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ON THE OSCILLATORY INTEGRATION OF SOME ORDINARY DIFFERENTIAL EQUATIONS

Octavian G. Mustafa

Abstract. Conditions are given for a class of nonlinear ordinary differential equations $x'' + a(t)w(x) = 0$, $t \geq t_0 \geq 1$, which includes the linear equation to possess solutions $x(t)$ with prescribed oblique asymptote that have an oscillatory pseudo-wronskian $x'(t) - \frac{x(t)}{t}$.

1. Introduction

A certain interest has been shown recently in studying the existence of bounded and positive solutions to a large class of elliptic partial differential equations which can be displayed as

$$\Delta u + f(x,u) + g(|x|)x \cdot \nabla u = 0\,, \qquad x \in G_R\,, \tag{1}$$

where $G_R = \{x \in \mathbb{R}^n : |x| > R\}$ for any $R \geq 0$ and $n \geq 2$. We would like to mention the contributions [3], [1], [8] – [11], [13, 14], [18] and their references in this respect.

It has been established, see [8, 9], that it is sufficient for the functions f, g to be Hölder continuous, respectively continuously differentiable in order to analyze the asymptotic behavior of the solutions to (1) by the comparison method [15]. In fact, given $\zeta > 0$, let us assume that there exist a continuous function $A\colon [R,+\infty) \to [0,+\infty)$ and a nondecreasing, continuously differentiable function $W\colon [0,\zeta] \to [0,+\infty)$ such that

$$0 \leq f(x,u) \leq A\big(|x|\big)W(u) \quad \text{for all} \quad x \in G_R,\, u \in [0,\zeta]$$

and $W(u) > 0$ when $u > 0$. Then we are interested in the positive solutions $U = U\big(|x|\big)$ of the elliptic partial differential equation

$$\Delta U + A\big(|x|\big)W(U) = 0\,, \qquad x \in G_R\,,$$

for the role of super-solutions to (1).

M. Ehrnström [13] noticed that, by imposing the restriction

$$x \cdot \nabla U(x) \leq 0\,, \qquad x \in G_R\,,$$

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upon the super-solutions U, an improvement of the conclusions from the literature is achieved for the special subclass of equations (1) where g takes only nonnegative values. Further developments of Ehrnström's idea are given in [3, 1, 11, 14].

Translated into the language of ordinary differential equations, the research about U reads as follows: given c_1, $c_2 \geq 0$, find (if any) a positive solution $x(t)$ of the nonlinear differential equation

$$x'' + a(t)w(x) = 0\,, \qquad t \geq t_0 \geq 1\,, \tag{2}$$

where the coefficient $a\colon [t_0, +\infty) \to \mathbb{R}$ and the nonlinearity $w\colon \mathbb{R} \to \mathbb{R}$ are continuous and given by means of A, W, such that

$$x(t) = c_1 t + c_2 + o(1) \quad \text{when} \quad t \to +\infty \tag{3}$$

and

$$\mathcal{W}(x,t) = \frac{1}{t}\begin{vmatrix} x'(t) & 1 \\ x(t) & t \end{vmatrix} = x'(t) - \frac{x(t)}{t} < 0\,, \qquad t > t_0\,. \tag{4}$$

The symbol $o(f)$ for a given functional quantity f has here its standard meaning. In particular, by $o(1)$ we refer to a function of t that decreases to 0 as t increases to $+\infty$.

The papers [2, 1, 22, 21, 20] present various properties of the functional quantity $\mathcal{W}$, which shall be called *pseudo-wronskian* in the sequel. Our aim in this note is to complete their conclusions by giving some sufficient conditions upon a and w which lead to the existence of a solution x to (2) that verifies (3) while having an *oscillatory pseudo-wronskian* (this means that there exist the unbounded from above sequences $(t_n^{\pm})_{n\geq 1}$ and $(t_n^0)_{n\geq 1}$ such that $t_{2n-1}^0 < t_n^+ < t_{2n}^0 < t_n^- < t_{2n+1}^0$ and $\mathcal{W}(t_n^+) > \mathcal{W}(t_n^0) = 0 > \mathcal{W}(t_n^-)$ for all $n \geq 1$). We answer thus to a question raised in [1, p. 371], see also the comment in [2, pp. 46–47].

2. The sign of $\mathcal{W}$

Let us start the discussion with a simple condition to settle the sign issue of the pseudo-wronskian.

