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ON ENDOMORPHISMS OF MULTIPLICATION AND COMULTIPLICATION MODULES

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Abstract. Let R be a ring with an identity (not necessarily commutative) and let M be a left R-module. This paper deals with multiplication and comultiplication left R-modules M having right $\operatorname{End}_R(M)$-module structures.

1. Introduction

Throughout this paper R will denote a ring with an identity (not necessarily commutative) and all modules are assumed to be left modules. Further "$\subset$" will denote the strict inclusion and $\mathbb{Z}$ will denote the ring of integers.

Let M be a left R-module and let $S := \operatorname{End}_R(M)$ be the endomorphism ring of M. Then M has a structure as a right S-module so that M is an $R-S$ bimodule. If $f\colon M \to M$ and $g\colon M \to M$, then $fg\colon M \to M$ defined by $m(fg) = (mf)g$. Also for a submodule N of M,

$$I^N := \{ f \in S : \operatorname{Im}(f) = Mf \subseteq N \}$$

and

$$I_N := \{ f \in S : N \subseteq \operatorname{Ker}(f) \}$$

are respectively a left and a right ideal of S. Further a submodule N of M is called ([3]) an open (resp. a closed) submodule of M if $N = N^\circ$, where $N^\circ = \sum_{f \in I^N} \operatorname{Im}(f)$ (resp. $N = \bar{N}$, where $\bar{N} = \cap_{f \in I_N} \operatorname{Ker}(f)$). A left R-module M is said to self-generated (resp. self-cogenerated) if each submodule of M is open (resp. is closed).

Let M be an R-module and let $S = \operatorname{End}_R(M)$. Recently a large body of researches has been done about multiplication left R-module having right S-module structures. An R-module M is said to be a multiplication R-module if for every submodule N of M there exists a two-sided ideal I of R such that $N = IM$.

In [2], H. Ansari-Toroghy and F. Farshadifar introduced the concept of a comultiplication R-module and proved some results which are dual to those of multiplication R-modules. An R-module M is said to be a *comultiplication R-module* if for every submodule N of M there exists a two-sided ideal I of R such that $N = (0 :_M I)$.

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This paper deals with multiplication and comultiplication left R-modules M having right $\mathrm{End}_R(M)$-modules structures. In section three of this paper, among the other results, we have shown that every comultiplication R-module is co-Hopfian and generalized Hopfian. Further if M is a comultiplication module satisfying ascending chain condition on submodules N such that M/N is a comultiplication R-module, then M satisfies Fitting's Lemma. Also it is shown that if R is a commutative ring and M is a multiplication R-module and S is a domain, then for every maximal submodule P of M, I^P is a maximal ideal of S.

2. Previous results

In this section we will provide the definitions and results which are necessary in the next section.

Definition 2.1.

(a) M is said to be (see [9]) a *multiplication R-module* if for any submodule N of M there exists a two-sided ideal I of R such that $N = IM$.

(b) M is said to be a *comultiplication R-module* if for any submodule N of M there exists a two-sided ideal I of R such that $N = (0 :_M I)$. For example if p is a prime number, then $\mathbb{Z}(p^\infty)$ is a comultiplication $\mathbb{Z}$-module but $\mathbb{Z}$ (as a $\mathbb{Z}$-module) is not a comultiplication module (see [2]).

(c) Let N be a non-zero submodule of M. Then N is said to be (see [1]) *large or essential* (resp. *small*) if for every non-zero submodule L of M, $N \cap L \neq 0$ (resp. $L + N = M$ implies that $L = M$).

(d) M is said to be (see [7]) *Hopfian* (resp. *generalized Hopfian* (*gH* for short)) if every surjective endomorphism f of M is an isomorphism (resp. has a small kernel).

(e) M is said to be (see [8]) *co-Hopfian* (resp. *weakly co-Hopfian*) if every injective endomorphism f of M is an isomorphism (resp. an essential homomorphism).

(f) An R-module M is said to satisfy *Fitting's Lemma* if for each $f \in \mathrm{End}_R(M)$ there exists an integer $n \geq 1$ such that $M = \mathrm{Ker}(f^n) \bigoplus \mathrm{Im}(f^n)$ (see [5]).

