mathring{u}}^{u} \Big(\sqrt{(m\sin\mu\tilde{u}+n)^3} + ac\sqrt{m\sin\mu\tilde{u}+n}\Big)\,\mathrm{d}\tilde{u}\,.
\end{aligned} \tag{39}
$$

Making the same substitutions as before (see (35)), we reduce the problem to the evaluation of two integrals whose integrands are powers of the Jacobian elliptic function dn:

$$
\mathcal{V}(A,B) = -\pi c^3 \int_{\mathring{t}}^{t} \mathrm{dn}^4(\tilde{t},k)\mathrm{d}\tilde{t} - \pi a c^2 \int_{\mathring{t}}^{t} \mathrm{dn}^2(\tilde{t},k)\mathrm{d}\tilde{t}\,. \tag{40}
$$

Since we have already evaluated the second integral (see (36) and (37)), it is simply a question of calculating the first. We obtain

$$
\begin{aligned}
\int_{\mathring{t}}^{t} \mathrm{dn}^4(\tilde{t},k)\mathrm{d}\tilde{t} = \frac{1}{3}\big[&k^2\mathrm{sn}(t,k)\mathrm{cn}(t,k)\mathrm{dn}(t,k) \\
&- (1-k^2)F(\mathrm{am}(t,k),k) + 2(2-k^2)E(\mathrm{am}(t,k),k)\big]\Big|_{\mathring{t}}^{t}\,.
\end{aligned}
$$

Going back to the original variable u and the geometrical parameters a and c, the right-hand side of the formula above can be rewritten as

$$
\frac{1}{3}\Big[\frac{c^2-a^2}{2c^3}\,z(u)\cos\mu u - \frac{a^2}{c^2}F(\frac{\pi-2\mu u}{4},k) + \frac{2(a^2+c^2)}{c^2}E(\frac{\pi-2\mu u}{4},k)\Big]\Big|_{\mathring{u}}^{u}\,.
$$

Therefore, we have

$$
\begin{aligned}
\mathcal{V}(A,B) = \Big[&\frac{\pi(2(a^2+c^2)c+3ac^2)}{3}E\Big(\frac{2\mu u-\pi}{4},k\Big) \\
&- \frac{\pi a^2 c}{3}F\Big(\frac{2\mu u-\pi}{4},k\Big) - \frac{\pi(c^2-a^2)}{6}\,z(u)\cos\mu u\Big]\Big|_{\mathring{u}}^{u}\,.
\end{aligned} \tag{41}
$$

We then simply evaluate the limits of integration to obtain

$$
\mathcal{V}(A,C) = \frac{\pi c}{3}\big((2a^2+3ac+2c^2)E(k) - a^2K(k)\big)\,, \tag{42}
$$

and note that we have

Theorem 3.5. *The volume $\mathcal{V}$ of one period is $\mathcal{V} = 2\,\mathcal{V}(A,C)$.*

Before we end this section, we want to make the following geometro-mechanical observation: The distance L along the X-axis between the points A' and D' (which the points A and D marking one complete period project to), is

$$
L = 2cE(k) + 2aK(k)\,. \tag{43}
$$

This equation, taken together with the equation for the volume, $\mathcal{V} = 2\mathcal{V}(A,C)$, leads to a linear system for the complete elliptic integrals $K(k)$ and $E(k)$. Hence, by making direct measurements, we are able to find the values of these integrals.

4. Parallel Surfaces of Unduloids and Maple

We can use the unduloid parametrization provided by Theorem 3.1 to not only plot unduloids, but to create certain surfaces associated to them.

Definition 4.1. If $\mathcal{M} \subseteq \mathbb{R}^3$ is a surface, then the *parallel surface* at t associated to $\mathcal{M}$ is the surface

$$\mathcal{M}^t = \mathcal{M} + t\mathcal{N}$$

where $\mathcal{N}$ is a unit normal (either inward or outward) of $\mathcal{M}$.

Recall the following standard result.

Proposition 4.2 (see [12, Exercise 3.2.6])**.** *If $\mathcal{M}^t = \mathcal{M} + t\mathcal{N}$ is a parallel surface of $\mathcal{M}$, then the Gauss and mean curvatures are given by the formulas*

$$K^t = \frac{K}{1 - 2Ht + Kt^2}, \qquad H^t = \frac{H - Kt}{1 - 2Ht + Kt^2}$$

where K and H are the Gauss and mean curvatures of $\mathcal{M}$. In particular, if $\mathcal{M}$ has constant mean curvature $H \neq 0$, then $\mathcal{M}^{1/(2H)}$ has constant Gauss curvature $K^{1/(2H)} = 4H^2$.