Lemma 1. *Given $x \in C^2\big([t_0, +\infty), \mathbb{R}\big)$, suppose that $x''(t) \leq 0$ for all $t \geq t_0$. Then $\mathcal{W}(x,\cdot)$ can change from being nonnegative-valued to being negative-valued at most once in $[t_0, +\infty)$. In fact, its set of zeros is an interval (possibly degenerate).*

Proof. Notice that

$$\frac{d^2}{dt^2}\big[x(t)\big] = \frac{1}{t} \cdot \frac{d}{dt}\big[t\mathcal{W}(x,t)\big]\,, \qquad t \geq t_0\,.$$

The function $t \mapsto t\mathcal{W}(x,t)$ being nonincreasing, it is clear that, if it has zeros, it has either a unique zero or an interval of zeros. □

The result has an obvious counterpart.

Lemma 2. *Given $x \in C^2\big([t_0, +\infty), \mathbb{R}\big)$, suppose that $x''(t) \geq 0$ for all $t \geq t_0$. Then, $\mathcal{W}(x,\cdot)$ can change from being nonpositive-valued to being positive-valued at most once in $[t_0, +\infty)$. Again, its set of zeros is an interval (possibly reduced to one point).*

Consider that x is a positive solution of equation (2) in the case where $a(t) \geq 0$ in $[t_0, +\infty)$ and $w(u) > 0$ for all $u > 0$. Then, we have

$$\frac{d\mathcal{W}}{dt} = -\frac{\mathcal{W}}{t} - a(t)w\big(x(t)\big)\,, \qquad t \geq t_0\,,$$

which leads to

$$\mathcal{W}(x,t) = \frac{1}{t}\Big[t_0\mathcal{W}_0 - \int_{t_0}^{t} sa(s)w\big(x(s)\big)\,ds\Big]\,, \qquad \mathcal{W}_0 = \mathcal{W}(x,t_0)\,, \tag{5}$$

throughout $[t_0, +\infty)$ by means of Lagrange's variation of constants formula.

The integrand in (5) being nonnegative-valued, we regain the conclusion of Lemma 1. In fact, if $T \in [t_0, +\infty)$ is a zero of $\mathcal{W}(x, \cdot)$ then it is a solution of the equation

$$t_0\mathcal{W}_0 = \int_{t_0}^{T} sa(s)w\big(x(s)\big)\,ds\,. \tag{6}$$

On the other hand, if the pseudo-wronskian of x is positive-valued throughout $[t_0, +\infty)$ then it is necessary to have

$$(t_0\mathcal{W}_0 \geq) \qquad \int_{t_0}^{+\infty} sa(s)w\big(x(s)\big)\,ds < +\infty\,. \tag{7}$$

It has become clear at this point that whenever the equation (2) has a positive solution x such that $\mathcal{W}_0 \leq 0$, the functional coefficient a is nonnegative-valued and has at most isolated zeros and $w(u) > 0$ for all $u > 0$, the pseudo-wronskian $\mathcal{W}$ satisfies the restriction (4). Now, returning to the problem stated in the Introduction, we can evaluate the main difficulty of the investigation: *if the positive solution x has prescribed asymptotic behavior, see formula* (3) *or a similar development, then we cannot decide upfront whether or not* $\mathcal{W}_0 \leq 0$. The formula (6) shows that there are also certain difficulties to estimate the zeros of the pseudo-wronskian.

3. The behavior of $\mathcal{W}$

Let us survey in this section some of the recent results regarding the pseudo-wronskian.

It has been established that its presence in the structure of a nonlinear differential equation

$$x'' + f(t, x, x') = 0\,, \qquad t \geq t_0 \geq 1\,, \tag{8}$$

where the nonlinearity $f\colon [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, allows for a remarkable flexibility of the hypotheses when searching for solutions with the asymptotic development (3) (or similar).

Theorem 1 ([22, p. 177]). *Assume that there exist the nonnegative-valued, continuous functions $a(t)$ and $g(s)$ such that $g(s) > 0$ for all $s > 0$ and $xg(s) \leq g(x^{1-\alpha}s)$, where $x \geq t_0$ and $s \geq 0$, for a certain $\alpha \in (0, 1)$. Suppose further that*

$$\big|f(t,x,x')\big| \leq a(t)g\Big(\Big|x' - \frac{x}{t}\Big|\Big) \quad \text{and} \quad \int_{t_0}^{+\infty} \frac{a(s)}{s^{\alpha}}\,ds < \int_{c+|\mathcal{W}_0|t_0^{1-\alpha}}^{+\infty} \frac{du}{g(u)}\,.$$

Then the solution of equation (8) *given by* (5) *exists throughout* $[t_0, +\infty)$ *and has the asymptotic behavior*

$$x(t) = c \cdot t + o(t)\,, \qquad x'(t) = c + o(1) \quad \text{when} \quad t \to +\infty \tag{9}$$

for some $c = c(x) \in \mathbb{R}$.

To compare this result with the standard conditions in asymptotic integration theory regarding the development (9), see the papers [2, 1, 24] and the monograph [19].