(g) Let M be an R-module and let I be an ideal of R. Then IM is called to be *idempotent* if $I^2M = IM$.

3. Main results

Lemma 3.1. *Let R be any ring. Every comultiplication R-module is co-Hopfian.*

Proof. Let M be a comultiplication R-module and let $f\colon M \to M$ be a monomorphism. There exists a two-sided ideal I of R such that $\mathrm{Im}(f) = (0 :_M I)$. Now let $m \in M$ so that $mf \in \mathrm{Im}(f)$. Then for each $a \in I$, we have $(am)f = a(mf) = 0$. It follows that $am \in \mathrm{Ker}(f) = 0$. This implies that $am = 0$ so that $m \in (0 :_M I) = Mf$. Hence we have $M \subseteq Mf$ so that f is epic. It follows that M is a co-Hopfian R-module. □

The following examples shows that not every comultiplication (resp. Artinian) R-module is an Artinian (resp. a comultiplication) R-module.

Example 3.2. Let p be a prime number. Then let R be the ring with underlying group

$$R = \mathrm{End}_{\mathbb{Z}}\left(\mathbb{Z}(p^{\infty})\right) \oplus \mathbb{Z}(p^{\infty}),$$

and with multiplication

$$(n_1, q_1) \cdot (n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1).$$

Osofsky has shown that R is a non-Artinian injective cogenerator (see [6, Exa. 24.34.1]). In fact R is a commutative ring. Hence R is a comultiplication R-module by [6, Prop. 23.13].

Example 3.3. Let F be a field, and let $M = \oplus_{i=1}^{n} F_i$, where $F_i = F$ for $i = 1, 2, \ldots, n$. Clearly M is an Artinian non-comultiplication F-module.

Theorem 3.4. *Let M be a comultiplication module satisfying ascending chain condition on submodules N such that M/N is a comultiplication R-module. Then M satisfies Fitting's Lemma.*

Proof. Let $f \in \mathrm{End}_R(M)$ and consider the sequence

$$\mathrm{Ker}\, f \subseteq \mathrm{Ker}\, f^2 \subseteq \cdots .$$

Since every submodule of a comultiplication R-module is a comultiplication R-module by [2], for each n we have $M/\mathrm{Ker}\, f^n \cong Imf^n$ implies that $M/\mathrm{Ker}\, f^n$ is a comultiplication R-module. Hence by hypothesis there exists a positive integer n such that $\mathrm{Ker}(f^n) = \mathrm{Ker}(f^{n+h})$ for all $h \geq 1$. Set $f_1^n = f^n \mid_{M(f^n)}$. Then $f_1^n \in \mathrm{End}_R(M(f^n))$. Further we will show that f_1^n is monic. To see this let $x \in \mathrm{Ker}(f_1^n)$. Then $x = y(f^n)$ for some $y \in M$ and we have $x(f^n) = 0$. It follows that $y(f^{2n}) = 0$ so that

$$y \in \mathrm{Ker}(f^{2n}) = \mathrm{Ker}(f^n).$$

Hence we have $x = 0$. But $(M)f^n$ is a comultiplication R-module and every comultiplication R-module is co-Hopfian by Lemma 3.1. So we conclude that f_1^n is an automorphism. In particular, $M(f^n) \cap \mathrm{Ker}(f^n) = 0$. Now let $x \in M$. Since f_1^n is epimorphism, then there exists $y \in M$ such that $x(f^n) = y(f^{2n})$. Hence $(x - y(f^n))(f^n) = 0$. It follows that $x - y(f^n) \in \mathrm{Ker}(f^n)$. Now the result follows from this because $x = y(f^n) + (x - y(f^n))$. □

Corollary 3.5. *Let M be an indecomposable comultiplication module satisfying ascending chain condition on submodules N such that M/N is a comultiplication R-module. Let $f \in \mathrm{End}_R(M)$. Then the following are equivalent.*

(i) *f is a monomorphism.*

(ii) *f is an epimorphism.*

(iii) *f is an automorphism.*

(iv) *f is not nilpotent.*

Proof. (i)$\Rightarrow$(ii). This is clear by Lemma 3.1.

