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NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION

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Abstract. Dealing with a question of Lahiri [6] we study the uniqueness problem of meromorphic functions concerning two nonlinear differential polynomials sharing a small function. Our results will not only improve and supplement the results of Lin-Yi [16], Lahiri Sarkar [12] but also improve and supplement a very recent result of the first author [1].

1. Introduction definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. A meromorphic function α is said to be a small function of f provided that $T(r,\alpha)=S(r,f)$, that is $T(r,\alpha)=o(T(r,f))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure. Clearly if f is rational then α is a constant and if f is transcendental then α is a nonconstant meromorphic function. We denote by $S(f)$ the set of all small functions of f.

If for some $\alpha\in S(f)\cap S(g)$, $f-\alpha$ and $g-\alpha$ have the same set of zeros with the same multiplicities, we say that f and g share α CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share α IM (ignoring multiplicities).

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside of a possible exceptional set of finite linear measure.

Let $N_E(r,\alpha;f,g)$ ($\overline{N}_E(r,\alpha;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-\alpha$ and $g-\alpha$ with the same multiplicities and $N_0(r,\alpha;f,g)$ ($\overline{N}_0(r,\alpha;f,g)$) be the counting function (reduced counting function) of all common zeros of $f-\alpha$ and $g-\alpha$ ignoring multiplicities.

If

$$\overline{N}(r,\alpha;f)+\overline{N}(r,\alpha;g)-2\overline{N}_E(r,\alpha;f,g)=S(r,f)+S(r,g)$$

then we say that f and g share α "CM".

On the other hand if

$$\overline{N}(r,\alpha;f)+\overline{N}(r,\alpha;g)-2\overline{N}_0(r,\alpha;f,g)=S(r,f)+S(r,g)$$

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then we say that f and g share α "IM".

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

In [6] Lahiri studied the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper [6] regarding the nonlinear differential polynomials Lahiri asked the following question. *What can be said if two nonlinear differential polynomials generated by two meromorphic functions share* 1 *CM?*

Naturally several authors investigate the possible answer to the above question and continuous efforts are being carried out to relax the hypothesis of the results. (cf. [1], [2], [3], [11], [12], [14], [15], [16]).

In 2002 Fang and Fang [2] and in 2004 Lin-Yi [15] independently proved the following result.

Theorem A. *Let f and g be two nonconstant meromorphic functions and $n\,(\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share* 1 *CM, then $f \equiv g$.*

In 2004 Lin-Yi [16] improved Theorem A by generalizing it in view of fixed point. Lin-Yi [16] proved the following result.

Theorem B. *Let f and g be two transcendental meromorphic functions and $n\,(\geq 13)$ be an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share z CM, then $f \equiv g$.*

In the same paper Lin-Yi [16] mentioned that in Theorem B z can be replaced by $\alpha(z)$.

In 2001 an idea of gradation of sharing of values was introduced in ([8], [9]) which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1.1 ([8, 9])**.** Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of value recently the first author [1] improved Theorem A as follows.

Theorem C ([1])**.** *Let f and g be two nonconstant meromorphic functions and $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$, is an integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $(1, 2)$ then $f \equiv g$.*

In the mean time Lahiri and Sarkar [12] also studied the uniqueness of meromorphic functions corresponding to nonlinear differential polynomials which are different from that of previously mentioned and proved the following.

Theorem D ([12])**.** *Let f and g be two nonconstant meromorphic functions such that $f^n(f^2-1)f'$ and $g^n(g^2-1)g'$ share $(1,2)$, where n (≥ 13) is an integer then either $f \equiv g$ or $f \equiv -g$. If n is an even integer then the possibility of $f \equiv -g$ does not arise.*

From the above discussion it will be a natural query to investigate the uniqueness of meromorphic functions when two non linear differential polynomials of more general form namely $f^n(af^2+bf+c)f'$ and $g^n(ag^2+bg+c)g'$ where $a \neq 0$ and $|b|+|c| \neq 0$ share a small function.

In this paper we will study the above problem with the notion of weakly weighted sharing which has recently been introduced by Lin and Lin [13] generalizing the idea of weighted sharing of values. We are now giving the definition.

Definition 1.2 ([13])**.** Let f g share α "IM" for $\alpha \in S(f) \cap S(g)$ and k is a positive integer or ∞.

(i) $\overline{N}^E(r,\alpha;f,g\mid\leq k)$ denotes the reduced counting function of those α-points of f whose multiplicities are equal to the corresponding α-points of g, both of their multiplicities are not greater than k.

(ii) $\overline{N}^0(r,\alpha;f,g\mid> k)$ denotes the reduced counting function of those α-points of f which are α-points of g, both of their multiplicities are not less than k.