Another result is concerned with the presence of the pseudo-wronskian in the function space $L^1((t_0, +\infty), \mathbb{R})$.

Theorem 2 ([1, p. 371])**.** *Assume that* f *does not depend explicitly of* x' *and there exists the continuous function* $F : [t_0, +\infty) \times [0, +\infty) \to [0, +\infty)$, *which is nondecreasing with respect to the second variable, such that*

$$\left|f(t,x)\right| \le F\Big(t, \frac{|x|}{t}\Big) \quad \text{and} \quad \int_{t_0}^{+\infty} t\Big[1 + \ln\Big(\frac{t}{t_0}\Big)\Big] F\Big(t, |c| + \frac{\varepsilon}{t_0}\Big)\, dt < \varepsilon$$

for certain numbers $c \neq 0$ *and* $\varepsilon > 0$. *Then there exists a solution* $x(t)$ *of equation* (8) *defined in* $[t_0, +\infty)$ *such that*

$$x(t) = c \cdot t + o(1) \quad \text{when} \quad t \to +\infty \quad \text{and} \quad \mathcal{W}(x, \cdot) \in L^1\,.$$

The effect of perturbations upon the pseudo-wronskian is investigated in the papers [2, 22, 21].

Theorem 3 ([22, p. 183])**.** *Consider the nonlinear differential equation*

$$x'' + f(t, x, x') = p(t)\,, \qquad t \ge t_0 \ge 1\,, \tag{10}$$

where the functions $f\colon [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ *and* $p\colon [t_0, +\infty) \to \mathbb{R}$ *are continuous and verify the hypotheses*

$$\left|f(t,x,x')\right| \le a(t)\Big|x' - \frac{x}{t}\Big|\,, \qquad \int_{t_0}^{+\infty} t a(t)\, dt < +\infty$$

and

$$\lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} s p(s)\, ds = C \in \mathbb{R} - \{0\}\,.$$

Then, given $x_0 \in \mathbb{R}$, *there exists a solution* $x(t)$ *of equation* (10) *defined in* $[t_0, +\infty)$ *such that*

$$x(t_0) = x_0 \quad \text{and} \quad \lim_{t \to +\infty} \mathcal{W}(x, t) = C\,.$$

In particular,

$$\lim_{t \to +\infty} \frac{x(t)}{t \ln t} = C\,.$$

A slight modification of the discussion in [21, Remark 3], see [2, p. 47], leads to the next result.

Theorem 4. *Assume that f in (10) does not depend explicitly of x' and there exists the continuous function $F\colon [t_0,+\infty)\times[0,+\infty)\to[0,+\infty)$, which is nondecreasing with respect to the second variable, such that*

$$\bigl|f(t,x)\bigr| \le F\bigl(t,|x|\bigr) \quad \textit{and} \quad \int_t^{+\infty} sF\Bigl(s,|P(s)|+\sup_{\tau\ge s}\{q(\tau)\}\Bigr)\,ds \le q(t)\,, \quad t\ge t_0\,,$$

for a certain positive-valued, continuous function $q(t)$ possibly decaying to 0 as $t\to+\infty$. Here, P is the twice continuously differentiable antiderivative of p, that is $P''(t)=p(t)$ for all $t\ge t_0$. Suppose further that

$$\limsup_{t\to+\infty}\Bigl[t\frac{\mathcal{W}(P,t)}{q(t)}\Bigr] > 1 \quad \textit{and} \quad \liminf_{t\to+\infty}\Bigl[t\frac{\mathcal{W}(P,t)}{q(t)}\Bigr] < -1\,.$$

Then equation (10) has a solution $x(t)$ throughout $[t_0,+\infty)$ such that

$$x(t)=P(t)+o(1) \quad \textit{when} \quad t\to+\infty$$

and $\mathcal{W}(x,\cdot)$ oscillates.

Finally, the presence of the pseudo-wronskian in the structure of a nonlinear differential equation can lead to multiplicity when searching for solutions with the asymptotic development (3).

Theorem 5 ([20, Theorem 1])**.** *Given the numbers x_0, x_1, $c\in\mathbb{R}$, with $c\ne 0$, and $t_0\ge 1$ such that $t_0x_1-x_0=c$, consider the Cauchy problem*

$$\begin{cases} x''=\frac{1}{t}\,g(tx'-x)\,, & t\ge t_0\ge 1\,,\\ x(t_0)=x_0\,, & x'(t_0)=x_1\,, \end{cases} \tag{11}$$

where the function $g\colon\mathbb{R}\to\mathbb{R}$ is continuous, $g(c)=g(3c)=0$ and $g(u)>0$ for all $u\ne c$. Assume further that

$$\int_{c+}^{2c}\frac{du}{g(u)} < +\infty \quad \textit{and} \quad \int_{2c}^{(3c)-}\frac{du}{g(u)} = +\infty\,.$$

Then problem (11) has an infinity of solutions $x(t)$ defined in $[t_0,+\infty)$ and developable as

$$x(t)=c_1t+c_2+o(1) \quad \textit{when} \quad t\to+\infty$$

for some $c_1=c_1(x)$ and $c_2=c_2(x)\in\mathbb{R}$.