(iii)$\Rightarrow$(ii). This is clear.

(iii)$\Rightarrow$(iv). Assume that f is an automorphism. Then $M = Mf$. Hence,

$$M = Mf = M(f^2) = \cdots .$$

If f were nilpotent, then M would be zero.

(ii)$\Rightarrow$(i). Assume that f is an epimorphism. Then $M = Mf$. Hence

$$M = Mf = M(f^2) = \cdots .$$

By Theorem 3.4, there is a positive integer n such that

$$M = \mathrm{Ker}(f^n) \oplus \mathrm{Im}(f^n) .$$

Hence $M = \mathrm{Ker}(f^n) \oplus M$, so $\mathrm{Ker}(f^n) = 0$. Thus, $\mathrm{Ker}(f) = 0$.

(ii)$\Rightarrow$(iii). This follows from (ii)$\Rightarrow$(i).

(iv)$\Rightarrow$(iii). Suppose that f is not nilpotent. By Theorem 3.4, there exists a positive integer n such that $M = Mf^n \bigoplus \mathrm{Ker}\, f^n$. Since M is indecomposable R-module, it follows that $\mathrm{Ker}\, f^n = 0$ or $Mf^n = 0$. Since f is not nilpotent, we must have $\mathrm{Ker}\, f^n = 0$. This implies that f is monic. This in turn implies that f is epic by Lemma 3.1. Hence the proof is completed. □

Example 3.6. Let $A = K[x, y]$ be the polynomial ring over a field K in two indeterminates x, y. Then $\overline{A} = A/(x^2, y^2)$ is a comultiplication $\overline{A}$-module. But $\overline{A}/\overline{A}\overline{xy}$ is not a comultiplication $\overline{A}$-module (see [6, Exa. 24.4]). Therefore, not every homomorphic image of a comultiplication module is a comultiplication module.

Remark 3.7. In the Corollary 3.5 the condition M satisfying ascending chain condition on submodules N such that M/N is a comultiplication R-module can not be omitted. For example $M = \mathbb{Z}(p^\infty)$ is an indecomposable comultiplication $\mathbb{Z}$-module but not satisfying ascending chain condition on submodules N such that M/N is a comultiplication $\mathbb{Z}$-module. Define $f \colon \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$ by $x \to px$. Clearly f is an epimorphism with $\mathrm{Ker}\, f = \mathbb{Z}(1/p + \mathbb{Z})$. Hence f is not a monomorphism.

Lemma 3.8. *Let M be a comultiplication R-module and let N be an essential submodule of M. If the right ideal I_N of $\mathrm{End}_R(M)$ is non-zero, then it is small in $\mathrm{End}_R(M)$.*

Proof. Let J be any right ideal of $S = \mathrm{End}_R(M)$ such that $I_N + J = S$. Then $1_M = f + j$ for some $f \in I_N$ and $j \in J$. Since $\mathrm{Ker}(1_M - f) \cap N = 0$ and N is an essential submodule of M, it follows that j is a monomorphism. Hence by Lemma 3.1, j is an automorphism so that $J = S$. Hence I_N is a small right ideal of S. □

Proposition 3.9. *Let M be a comultiplication R-module and let N be a submodule of M such that M/N is a faithful R-module. Then M/N is a co-Hopfian R-module.*

Proof. Let $f \colon M/N \to M/N$ be an R-monomorphism and $(M/N)f = K/N$, with $N \subseteq K \subseteq M$. Since M is a comultiplication R-module there exists a two-sided ideal I of R such that $K = (0 :_M I)$. Now

$$(I(M/N))f = I(M/N)f = I(K/N) = 0 .$$

Since f is monic, it follows that $I(M/N) = 0$. This in turn implies that $I \subseteq \mathrm{Ann}_R(M/N) = 0$. Hence we have $K = M$ so that f is an epimorphism. $\square$