Definition 1.3 ([13])**.** For $\alpha \in S(f) \cap S(g)$, if k is a positive integer or ∞ and

$$\overline{N}(r,\alpha;f\mid\leq k) - \overline{N}^E(r,\alpha;f,g\mid\leq k) = S(r,f)\,,$$
$$\overline{N}(r,\alpha;g\mid\leq k) - \overline{N}^E(r,\alpha;f,g\mid\leq k) = S(r,g)\,,$$
$$\overline{N}(r,\alpha;f\mid\geq k+1) - \overline{N}^0(r,\alpha;f,g\mid\geq k+1) = S(r,f)\,,$$
$$\overline{N}(r,\alpha;g\mid\geq k+1) - \overline{N}^0(r,\alpha;f,g\mid\geq k+1) = S(r,g)$$

or if $k=0$ and

$$\overline{N}(r,\alpha;f) - \overline{N}_0(r,\alpha;f,g) = S(r,f)\,,$$
$$\overline{N}(r,\alpha;g) - \overline{N}_0(r,\alpha;f,g) = S(r,g)\,,$$

then we say f, g weakly share α with weight k. Here we write f, g share "(α,k)" to mean that f, g weakly share α with weight k.

Obviously if f, g share "(α,k)", then f, g share "(α,p)" for any integer p, $0 \leq p < k$. Also we note that f, g share α "IM" or "CM" if and only if f, g share "$(\alpha,0)$" or "(α,∞)" respectively.

We now state the following theorem which is the main result of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions such that $f^n(af^2+bf+c)f'$ and $g^n(ag^2+bg+c)g'$ where $a \neq 0$ and $|b|+|c| \neq 0$ share "$(\alpha,2)$". Then the following holds.*

(i) *If* $b \neq 0$, $c = 0$ *and* $n > \max\big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}, \frac{4}{\Theta(\infty;f)+\Theta(\infty;g)} - 2\big]$*, be an integer, where* $\Theta(\infty; f) + \Theta(\infty; g) > 0$*, then* $f \equiv g$.

(ii) *If* $b \neq 0$, $c \neq 0$, $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$*, the roots of the equation* $az^2 + bz + c = 0$ *are distinct and one of* f *and* g *is non entire meromorphic function having only multiple poles, then* $f \equiv g$.

(iii) *If* $b \neq 0$, $c \neq 0$, $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *and the roots of the equation* $az^2 + bz + c = 0$ *coincides, then* $f \equiv g$.

(iv) $b = 0$, $c \neq 0$, $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$*, then either* $f \equiv g$ *or* $f \equiv -g$*. If* n *is an even integer then the possibility* $f \equiv -g$ *does not arise.*

From Theorem 1.1 we can immediately deduce the following corollaries.

Corollary 1.1. *Let* f *and* g *be two transcendental meromorphic functions such that* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$*, and* $n\,(\geq 13)$ *be an integer. If* $f^n(af^2 + bf)f'$ *and* $g^n(ag^2 + bg)g'$ *share "*$(\alpha, 2)$*" then* $f \equiv g$.

Corollary 1.2. *Let* f *and* g *be two transcendental meromorphic functions and one of* f *and* g *is non entire meromorphic function having only multiple poles, such that* $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *be an integer. If* $af^n(f - \beta_1)(f - \beta_2)f'$ *and* $ag^n(g - \beta_1)(g - \beta_2)g'$ *share "*$(\alpha, 2)$*", where* β_1 *and* β_2 *are the distinct roots of the equation* $az^2 + bz + c = 0$ *with* $|\beta_1| \neq |\beta_2|$*, then* $f \equiv g$.

Corollary 1.3. *Let* f *and* g *be two transcendental meromorphic functions such that* $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *be an integer. If* $af^n(f + k)^2 f'$ *and* $ag^n(g + k)^2 g'$ *share "*$(\alpha, 2)$*" where* k *is a nonzero constant then* $f \equiv g$.

Corollary 1.4. *Let* f *and* g *be two transcendental meromorphic functions such that* $n > \big[12 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\Theta(\infty; f), \Theta(\infty; g)\}\big]$ *be an integer. If* $f^n(af^2 + c)f'$ *and* $g^n(ag^2 + c)g'$ *share "*$(\alpha, 2)$*" then* $f \equiv g$ *or* $f \equiv -g$*. If* n *is an even integer then the possibility* $f \equiv -g$ *does not arise.*

Though we use the standard notations and definitions of the value distribution theory available in [5], we explain some definitions and notations which are used in the paper.

Definition 1.4 ([7])**.** For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)$ $\big(N(r, a; f \mid\geq m)\big)$ the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $\big(\overline{N}(r, a; f \mid\geq m)\big)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.5 ([9], cf.[20])**.** We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;\ f\mid\geq 2)$.