The asymptotic analysis of certain functional quantities attached to the solutions of equations (2), (8) and (10), as in our case the pseudo-wronskian, might lead to some surprising consequences. Among the functional quantities that gave the impetus to spectacular developments in the qualitative theory of linear/nonlinear ordinary differential equations we would like to refer to

$$\mathcal{K}(x)(t)=x(t)x'(t)\,, \qquad t\ge t_0\,,$$

employed in the theory of *Kneser-solutions*, see the papers [6, 7] for the linear and respectively the nonlinear case and the monograph [19], and

$$\mathcal{HW}(x)=\int_{t_0}^{+\infty} x(s)w\bigl(x(s)\bigr)\,ds\,.$$

The latter quantity is the core of the nonlinear version of *Hermann Weyl's limit-point/limit-circle classification* designed for equation (2), see the well-documented monograph [5] and the paper [23].

4. The negative values of $\mathcal{W}$

We shall assume in the sequel that the nonlinearity w of equation (2) verifies some of the hypotheses listed below:

$$\big|w(x)-w(y)\big| \le k|x-y|\,, \quad \text{where} \quad k>0\,, \tag{12}$$

and

$$w(0)=0\,, \quad w(x)>0 \quad \text{when} \quad x>0\,, \qquad \big|w(xy)\big| \le w\big(|x|\big)w\big(|y|\big) \tag{13}$$

for all x, $y \in \mathbb{R}$. We notice that restriction (13) implies the existence of a majorizing function F, as in Theorem 2, given by the estimates

$$\big|f(t,x)\big| = \big|a(t)\,w(x)\big| \le \big|a(t)\big| \cdot w(t)\,w\Big(\frac{|x|}{t}\Big) = F\Big(t,\frac{|x|}{t}\Big)\,.$$

We can now use the paper [24] to recall the main conclusions of an asymptotic integration of equation (2). It has been established that whenever $\int\limits_{t_0}^{+\infty} tw(t)\big|a(t)\big|\,dt < +\infty$, all the solutions of (2) have asymptotes (3) and their first derivatives are developable as

$$x'(t) = c_1 + o\big(t^{-1}\big) \quad \text{when} \quad t\to+\infty\,. \tag{14}$$

Consequently, $\mathcal{W}(x,t) = -c_2t^{-1}+o(t^{-1})$ for all large t's. In this case (the functional coefficient a has varying sign), when dealing with the sign of the pseudo-wronskian, of interest would be the subcase where $c_2=0$. Here, the asymptotic development does not even ensure that $\mathcal{W}$ is eventually negative. Enlarging the family of coefficients to the ones subjected to the restriction $\int\limits_{t_0}^{+\infty} t^{\varepsilon}w(t)\big|a(t)\big|\,dt<+\infty$, where $\varepsilon\in[0,1)$, the developments (3), (14) become

$$x(t) = ct + o\big(t^{1-\varepsilon}\big)\,, \quad x'(t) = c + o\big(t^{-\varepsilon}\big)\,, \quad c\in\mathbb{R}\,, \tag{15}$$

yielding the less precise estimate $\mathcal{W}(x,t) = o(t^{-\varepsilon})$ when $t\to+\infty$. We have again a lack of precision in the asymptotic development of $\mathcal{W}(x,\cdot)$ with respect to the sign issue. We also deduce on the basis of (3), (15) that some of the coefficients a in these classes verify (7), a fact that complicates the discussion.

The next result establishes the existence of a positive solution to (2) subjected to (4), (15) for the largest class of functional coefficients: $\varepsilon=0$. By taking into account Lemmas 1, 2 and the non-oscillatory character of equation (2) when the nonlinearity w verifies (13), we conclude that *for an investigation within this class of coefficients a of the solutions with oscillatory pseudo-wronskian it is necessary that a itself oscillates.* Also, when a is non-negative valued we recall that the condition

$$\int_{t_0}^{+\infty} a(t)\,dt < +\infty$$

is necessary for the linear case of equation (2) to be non-oscillatory, see [16], while in the case given by $w(x) = x^\lambda$, $x \in \mathbb{R}$, with $\lambda > 1$ (such an equation is usually called an *Emden-Fowler equation*, see the monograph [19]) the condition

$$\int_{t_0}^{+\infty} ta(t)\, dt = +\infty \tag{16}$$

is necessary and sufficient for oscillation, see [4]. In the case of Emden-Fowler equations with $\lambda \in (0, 1)$ and a continuously differentiable coefficient a such that $a(t) \geq 0$ and $a'(t) \leq 0$ throughout $[t_0, +\infty)$, another result establishes that equation (2) has no oscillatory solutions provided that condition (16) fails, see [17].