We have seen that the parametrization derived from Theorem 3.1 leads to constant mean curvature $H = 1/(a+c)$, so if we take the parallel surface with $t = (a+c)/2$, we will obtain a surface of constant Gauss curvature $K^t = 4/(a+c)^2$. While this abstract result is interesting, we are unaware of any explicit depictions of this phenomena. Here we will see how *Maple* can be used to see this connection between constant mean and Gauss curvature surfaces in all its beauty.

```
> with(LinearAlgebra):with(plots):
```

Maple 10 has a flaw in its EllipticE procedure. The correction is as follows. Just click on it.

```
> `evalf/Elliptic/Ell_E`:= parse(StringTools:
-Substitute(convert(eval(`evalf/Elliptic/Ell_E`),
string),"F_0","E_0")):
```

We need the following procedures to create the unit normal for a given parametrization.

```
> EFG := proc(X)
local Xu,Xv,E,F,G;
Xu := <diff(X[1],u),diff(X[2],u),diff(X[3],u)>;
Xv := <diff(X[1],v),diff(X[2],v),diff(X[3],v)>;
E := DotProduct(Xu,Xu,conjugate=false);
F := DotProduct(Xu,Xv,conjugate=false);
G := DotProduct(Xv,Xv,conjugate=false);
simplify([E,F,G]);
end:
```

```
> UN := proc(X)
local Xu,Xv,Z,s;
Xu := <diff(X[1],u),diff(X[2],u),diff(X[3],u)>;
```

```
Xv := <diff(X[1],v),diff(X[2],v),diff(X[3],v)>;
Z := CrossProduct(Xu,Xv);
s:=VectorNorm(Z,Euclidean,conjugate=false);
simplify(<Z[1]/s|Z[2]/s|Z[3]/s>,sqrt,trig,symbolic);
end:
```

Now let's make a parallel surface to the unduloid with constant Gauss curvature. Note that $\mu = \frac{1}{2H}$, so we just need to multiply the unit normal by μ and add this to the unduloid. Of course, we can add or subtract, so the procedure allows the user to input +1 or -1 for the input pm. The input +1 takes the unduloid to a surface of constant Gauss curvature inside the unduloid, while -1 takes the unduloid to a surface of constant Gauss curvature outside the unduloid.

```
> und_par:=proc(a,c,pm,orient1,orient2)
local sty1,sty2,k,mu,m,n,phi,ulimup,ulimdown,undy,
undyplot,parallelsurf;
if pm=1 then sty1:=`wireframe`;sty2:=`patch`;fi;
if pm=-1 then sty2:=`wireframe`;sty1:=`patch`;fi;
k:=sqrt(c^2-a^2)/c;
mu:=2/(a+c);
m:=(c^2-a^2)/2;
n:=(c^2+a^2)/2;
phi:=mu*u/2-Pi/4;
ulimup:=fsolve(sin(phi)=1,u);
ulimdown:=fsolve(sin(phi)=0,u);
undy:=<(a*EllipticF(sin(phi),k)+c*EllipticE(sin(phi),k))|
sqrt(m*sin(mu*u)+n)*cos(v)|sqrt(m*sin(mu*u)+n)*sin(v)>;
undyplot:=plot3d(subs({u=u(t),v=v(t)},convert(undy,list)),
u=-ulimdown..ulimup,v=0..2*Pi,scaling=constrained,
style=sty1);
parallelsurf:=plot3d(subs({u=u(t),v=v(t)},
convert(undy+pm*1/mu*UN(undy),list)),u=-ulimdown..ulimup,
v=0..2*Pi,scaling=constrained,style=sty2);
display(undyplot,parallelsurf,orientation=[orient1,
orient2]);
end:
```

Here are some examples of unduloids and their parallel surfaces of constant Gauss curvature. Compare with the plots of constant Gauss curvature surfaces in [12, Chapter 3].

```
> und_par(0.2,0.4,1,48,73);
> und_par(0.2,1.2,1,43,78);
> und_par(0.2,0.4,-1,48,73);
> und_par(0.2,1.2,-1,43,78);
```

The final two commands show spheres and their parallel surfaces (which, of course, are also spheres!). We leave the actual pictures to the reader.

```
> und_par(0,1.2,1,43,78);
> und_par(0,1.2,-1,43,78);
```

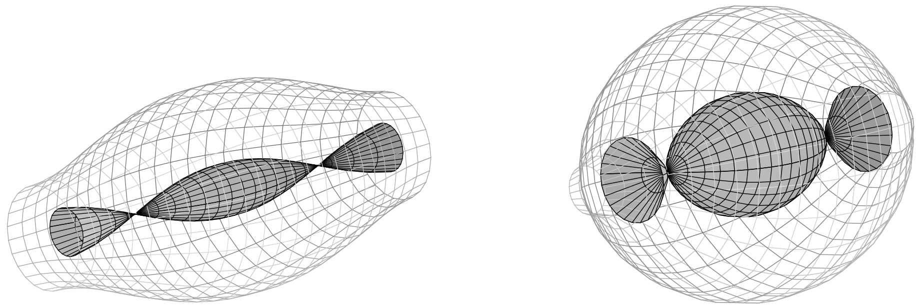

Figure 5: Unduloids and their +1 parallel constant K surfaces.