Definition 1.6 ([9])**.** Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g for which $p>q$, each point in this counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$.

Definition 1.7 ([10])**.** Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid\ g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.

Definition 1.8 ([10])**.** Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f\mid\ g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let f, g, F_1, G_1 be four nonconstant meromorphic functions. Henceforth we shall denote by h and H the following two functions.

$$h=\Big(\frac{f^{''}}{f'}-\frac{2f'}{f-1}\Big)-\Big(\frac{g^{''}}{g'}-\frac{2g'}{g-1}\Big)$$

and

$$H=\Big(\frac{F_1^{''}}{F_1'}-\frac{2F_1'}{F_1-1}\Big)-\Big(\frac{G_1^{''}}{G_1'}-\frac{2G_1'}{G_1-1}\Big).$$

Lemma 2.1. *If f, g be share "$(1,1)$" and $h\not\equiv 0$. Then*

$$N(r,1;f\mid\leq 1)\leq N(r,0;h)+S(r,f)\leq N(r,\infty;h)+S(r,f)+S(r,g).$$

Proof. Since f, g share "$(1,1)$" it follows that if z_0 be a common simple 1-point of f and g, then in some neighborhoods of z_0 we have $h=(z-z_0)\phi(z)$, where $\phi(z)$ is analytic at z_0. Hence by the first fundamental theorem and Milloux theorem (p. 55 [5]) we get

$$\begin{aligned}N(r,1;f\mid\leq 1)&=N^E(r,1;f,g\mid\leq 1)+S(r,f)\\&\leq N(r,0;h)+S(r,f)\leq N(r,\infty;h)+S(r,f)+S(r,g)\end{aligned}$$ □

Lemma 2.2. *If f, g share "$(1,1)$" and $h\not\equiv 0$. Then*

$$\begin{aligned}N(r,\infty;h)\leq\ &\overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,0;g\mid\geq 2)\\&+\overline{N}(r,\infty;f\mid\geq 2)+\overline{N}(r,\infty;g\mid\geq 2)\\&+\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f-1)$ and $\overline{N}_0(r,0;g')$ is similarly defined.

Proof. We can easily verify that possible poles of h occur at (i) multiple zeros of f and g, (ii) multiple poles of f and g, (iii) the common zeros of $f-1$ and $g-1$ whose multiplicities are different, (iii) those 1-points of f (g) which are not the 1-points of g (f), (iv) zeros of f' which are not the zeros of $f(f-1)$, (v) zeros of g' which are not zeros of $g\,(g-1)$. Since all the poles of h are simple the lemma follows from above. This proves the lemma. □

Lemma 2.3. *If for a positive integer k, $N_k(r,0;f' \mid f\neq 0)$ denotes the counting function of those zeros of of f' which are not the zeros of f, where a zero of f' with multiplicity m is counted m times if $m\le k$ and k times if $m>k$ then*

$$N_k(r,0;f' \mid f\neq 0)\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-\sum_{p=k+1}^{\infty}\overline{N}\Big(r,0;\frac{f'}{f}\mid\ge p\Big)+S(r,f)\,.$$

Proof. By the first fundamental theorem and Milloux theorem (p. 55 [5]) we get

$$\begin{aligned}N(r,0;f' \mid f\neq 0)&=N\Big(r,0;\frac{f'}{f}\Big)\le N\Big(r,\infty;\frac{f'}{f}\Big)+S(r,f)\\&=\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\,.\end{aligned}$$

Now

$$\begin{aligned}N_k\Big(r,0;\frac{f'}{f}\Big)+\sum_{p=k+1}^{\infty}\overline{N}\Big(r,0;\frac{f'}{f}\mid\ge p\Big)&=N\Big(r,0;f' \mid f\neq 0\Big)\\&\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f)\,.\end{aligned}$$

The lemma follows from above as $N_k\big(r,0;\frac{f'}{f}\big)=N_k(r,0;f' \mid f\neq 0)$. □

Lemma 2.4. *Let f, g share "$(1,2)$" and $h\not\equiv 0$. Then*

$$\begin{aligned}T(r,f)\le{}& N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\&-\sum_{p=3}^{\infty}\overline{N}\Big(r,0;\frac{g'}{g}\mid\ge p\Big)+S(r,f)+S(r,g)\,.\end{aligned}$$