Regardless of the oscillation of a, it is known [1, p. 360] that the linear case of equation (2) has bounded and positive solutions with eventually negative pseudo-wronskian.

Theorem 6. *Assume that the nonlinearity w verifies hypothesis* (13) *and is non-decreasing. Given c, $d > 0$, suppose that the functional coefficient a is nonnegative--valued, with eventual isolated zeros, and*

$$\int_{t_0}^{+\infty} w(t)\, a(t)\, dt \leq \frac{d}{w(c+d)}\,.$$

Then, the equation (2) *has a solution x such that $\mathcal{W}_0 = 0$,*

$$c - d \leq x'(t) < \frac{x(t)}{t} \leq c + d \quad \textit{for all} \quad t > t_0 \tag{17}$$

and

$$\lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} \frac{x(t)}{t} = c\,. \tag{18}$$

Proof. We introduce the set D given by

$$D = \bigl\{u \in C\bigl([t_0, +\infty), \mathbb{R}\bigr) : ct \leq u(t) \leq (c+d)t \text{ for every } t \geq t_0\bigr\}\,.$$

A partial order on D is provided by the usual pointwise order "$\leq$", that is, we say that $v_1 \leq v_2$ if and only if $v_1(t) \leq v_2(t)$ for all $t \geq t_1$, where v_1, $v_2 \in D$. It is not hard to see that $(D, \leq)$ is a complete lattice.

For the operator $V \colon D \to C([t_0, +\infty), \mathbb{R})$ with the formula

$$V(u)(t) = t\Bigl\{c + \int_t^{+\infty} \frac{1}{s^2} \int_{t_0}^{s} \tau a(\tau)\, w\bigl(u(\tau)\bigr)\, d\tau\, ds\Bigr\}\,, \qquad u \in D\,,\ t \geq t_0\,,$$

the next estimates hold

$$\begin{aligned}
c \le \frac{V(u)(t)}{t} &= c + \int_t^{+\infty} \frac{1}{s^2}\int_{t_0}^{s} \tau a(\tau)\cdot w(\tau)\, w\Big(\frac{u(\tau)}{\tau}\Big)\, d\tau\, ds \\
&\le c + \sup_{\xi\in[0,c+d]}\{w(\xi)\}\cdot \int_t^{+\infty} \frac{1}{s^2}\int_{t_0}^{s} \tau w(\tau)\, a(\tau)\, d\tau\, ds \\
&= c + w(c+d)\Big[\frac{1}{t}\int_{t_0}^{t} \tau w(\tau)\, a(\tau)\, d\tau + \int_t^{+\infty} w(\tau)\, a(\tau)\, d\tau\Big] \\
&\le c + w(c+d)\int_{t_0}^{+\infty} w(\tau)\, a(\tau)\, d\tau \le c+d
\end{aligned}$$

by means of (13). These imply that $V(D)\subseteq D$.

Since $c\cdot t \le V(c\cdot t)$ for all $t\ge t_0$, by applying the Knaster-Tarski fixed point theorem [12, p. 14], we deduce that the operator V has a fixed point u_0 in D. This is the pointwise limit of the sequence of functions $\big(V^n(c\cdot \mathrm{Id}_I)\big)_{n\ge 1}$, where $V^1 = V$, $V^{n+1} = V^n\circ V$ and $I=[t_0,+\infty)$.

We deduce that

$$u_0'(t) = \big[V(u_0)\big]'(t) = \frac{u_0(t)}{t} - \frac{1}{t}\int_{t_0}^{t} \tau a(\tau)\, w\big(u_0(\tau)\big)\, d\tau < \frac{u_0(t)}{t}\,,$$

when $t>t_0$, and thus (17), (18) hold true.

The proof is complete. □

5. The oscillatory integration of equation (2)

Let the continuous functional coefficient a with varying sign satisfy the restriction

$$\int_{t_0}^{+\infty} t^2|a(t)|\, dt < +\infty\,.$$

We call the problem studied in the sequel *an oscillatory* (*asymptotic*) *integration* of equation (2).