Lemma 3.10. *Every comultiplication R-module is gH.*

Proof. Let M be comultiplication R-module and let $f\colon M \to M$ be an epimorphism and assume that $\mathrm{Ker}(f) + K = M$, where K is a submodule of M. So $Kf = Mf = M$. Since M is a comultiplication module, there exists a two-sided ideal J of R such that $K = (0 :_M J)$. Now

$$0 = 0f = \big(J(0 :_M J)\big)f = J(Kf) = JM\,.$$

It follows that $J \subseteq \mathrm{Ann}_R(M)$. Hence we have $K = (0 :_M J) = M$. This shows that $\mathrm{Ker}(f)$ is a small submodule of M. So the proof is completed. $\square$

Proposition 3.11.

(a) *Assume that whenever $f, g \in \mathrm{End}_R(M)$ with $fg = 0$ then we have $gf = 0$. If M is a self-generated (resp. self-cogenerated) R-module, then M is Hopfian (resp. co-Hopfian).*

(b) *Let M be a self-generated (resp. self-cogenerated) R-module and let S be a left Noetherian (resp. right Artinian) ring. Then M is a Noetherian S-module.*

Proof. (a) Let $S = \mathrm{End}_R(M)$ and let $g\colon M \to M$ be an epimorphism. Let f be any element of $I^{\mathrm{Ker}(g)}$. Then $Mf \subseteq \mathrm{Ker}(g)$, so $M(fg) = (Mf)g = 0$. Hence, $fg = 0$. By our assumption, $gf = 0$. Since g is an epimorphism, we have

$$Mf = (Mg)f = M(gf) = 0\,.$$

Thus, if M is self-generated,

$$\mathrm{Ker}(g) = \sum_{f \in I^{\mathrm{Ker}(g)}} \mathrm{Im}(f) = 0\,.$$

Hence M is a Hopfian R-module. The proof is similar when M is a self-cogenerated R-module.

(b) Let

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$$

be an ascending chain of S-submodules of M. This induces the sequence

$$I^{N_1} \subseteq I^{N_2} \subseteq \cdots \subseteq I^{N_k} \subseteq \cdots\,.$$

Now there exists a positive integer s such that for each $0 \leq i$, $I^{N_s} = I^{N_{i+s}}$. Since M is a self-generated R-module, we have $N_s = MI^{N_s} = MI^{N_{i+s}} = N_{i+s}$ for every $0 \leq i$. Thus M is a Noetherian S-module. For right Artinian case when M is a self-cogenerator R-module, the proof is similar. So the proof is completed. $\square$

Theorem 3.12. *Let M be a multiplication R-module and let N be a submodule of M.*

(a) *If R is a commutative ring, and I is an ideal of R such that IM is an idempotent submodule of M, then IM is gH.*

(b) *If R is a commutative ring and N is faithful, then N is weakly co-Hopfian.*

(c) *If M is a quasi-injective, N is gH.*

Proof. (a) Let I be an ideal of R such that IM be an idempotent submodule of M. Let $f : IM \to IM$ be an epimorphism and assume that $\mathrm{Ker}(f) + L = IM$, where L is a submodule of IM. Then we have $I\big(\mathrm{Ker}(f)\big) + IL = IM$. Let $\mathrm{Ker}(f) = JM$ for some ideal J of R. Since R is a commutative ring, we have

$$0 = I\big(\mathrm{Ker}(f)\big)f = (IJM)f = J(IM)f = JIM = IJM = I\big(\mathrm{Ker}(f)\big)\,.$$

Thus by the above arguments, $IL = IM$ so that $IM \subseteq L$. It follows that $IM = L$ so that IM is a generalized Hopfian R-module.