Proof. Since f and g share "$(1,2)$" it follows that f and g share "$(1,1)$". Also we note that $\overline{N}_L(r,1;f)+\overline{N}_L(r,1;g)\le\overline{N}(r,1;g\mid\ge 3)$. So by the second fundamental theorem Lemmas 2.1, 2.2 and 2.3 we get

$$\begin{aligned}T(r,f)&\le\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;f)-N_0(r,0;f')+S(r,f)\\&\le\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+N(r,1;f\mid\le 1)+\overline{N}(r,1;f\mid\ge 2)-N_0(r,0;f')\\&\le N_2(r,0;f)+N_2(r,\infty;f)+\overline{N}(r,0;g\mid\ge 2)+\overline{N}(r,\infty;g\mid\ge 2)\\&\quad+\overline{N}(r,1;g\mid\ge 2)+\overline{N}(r,1;g\mid\ge 3)+S(r,f)+S(r,g)\end{aligned}$$

$$\begin{aligned}
&\le N_2(r,0;f)+N_2(r,\infty;f)+\overline{N}(r,0;g\mid\ge 2)+\overline{N}(r,\infty;g\mid\ge 2)\\
&\quad + N_2(r,0;g'\mid g\ne 0)+S(r,f)+S(r,g)\\
&\le N_2(r,0;f)+N_2(r,\infty;f)+N_2(r,0;g)+N_2(r,\infty;g)\\
&\quad -\sum_{p=3}^{\infty}\overline{N}\Big(r,0;\frac{g'}{g}\mid\ge p\Big)+S(r,f)+S(r,g)\,.
\end{aligned}$$

□

Lemma 2.5 ([17]). *Let f be a nonconstant meromorphic function and let*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n\ne 0$ and $b_m\ne 0$. Then

$$T\big(r,R(f)\big)=dT(r,f)+S(r,f)\,,$$

where $d=\max\{n,m\}$.

Lemma 2.6. *Let $F_1=\frac{f^n(af^2+bf+c)f'}{\alpha}$ and $G_1=\frac{g^n(ag^2+bg+c)g'}{\alpha}$, where $a\ne 0$ and $|b|+|c|\ne 0$. Then $S(r,F_1)=S(r,f)$ and $S(r,G_1)=S(r,g)$.*

Proof. Using Lemma 2.5 we see that

$$T(r,F_1)\le (n+2)T(r,f)+T(r,f')+S(r,f)=(n+4)\,T(r,f)+S(r,f)$$

and

$$(n+2)T(r,f)=T(r,f^n(af^2+bf+c))+0(1)\le T(r,F_1)+T(r,f')+S(r,f)\,,$$

that is,

$$T(r,F_1)\ge n\,T(r,f)+S(r,f)\,.$$

Hence $S(r,F_1)=S(r,f)$. In the same way we can prove $S(r,G_1)=S(r,g)$. □

Lemma 2.7 ([21]). *If $h\equiv 0$ and*

$$\limsup_{r\to\infty}\frac{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)}{T(r)}<1\,,\quad r\in I$$

then $f\equiv g$ or $f\cdot g\equiv 1$.

Lemma 2.8. *Let f, g be two nonconstant meromorphic functions. Then*

$$f^n(af^2+bf+c)f'g^n(ag^2+bg+c)g'\not\equiv\alpha^2\,,$$

where $a\ne 0$ and $|b|+|c|\ne 0$ and $n\ (>7)$ is an integer.

Proof. If possible, let

$$f^n(af^2+bf+c)f'g^n(ag^2+bg+c)g'\equiv\alpha^2\,. \tag{2.1}$$

We consider the following cases.

Case 1. The roots of the equation $az^2+bz+c=0$ are distinct and suppose they are β_1 and β_2.

Subcase 1.1. One of β_1 and β_2 say $\beta_2 = 0$. Then (2.1) reduces to

$$a^2 f^{n+1}(f-\beta_1)f' g^{n+1}(g-\beta_1)g'' \equiv \alpha^2 .$$

Let z_0 be a zero of f with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Clearly z_0 is a pole of g with multiplicity $q\,(\geq 1)$ such that

$$(n+1)p + p - 1 = (n+2)q + q + 1\,, \tag{2.2}$$

i.e.

$$q = (n+2)(p-q) - 2 \geq n\,.$$

Again from (2.2) we get

$$(n+2)p = (n+3)q + 2 = (n+2)q + q + 2 \geq (n+1)(n+2)\,, \quad \text{i.e.,} \quad p \geq n+1\,.$$