Theorem 7. *Assume that w verifies* (12)*, $w(0)=0$ and there exists $c>0$ such that*

$$L_+^c > 0 > L_-^c\,, \tag{19}$$

where

$$L_+^c = \limsup_{t\to+\infty} \frac{t\int_t^{+\infty} s w(cs)\, a(s)\, ds}{\int_t^{+\infty} s^2|a(s)|\, ds}\,,\qquad L_-^c = \liminf_{t\to+\infty} \frac{t\int_t^{+\infty} s w(cs)\, a(s)\, ds}{\int_t^{+\infty} s^2|a(s)|\, ds}\,.$$

Then the equation (2) *has a solution $x(t)$ with oscillatory pseudo-wronskian such that*

$$x(t) = c\cdot t + o(1) \quad \textit{when} \quad t\to+\infty\,. \tag{20}$$

Proof. There exist $\eta > 0$ such that $L_+^c > \eta$, $L_-^c < -\eta$ and two increasing, unbounded from above sequences $(t_n)_{n\geq 1}$, $(t^n)_{n\geq 1}$ of numbers from $(t_0, +\infty)$ such that $t^n \in (t_n, t_{n+1})$ and

$$t_n \int_{t_n}^{+\infty} sw(cs)\, a(s)\, ds + k\eta \int_{t_n}^{+\infty} s^2 |a(s)|\, ds < 0 \tag{21}$$

and

$$t^n \int_{t^n}^{+\infty} sw(cs)\, a(s)\, ds - k\eta \int_{t^n}^{+\infty} s^2 |a(s)|\, ds > 0 \tag{22}$$

for all $n \geq 1$.

Assume further that

$$\int_{t_0}^{+\infty} \tau^2 |a(\tau)|\, d\tau \leq \frac{\eta}{k(c+\eta)}$$

and introduce the complete metric space $S = (D, \delta)$ given by

$$D = \big\{ y \in C([t_0, +\infty), \mathbb{R}) : t|y(t)| \leq \eta \text{ for every } t \geq t_0 \big\}$$

and

$$\delta(y_1, y_2) = \sup_{t \geq t_0} \big\{ t|y_1(t) - y_2(t)| \big\}, \quad y_1, y_2 \in D.$$

For the operator $V\colon D \to C\big([t_0, +\infty), \mathbb{R}\big)$ with the formula

$$V(y)(t) = \frac{1}{t} \int_t^{+\infty} sa(s)\, w\Big(s\Big[c - \int_s^{+\infty} \frac{y(\tau)}{\tau}\, d\tau\Big]\Big)\, ds, \quad y \in D,\, t \geq t_0,$$

the next estimates hold (notice that $|w(x)| \leq k|x|$ for all $x \in \mathbb{R}$)

$$t\big|V(y)(t)\big| \leq k \int_t^{+\infty} s^2 \big|a(s)\big| \Big[c + \eta \int_s^{+\infty} \frac{d\tau}{\tau^2}\Big]\, ds \leq \eta \tag{23}$$

and

$$\begin{aligned} t\big|V(y_2)(t) - V(y_1)(t)\big| &\leq k \int_t^{+\infty} s^2 \big|a(s)\big| \Big(\int_s^{+\infty} \frac{d\tau}{\tau^2}\Big)\, ds \cdot \delta(y_1, y_2) \\ &\leq \frac{k}{t_0} \int_t^{+\infty} s^2 \big|a(s)\big|\, ds \leq \frac{\eta}{c+\eta} \cdot \delta(y_1, y_2). \end{aligned}$$

These imply that $V(D) \subseteq D$ and thus $V\colon S \to S$ is a contraction.

From the formula of operator V we notice also that

$$\lim_{t \to +\infty} tV(y)(t) = 0 \quad \text{for all} \quad y \in D. \tag{24}$$

Given $y_0 \in D$ the unique fixed point of V, one of the solutions to (2) has the formula $x_0(t) = t\Big[c - \int\limits_t^{+\infty} \frac{y_0(s)}{s}\, ds\Big]$ for all $t \geq t_0$. Via (24) and L'Hospital's rule, we provide also an asymptotic development for this solution, namely

$$\lim_{t\to+\infty}[x_0(t)-c\cdot t] = -\lim_{t\to+\infty} t\int_t^{+\infty}\frac{y_0(s)}{s}\,ds = -\lim_{t\to+\infty} ty_0(t)$$
$$= -\lim_{t\to+\infty} tV(y_0)(t) = 0.$$

The estimate

$$\Big|ty_0(t) - \int_t^{+\infty} sw(cs)\,a(s)\,ds\Big| \le k\int_t^{+\infty} s^2\big|a(s)\big|\Big[\int_s^{+\infty}\frac{|y_0(\tau)|}{\tau}\,d\tau\Big]\,ds$$
$$\le k\eta\cdot\frac{1}{t}\int_t^{+\infty} s^2\big|a(s)\big|\,ds\,, \qquad t\ge t_0\,,$$

accompanied by (21), (22), leads to

$$y_0(t_n) = \mathcal{W}(x_0,t_n) < 0 \quad \text{and} \quad y_0(t^n) = \mathcal{W}(x_0,t^n) > 0\,. \tag{25}$$