(b) Let I be an ideal of R such that $N = IM$. Let $f\colon N \to N$ be an injective homomorphism and assume that $Nf \cap K = 0$, where K is a submodule of N. Then there exist ideals J_1 and J_2 of R such that $Nf = J_1M$ and $K = J_2M$. Then we have

$$0 = K \cap Nf = K \cap (IM)f = (J_2M) \cap (IM)f = J_2M \cap J_1M \supseteq J_2J_1M\,.$$

Hence $J_2J_1M = 0$. Now we have

$$(IJ_2M)f = J_2(IM)f = J_2J_1M = 0\,.$$

Since f is monic, $J_2N = IJ_2M = 0$. Since N is a faithful R-module, we have $J_2 = 0$ so that $K = 0$. Hence Nf is essential in N. It implies that N is a weakly co-Hopfian R-module as desired.

(c) Let $f\colon N \to N$ be an epimorphism and let $\mathrm{Ker}(f) + K = N$, where K is a submodule of N. Since M is quasi-injective, we can extend f to $g\colon M \to M$. But as M is a multiplication module, $Kg \subseteq K$, therefore $Kf \subseteq K$. On the other hand, $Kf = N$ since f is epimorphism. Therefore $K = N$. Hence N is a generalized Hopfian R-module as desired. □

Proposition 3.13. *Let R be a commutative ring and let M be a multiplication R-module. Let $S = \mathrm{End}_R(M)$ be a domain. Then the following assertions hold.*

(a) *Each non-zero element of S is a monomorphism.*

(b) *If I and J are ideals of S such that $I \neq J$, then $MI \neq MJ$.*

Proof. (a) Assume that $0 \neq g \in S$. Then there exist ideals I and J of R such that $\mathrm{Im}(g) = JM$ and $\mathrm{Ker}(g) = IM$. Now we have

$$0 = \big(\mathrm{Ker}(g)\big)g = (IM)g = I(Mg) = IJM\,.$$

It implies that $IJ \subseteq \mathrm{Ann}_R(M)$. Since S is a domain, $\mathrm{Ann}_R(M)$ is a prime ideal of R by [2, 2.3]. Hence $I \subseteq \mathrm{Ann}_R(M)$ or $J \subseteq \mathrm{Ann}_R(M)$ so that $IM = 0$ or $JM = 0$. It turns out that $\mathrm{Ker}(g) = 0$ as desired.

(b) Since R is a commutative ring, M is a multiplication S-module. Hence for $0 \neq m \in M$ there exists an ideal K of S such that $mS = MK$. Now we assume that $MI = MJ$. Since R is a commutative ring, S is a commutative ring by [4]. Hence

$$mI = mSI = (MK)I = (MI)K = (MJ)K = (MK)J = mSJ = mJ\,.$$

Choose $f \in I \setminus J$. Then since $mf \in mI = mJ$, there exists $h \in J$ such that $mh = mf$. Thus we have $m(h - f) = 0$. Further $h - f \neq 0$. So by using part (a), we have $m \in \mathrm{Ker}(h - f) = 0$. But this is a contradiction and the proof is completed. □

Corollary 3.14. *Let R be a commutative ring and M be a multiplication R-module. Set $S = \mathrm{End}_R(M)$ and $\mathrm{Im}(J) = \sum_{f \in J} \mathrm{Im}(f)$, where J is an ideal of S. If J is a proper ideal of a domain S, then $\mathrm{Im}(J)$ is a proper submodule of M.*

Proof. This is an immediate consequence of Proposition 3.13 (b). □

Theorem 3.15. *Let R be a commutative ring and let M be a multiplication R-module such that $S = \mathrm{End}_R(M)$ is a domain. Then for every maximal submodule P of M, I^P is a maximal ideal of S.*

Proof. Since $\mathrm{Id}_M \in S$ and $\mathrm{Id}_M \notin I^P$, we have $I^P \neq S$. Now assume that U is an ideal of S such that $I^P \subseteq U \subseteq S$. Then if $MU = M$, then by Corollary 3.14, $U = S$. If $MU = P$, then $U \subseteq I^P$, so $U = I^P$. Hence I^P is a maximal ideal of S and the proof is completed. □

Example 3.16. Let R be a commutative ring and let P be a prime ideal of R. Set $M = R/P$. Then M is a multiplication R-module and $S = \mathrm{End}_R(M)$ is a domain. Hence by Theorem 3.15, for every maximal submodule N of M, I^N is a maximal ideal of S.

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