Noting that α is a small function we obtain

$$N(r,0;f) \geq (n+1)\overline{N}(r,0;f) + S(r,f)\,.$$

Next suppose z_1 be a zero of $f-\beta_1$ with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Then z_1 be a pole of g with multiplicity $q\,(\geq 1)$ such that

$$2p - 1 = (n+1)q + 2q + 1 \quad \text{i.e.,} \quad p \geq \frac{n+5}{2}\,.$$

Let $\overline{N}_{\otimes}(r,0;f')$ $(\overline{N}_{\otimes}(r,0;g'))$ denotes the reduced counting function of those zeros of f' (g') which are not the zeros of $f(f-\beta_1)$ $(g(g-\beta_1))$. Since a pole of f is either a zero of $g(g-\beta_1)$ or a zero of g' or a zero or pole of α we note that

$$\begin{aligned}
\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,\beta_1;g) + \overline{N}_{\otimes}(r,0;g') + S(r) \\
&\leq \frac{1}{n+1}N(r,0;g) + \frac{2}{n+5}N(r,\beta_1;g) + \overline{N}_{\otimes}(r,0;g') + S(r) \\
&\leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,g) + \overline{N}_{\otimes}(r,0;g') + S(r)\,.
\end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned}
T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,\beta_1;f) + \overline{N}(r,\infty;f) - \overline{N}_{\otimes}(r,0;f') + S(r,f) \\
&\leq \frac{1}{n+1}N(r,0;f) + \frac{2}{n+5}N(r,\beta_1;f) + \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,g) \\
&\quad + \overline{N}_{\otimes}(r,0;g') - \overline{N}_{\otimes}(r,0;f') + S(r)\,,
\end{aligned}$$

i.e.,

$$\begin{aligned}
\left(1 - \frac{1}{n+1} - \frac{2}{n+5}\right) T(r,f) &\leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,g) \\
&\quad + \overline{N}_{\otimes}(r,0;g') - \overline{N}_{\otimes}(r,0;f') + S(r)\,.
\end{aligned} \tag{2.3}$$

In a similar manner we get

$$\begin{aligned}
\left(1 - \frac{1}{n+1} - \frac{2}{n+5}\right) T(r,g) &\leq \left(\frac{1}{n+1} + \frac{2}{n+5}\right) T(r,f) \\
&\quad + \overline{N}_{\otimes}(r,0;f') - \overline{N}_{\otimes}(r,0;g') + S(r)\,.
\end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4) we get

$$\left(1-\frac{2}{n+1}-\frac{4}{n+5}\right)\{T(r,f)+T(r,g)\}\leq S(r)\,,$$

which is a contradiction for $n>7$. Hence this subcase does not hold.

Subcase 1.2. Both the roots β_1 and β_2 are non zero.
Let z_0 be a zero of f with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Then from (2.1) we get z_0 is a pole of g with multiplicity $q\,(\geq 1)$ such that

$$np+p-1=(n+3)q+1 \tag{2.5}$$

i.e., $q\geq\frac{n-1}{2}$. So from (2.5) we get

$$(n+1)p\geq\frac{(n+3)(n-1)+4}{2}\,,\quad\text{i.e.,}\quad p\geq\frac{n+1}{2}\,.$$

So from above we have

$$N(r,0;f)\geq\frac{n+1}{2}\overline{N}(r,0;f)+S(r,f)\,,\quad\text{and so}\quad\Theta(0;f)\geq 1-\frac{2}{n+1}\,.$$

Next suppose z_1 be a zero of $f-\beta_1$ with multiplicity $p\,(\geq 1)$ and it is not a zero or pole of α. Then z_1 be a pole of g with multiplicity $q\,(\geq 1)$ such that

$$2p-1=(n+3)q+1\,,\quad\text{i.e.,}\quad p=\frac{(n+3)q+2}{2}\geq\frac{n+5}{2}\,.$$

$$N(r,\beta_1;f)\geq\frac{n+5}{2}\overline{N}(r,0;f)+S(r,f)\,,\quad\text{and so}\quad\Theta(\beta_1;f)\geq 1-\frac{2}{n+5}\,.$$

Similarly we can deduce that

$$\Theta(\beta_2;f)\geq 1-\frac{2}{n+5}\,.$$

Since $\Theta(0;f)+\Theta(\beta_1;f)+\Theta(\beta_2;f)\leq 2$, it follows that

$$3-\frac{4}{n+5}-\frac{2}{n+1}\leq 2\,,\quad\text{or}\quad\frac{4}{n+5}+\frac{2}{n+1}\geq 1$$

which is a contradiction for $n>7$. Hence this subcase also does not hold.