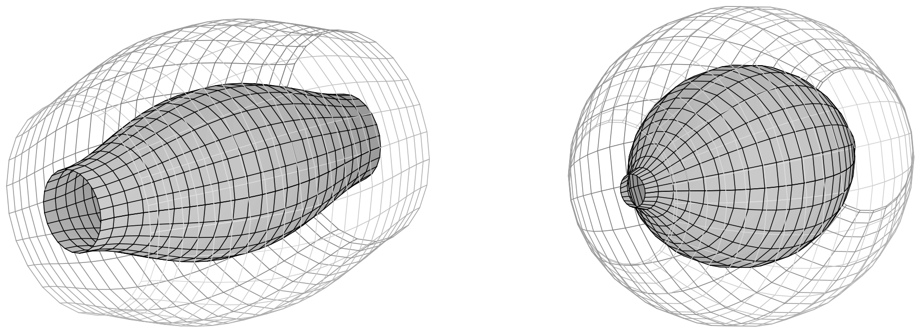

Figure 6: Unduloids and their -1 parallel constant K surfaces.

We have seen in [11] that forces in nature can produce shapes such as unduloids. It is also true that pressure-like forces tend to push out (or in) along the normal to a surface. Therefore, we might ask whether there are instances in nature where constant mean curvature surfaces spontaneously transform themselves into surfaces of constant Gauss curvature?

5. Appendix: Maple Proof of K Formula

In Theorem 3.2, we gave a formula for the Gauss curvature of the unduloid in terms of the function $z(u)$. We verify this now by using *Maple*. The following give the principal curvatures for the unduloid.

```
>  k[mu]:=(c-a)*(c-a+(c+a)*sin(2*u/(c+a)))
/((c+a)*(c^2+a^2+(c^2-a^2)*sin(2*u/(c+a))));
```

$$k_\mu = \frac{(c-a)\,(c-a+(c+a)\,\sin(\frac{2\,u}{c+a}))}{(c+a)\,(c^2+a^2+(c^2-a^2)\,\sin(\frac{2\,u}{c+a}))}$$

```
> k[pi]:=(c+a+(c-a)*sin(2*u/(c+a)))/
(c^2+a^2+(c^2-a^2)*sin(2*u/(c+a)));
```

$$k_\pi = \frac{c + a + (c - a)\,\sin(\frac{2\,u}{c+a})}{c^2 + a^2 + (c^2 - a^2)\,\sin(\frac{2\,u}{c+a})}$$

We compute the Gauss curvature two ways: first by taking the product of the principal curvatures and second, by the formula for Gauss curvature given Theorem 3.2.

```
> K:=simplify(k[mu]*k[pi],symbolic,trig);
```

$$K = \frac{(c - a)\,(c - a + (c + a)\,\sin(\frac{2\,u}{c+a}))\,(c + a + (c - a)\,\sin(\frac{2\,u}{c+a}))}{(c + a)\,(c^2 + a^2 + (c^2 - a^2)\,\sin(\frac{2\,u}{c+a}))^2}$$

```
> K2:=simplify(eval((1-(a*c/z^2)^2)/(a+c)^2,
z=sqrt((c^2-a^2)/2*sin(2*u/(c+a))+(c^2+a^2)/2)));
```

$$\begin{aligned}(44)\quad K2 = {}& (a^3 - 2\,a^3\sin(\frac{2\,u}{c+a}) + a^3\sin(\frac{2\,u}{c+a})^2 \\ & - a^2\,c + 2\,a^2\sin(\frac{2\,u}{c+a})\,c - a^2\sin(\frac{2\,u}{c+a})^2\,c - a\,c^2 \\ & - a\sin(\frac{2\,u}{c+a})^2\,c^2 - 2\,a\sin(\frac{2\,u}{c+a})\,c^2 + c^3 + \sin(\frac{2\,u}{c+a})^2\,c^3 \\ & + 2\sin(\frac{2\,u}{c+a})\,c^3) \Big/ ((c + a)\,(-c^2 - a^2 - \sin(\frac{2\,u}{c+a})\,c^2 + \sin(\frac{2\,u}{c+a})\,a^2)^2)\end{aligned}$$

To verify that these are the same, we compute the following to be zero.

```
> simplify(K-K2);
```

$$0$$

Therefore, since the difference in the expressions for K vanishes, we see that the beautiful formula for Gauss curvature,

$$K = \frac{1 - \left(\dfrac{a\,c}{z^2(u)}\right)^2}{(a + c)^2} \tag{45}$$

in Theorem 3.2 holds.

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