The proof is complete. □

Remark 1. When Equation (2) is linear, that is $w(x) = x$ for all $x\in\mathbb{R}$, the formula (19) can be recast as

$$L_+ = \limsup_{t\to+\infty}\frac{t\int_t^{+\infty} s^2a(s)\,ds}{\int_t^{+\infty} s^2|a(s)|\,ds} > 0 > \liminf_{t\to+\infty}\frac{t\int_t^{+\infty} s^2a(s)\,ds}{\int_t^{+\infty} s^2|a(s)|\,ds} = L_-\,.$$

We claim that *for all $c\neq 0$ there exists a solution $x(t)$ with oscillatory pseudo-wronskian which verifies* (20). In fact, replace c with c_0 in the formulas (21), (22) for a certain c_0 subjected to the inequality $\min\{L_+,-L_-\} > \frac{\eta}{c_0}$. It is obvious that, when $L_+ = -L_- = +\infty$, formulas (21), (22) hold for all $c_0,\eta>0$. Given $c\in\mathbb{R}-\{0\}$, there exists $\lambda\neq 0$ such that $c=\lambda c_0$. The solution of Equation (2) that we are looking for has the formula $x=\lambda\cdot x_0$, where $x_0(t) = t\Big[c_0 - \int\limits_t^{+\infty}\frac{y_0(s)}{s}\,ds\Big]$ for all $t\ge t_0$ and y_0 is the fixed point of operator V in D. Its pseudo-wronskian oscillates as a consequence of the obvious identity

$$\lambda\cdot\mathcal{W}(x_0,t) = \mathcal{W}(x,t)\,, \qquad t\ge t_0\,.$$

Example 1. An immediate example of functional coefficient a for the problem of linear oscillatory integration is given by $a(t) = t^{-2}e^{-t}\cos t$, where $t\ge 1$.

We have

$$\int_t^{+\infty} s^2a(s)\,ds = \frac{1}{\sqrt{2}}\cos\Big(t+\frac{\pi}{4}\Big)e^{-t} \quad \text{and} \quad \int_t^{+\infty} s^2|a(s)|\,ds \le e^{-t}$$

throughout $[1,+\infty)$ which yields $L_+ = +\infty$, $L_- = -\infty$.

Sufficient conditions are provided now for an oscillatory pseudo-wronskian to be in $L^p\big((t_0,+\infty),\mathbb{R}\big)$, where $p>0$. Since $\lim\limits_{t\to+\infty}\mathcal{W}(x,t) = 0$ for any solution $x(t)$ of equation (2) with the asymptotic development (20), (14), we are interested in the case $p\in(0,1)$.

Theorem 8. *Assume that, in the hypotheses of Theorem 7, the coefficient a verifies the condition*

$$\int_{t_0}^{+\infty}\left[\frac{t}{\int_t^{+\infty}s^2|a(s)|\,ds}\right]^{1-p}t^2|a(t)|\,dt<+\infty \quad \textit{for some} \quad p\in(0,1)\,. \tag{26}$$

Then the equation (2) *has a solution $x(t)$ with an oscillatory pseudo-wronskian in L^p and the asymptotic expansion* (20)*.*

Proof. Recall that y_0 is the fixed point of operator V. Then, formula (23) implies that

$$\big|y_0(t)\big|\le k(c+\eta)\cdot\frac{1}{t}\int_t^{+\infty}s^2\big|a(s)\big|\,ds,\qquad t\ge t_0\,.$$

Via an integration by parts, we have

$$\frac{1}{[k(c+\eta)]^p}\int_t^T\big|y_0(s)\big|^p\,ds\le\frac{T^{1-p}}{1-p}\Big[\int_T^{+\infty}s^2\big|a(s)\big|\,ds\Big]^p$$
$$+\frac{p}{1-p}\int_t^T\left[\frac{s}{\int_s^{+\infty}\tau^2|a(\tau)|\,d\tau}\right]^{1-p}s^2\big|a(s)\big|\,ds$$

for all $T\ge t\ge t_0$.

The estimates

$$\frac{T^{1-p}}{1-p}\Big[\int_T^{+\infty}s^2\big|a(s)\big|\,ds\Big]^p=\frac{T^{1-p}}{1-p}\int_T^{+\infty}\left[\frac{1}{\int_T^{+\infty}\tau^2|a(\tau)|\,d\tau}\right]^{1-p}s^2\big|a(s)\big|\,ds$$
$$\le\frac{1}{1-p}\int_T^{+\infty}\left[\frac{s}{\int_s^{+\infty}\tau^2|a(\tau)|\,d\tau}\right]^{1-p}s^2\big|a(s)\big|\,ds$$

allow us to establish that

$$\frac{1}{[k(c+\eta)]^p}\int_t^T\big|y_0(s)\big|^p\,ds\le\frac{1+p}{1-p}\int_t^{+\infty}\left[\frac{s}{\int_s^{+\infty}\tau^2|a(\tau)|\,d\tau}\right]^{1-p}s^2\big|a(s)\big|\,ds\,.$$

The conclusion follows by letting $T\to+\infty$.