Case 2. The roots of the equation $az^2+bz+c=0$ are equal say $\beta_1=\beta_2=\beta$.
Let z_0 be a zero of f with multiplicity $p\,(\geq 1)$ which is not a zero or pole of α. Then z_0 is a pole of g with multiplicity $q\,(\geq 1)$ such that $np+p-1=(n+3)q+1$, i.e.

$$q\geq\frac{n-1}{2}\quad\text{and so}\quad p\geq\frac{n+1}{2}\,.$$

Hence

$$N(r,0;f)\geq\frac{n+1}{2}\overline{N}(r,0;f)+S(r,f)\,.$$

Next suppose z_1 be a zero of $f-\beta$ with multiplicity $p(\geq 1)$ which is not a zero or pole of α. Then z_1 be a pole of g with multiplicity $q(\geq 1)$ such that

$$3p-1=(n+3)q+1\geq n+4\,,\quad\text{i.e.,}\quad p\geq\frac{n+5}{3}\,.$$

Let $\overline{N}_{\oplus}(r,0;f')$ ($\overline{N}_{\oplus}(r,0;g')$) denotes the reduced counting function of those zeros of f' (g') which are not the zeros of $f(f-\beta)$ ($g(g-\beta)$). Now proceeding in the same way as done in Subcase 1.1 we note that

$$\overline{N}(r,\infty;f) \le \Big(\frac{2}{n+1}+\frac{3}{n+5}\Big)\, T(r,g) + \overline{N}_{\oplus}(r,0;g') + S(r)\,.$$

By the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,\beta;f) + \overline{N}(r,\infty;f) - \overline{N}_{\oplus}(r,0;f') + S(r,f) \\ &\le \frac{2}{n+1} N(r,0;f) + \frac{3}{n+5} N(r,\beta;f) + \Big(\frac{2}{n+1}+\frac{3}{n+5}\Big)\, T(r,g) \\ &\quad + \overline{N}_{\oplus}(r,0;g') - \overline{N}_{\oplus}(r,0;f') + S(r)\,, \end{aligned}$$

i.e.,

$$\begin{aligned} \Big(1-\frac{2}{n+1}-\frac{3}{n+5}\Big)\, T(r,f) &\le \Big(\frac{2}{n+1}+\frac{3}{n+5}\Big)\, T(r,g) \\ &\quad + \overline{N}_{\oplus}(r,0;g') - \overline{N}_{\oplus}(r,0;f') + S(r)\,. \end{aligned} \tag{2.6}$$

In a similar manner we get

$$\begin{aligned} \Big(1-\frac{2}{n+1}-\frac{3}{n+5}\Big)\, T(r,g) &\le \Big(\frac{2}{n+1}+\frac{3}{n+5}\Big)\, T(r,f) \\ &\quad + \overline{N}_{\oplus}(r,0;f') - \overline{N}_{\oplus}(r,0;g') + S(r)\,. \end{aligned} \tag{2.7}$$

Adding (2.6) and (2.7) we get

$$\Big(1-\frac{4}{n+1}-\frac{6}{n+5}\Big)\, \{T(r,f)+T(r,g)\} \le S(r)\,,$$

which is a contradiction for $n>7$. This proves the lemma. □

Lemma 2.9. *Let $F = f^{n+1}\big[\frac{af^2}{n+3}+\frac{bf}{n+2}+\frac{c}{n+1}\big]$ and $G = g^{n+1}\big[\frac{ag^2}{n+3}+\frac{bg}{n+2}+\frac{c}{n+1}\big]$, where $n(\ge 5)$ is an integer $a \ne 0$, $|b|+|c| \ne 0$. Then $F' \equiv G'$ implies $F \equiv G$.*

Proof. Let $F' \equiv G'$, then $F = G + d$ where d is a constant. If possible let $d \ne 0$. Then by the second fundamental theorem and Lemma 2.5 we get

$$\begin{aligned} (n+3)T(r,f) &\le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,d;F) + S(r,F) \\ &\le \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,\beta_1;f) + \overline{N}(r,\beta_2;f) \\ &\quad + \overline{N}(r,0;g) + \overline{N}(r,\beta_1;g) + \overline{N}(r,\beta_2;g) + S(r,f) \\ &\le 4T(r,f) + 3T(r,g) + S(r,f)\,, \end{aligned} \tag{2.8}$$

where β_1 and β_2 are the roots of the equation $az^2+bz+c=0$. In a similar manner we get

$$(n+3)T(r,g) \le 3T(r,f) + 4T(r,g) + S(r,g). \tag{2.9}$$

Adding (2.8) and (2.9) we get

$$(n-4)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which is a contradiction for $n \ge 5$. So $d=0$ and the lemma follows. □

Lemma 2.10 ([4])**.** *Let*

$$Q(\omega) = (n-1)^2(\omega^n - 1)(\omega^{n-2} - 1) - n(n-2)\,(\omega^{n-1} - 1)^2\,,$$

then

$$Q(\omega) = (\omega - 1)^4(\omega - \beta_1)\,(\omega - \beta_2)\ldots(\omega - \beta_{2n-6})\,,$$

where $\beta_j \in C \setminus \{0,1\}$ $(j = 1, 2, \ldots, 2n-6)$, *which are distinct respectively.*

Lemma 2.11. *Let* F *and* G *be given as in Lemma 2.9 and* $n\,(\geq 3)$ *be an integer. Suppose* $F \equiv G$. *Then the following holds.*

(i) *If* $b \neq 0$, $c = 0$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$ *then* $f \equiv g$.

(ii) *If* $b \neq 0$, $c \neq 0$, *and the roots of the equation* $az^2 + bz + c = 0$ *are distinct and one of* f *and* g *is non entire meromorphic functions having only multiple poles then* $f \equiv g$.

(iii) *If* $b \neq 0$, $c \neq 0$, *and the roots of the equation* $az^2 + bz + c = 0$ *coincides then* $f \equiv g$.

(iv) *If* $b = 0$, $c \neq 0$ *then either* $f \equiv g$ *or* $f \equiv -g$.
If n *is an even integer then the possibility* $f \equiv -g$ *does not arise.*

Proof. We consider the following cases.

Case 1. Suppose $c = 0$ and $b \neq 0$. Then $F \equiv G$ implies

$$f^{n+2}(\frac{a}{n+3}f + \frac{b}{n+2}) \equiv g^{n+2}(\frac{a}{n+3}g + \frac{b}{n+2})\,. \tag{2.10}$$

Let us assume $f \not\equiv g$. We consider two cases:

Subcase 1.1. Let $y = \frac{g}{f}$ be a constant. Since $y \neq 1$, from (2.10) it follows that $y^{n+2} \neq 1$, $y^{n+3} \neq 1$ and $f \equiv -\frac{b(n+3)(1-y^{n+2})}{a(n+2)(1-y^{n+3})}$, a constant, which is impossible.

Subcase 1.2. Let $y = \frac{g}{f}$ be nonconstant. Noting that $f \not\equiv g$ clearly the poles of f comes from the zeros of $y - u_k$ where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \ldots, n+2$. So we have

$$\sum_{k=1}^{n+2} \overline{N}(r, u_k; y) \leq \overline{N}(r, \infty; f)\,.$$

By the second fundamental theorem and Lemma 2.5 we get

$$\begin{aligned} n\,T(r,y) &\leq \sum_{k=1}^{n+2} \overline{N}(r, u_k; y) + S(r,y) \leq \overline{N}(r,\infty; f) + S(r,y) \\ &\leq (1 - \Theta(\infty; f) + \varepsilon)\,T(r,f) + S(r,y) \\ &= (n+2)\,(1 - \Theta(\infty; f) + \varepsilon)\,T(r,y) + S(r,y)\,, \end{aligned}$$

i.e.,

$$\Big[\frac{n}{n+2} - 1 + \Theta(\infty; f) - \varepsilon\Big]T(r,y) \leq S(r,y)\,, \tag{2.11}$$

where $\varepsilon > 0$ be arbitrary. In a similar manner we can obtain

$$\left[\frac{n}{n+2} - 1 + \Theta(\infty; g) - \varepsilon\right] T(r, y) \leq S(r, y)\,. \tag{2.12}$$

Adding (2.11) and (2.12) we get

$$\left(\Theta(\infty; f) + \Theta(\infty; g) - \frac{4}{n+2} - 2\varepsilon\right) T(r, y) \leq S(r, y)\,. \tag{2.13}$$

Since $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+2}$ we can choose a $\delta > 0$ such that

$$\Theta(\infty; f) + \Theta(\infty; g) = \frac{4}{n+2} + \delta\,.$$

So for $0 < \varepsilon < \frac{\delta}{2}$ from (2.13) we can deduce a contradiction. Hence $f \equiv g$.

Case 2. Suppose $b \neq 0$ and $c \neq 0$. Then $F \equiv G$ implies

$$Af^{n+3} + Bf^{n+2} + Cf^{n+1} \equiv Ag^{n+3} + Bg^{n+2} + Cg^{n+1}\,, \tag{2.14}$$

where $A = \frac{a}{n+3}$, $B = \frac{b}{n+2}$ and $C = \frac{c}{n+1}$.

Let us assume $f \not\equiv g$.