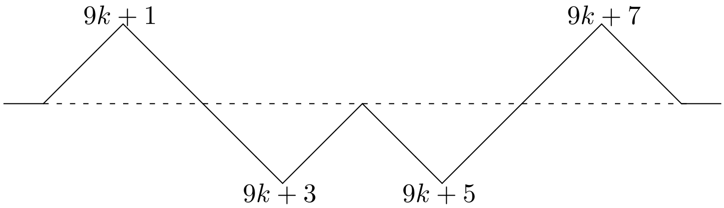
The proof is complete. □

Example 2. An example of functional coefficient a in the linear case that verifies the hypotheses of Theorem 8 is given by the formula

$$t^2a(t)=b(t)=\begin{cases}a_k(t-9k)\,, & t\in[9k,9k+1]\,,\\ a_k(9k+2-t)\,, & t\in[9k+1,9k+3]\,,\\ a_k(t-9k-4)\,, & t\in[9k+3,9k+4]\,,\\ a_k(9k+4-t)\,, & t\in[9k+4,9k+5]\,,\\ a_k(t-9k-6)\,, & t\in[9k+5,9k+7]\,,\\ a_k(9k+8-t)\,, & t\in[9k+7,9k+8]\,,\\ 0\,, & t\in[9k+8,9(k+1)]\,,\end{cases}\qquad k\ge1\,.$$

Here, we take $a_k = k^{-\alpha} - (k+1)^{-\alpha}$ for a certain integer $\alpha > \frac{2-p}{p}$.

To help the computations, the k-th "cell" of the function b can be visualized next.



It is easy to observe that

$$\int_{9k}^{9k+4} b(t)\,dt = \int_{9k+4}^{9k+8} b(t)\,dt = 0 \quad \text{for all} \quad k \geq 1\,.$$

We have

$$\int_{9k+2}^{+\infty} b(t)\,dt = \int_{9k+2}^{9k+4} b(t)\,dt = -a_k\,, \qquad \int_{9k+6}^{+\infty} b(t)\,dt = \int_{9k+6}^{9k+8} b(t)\,dt = a_k$$

and respectively

$$\int_{9k+2}^{+\infty} \left|b(t)\right| dt = 3a_k + 4\sum_{m=k+1}^{+\infty} a_m\,, \qquad \int_{9k+6}^{+\infty} \left|b(t)\right| dt = a_k + 4\sum_{m=k+1}^{+\infty} a_m\,.$$

By noticing that

$$L_+ = \lim_{k\to+\infty} \frac{(9k+6)\int_{9k+6}^{+\infty} b(t)\,dt}{\int_{9k+6}^{+\infty} |b(t)|\,dt}\,, \qquad L_- = \lim_{k\to+\infty} \frac{(9k+2)\int_{9k+2}^{+\infty} b(t)\,dt}{\int_{9k+2}^{+\infty} |b(t)|\,dt}\,,$$

we obtain $L_+ = \frac{9\alpha}{4}$ and $L_- = -\frac{9\alpha}{4}$.

To verify the condition (26), notice first that

$$\begin{aligned} I_k &= \int_{9k}^{9(k+1)} \left[\frac{t}{\int_t^{+\infty} |b(s)|\,ds}\right]^{1-p} t^2 \left|a(t)\right| dt \\ &\leq \int_{9k}^{9(k+1)} \left[\frac{9(k+1)}{\int_{9(k+1)}^{+\infty} |b(s)|\,ds}\right]^{1-p} a_k\,dt, \qquad k \geq 1\,. \end{aligned}$$

The elementary inequality $a_k \leq (2^\alpha - 1)(k+1)^{-\alpha}$ implies that

$$I_k \leq \frac{c_\alpha}{(k+1)^{(1+\alpha)p-1}}\,, \quad \text{where} \quad c_\alpha = 9\left(\frac{9}{4}\right)^{1-p}(2^\alpha - 1)\,,$$

and the conclusion follows from the convergence of the series $\sum\limits_{k\geq 1}(k+1)^{1-(1+\alpha)p}$.

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Corresponding address: Str. Tudor Vladimirescu, Nr. 26
200534 Craiova, Dolj, Romania
Faculty of Mathematics, D.A.L.
University of Craiova, Romania
E-mail: octaviangenghiz@yahoo.com