Subcase 2.1. Suppose the roots of the equation $az^2 + bz + c = 0$ are distinct. Since (2.14) implies f, g share (∞, ∞) without loss of generality we may assume that g has some multiple poles. Putting $\eta = \frac{f}{g}$ in (2.14) we get

$$Ag^2(\eta^{n+3} - 1) + Bg(\eta^{n+2} - 1) + C(\eta^{n+1} - 1) \equiv 0\,,$$

i.e.,

$$Ag^2 \equiv -Bg\ \frac{\eta^{n+2} - 1}{\eta^{n+3} - 1} - C\ \frac{\eta^{n+1} - 1}{\eta^{n+3} - 1}\,. \tag{2.15}$$

Let z_0 be a pole of g which is not a root of $\eta - u_k = 0$, where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \ldots, n+2$ with multiplicity p. Then from (2.15) we have

$$2p = p \quad \text{i.e.,} \quad p = 0\,,$$

which is impossible. The other poles of the right hand side of (2.15) are the roots of $\eta - u_k = 0$ where $u_k = \exp(\frac{2k\pi i}{n+3})$, $k = 1, 2, \ldots, n+2$. Suppose z_1 is a zero of $\eta - u_k$ of multiplicity r. From (2.15) we see that z_1 is a pole of g with multiplicity s (say) such that

$$2s = r + s \quad \text{i.e.,} \quad r = s\,.$$

Since g has no simple pole it follows that $\eta - u_k$ has no simple zero for $k = 1, 2, \ldots, n+2$. Hence

$$\Theta(u_k; \eta) \geq \frac{1}{2}$$

for $k = 1, 2, \ldots, n+2$. Since $\sum\limits_{k=1}^{n+2} \Theta(u_k; \eta) \leq 2$ and $n \geq 3$ we arrive at a contradiction.

Subcase 2.2. Suppose the roots of the equation $az^2 + bz + c = 0$ coincides and so we obtain $b^2 = 4ac$. Putting $\eta = \frac{f}{g}$ in (2.14) we get

$$\begin{aligned} &a(n+2)(n+1)g^2(\eta^{n+3}-1) + b(n+3)(n+1)g(\eta^{n+2}-1) \\ &\qquad\qquad + c(n+3)(n+2)(\eta^{n+1}-1) \equiv 0\,. \end{aligned} \tag{2.16}$$

Since η is not constant using Lemma 2.10 we get from (2.16) that

$$\begin{aligned} &\left[(n+2)(n+1)g(\eta^{n+3}-1) + \frac{b}{2a}(n+3)(n+1)(\eta^{n+2}-1)\right]^2 \\ &= -\,(n+3)(n+1)\left[\frac{c}{a}(n+2)^2(\eta^{n+3}-1)(\eta^{n+1}-1)\right. \\ &\quad \left. - \frac{b^2}{4a^2}(n+3)(n+1)(\eta^{n+2}-1)^2\right] = -\frac{c}{a}(n+3)(n+1)Q(\eta)\,, \end{aligned}$$

where $Q(\eta) = (\eta-1)^4(\eta-\beta_1)\ (\eta-\beta_2)\dots(\eta-\beta_{2n})$ and $\beta_j \in C \setminus \{0,1\}$ $(j = 1,2,\dots,2n)$ which are distinct. This implies that every zero of $\eta - \beta_j$ $(j = 1,2,\dots,2n)$ has a multiplicity of at least 2, i.e., $\Theta(\beta_j;\eta) \geq \frac{1}{2}$ for $(j = 1,2,\dots,2n)$. But $\sum\limits_{j=1}^{2n} \Theta(\beta_j;\eta) \leq 2$ which implies $n \leq 2$. This is a contradiction. So η is constant and from (2.15) we have $(\eta^{n+1}-1) = 0$ and $(\eta^{n+2}-1) = 0$ which implies $\eta = 1$ and so $f \equiv g$.

Case 3. Suppose $b = 0$ and $c \neq 0$. Then (2.14) reduces to

$$\left[\frac{a}{n+3}f^2 + \frac{c}{n+1}\right] f^{n+1} \equiv \left[\frac{a}{n+3}g^2 + \frac{c}{n+1}\right] g^{n+1}\,.$$

Now proceeding in the line of Lemma 2.4 in [12] we can prove $f \equiv g$ and $f \equiv -g$ and if n is an even integer then the possibility of $f \equiv -g$ does not arise. □

Lemma 2.12 ([19]). *Let f be a nonconstant meromorphic function. Then*

$$N(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N(r,0;f) + S(r,f)\,.$$

Lemma 2.13. *Let F and G be given as in Lemma 2.9 and F_1, G_1 be given by Lemma 2.6. If γ_1, γ_2 are the roots of $\frac{a}{n+3}z^2 + \frac{b}{n+2}z + \frac{c}{n+1} = 0$ and β_1, β_2 are the roots of $az^2 + bz + c = 0$. Then*

$$\begin{aligned} T(r,F) \leq\ & T(r,F_1) + N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) \\ & - N(r,\beta_1;f) - N(r,\beta_2;f) - N(r,0;f') + S(r)\,. \end{aligned}$$

Proof. Clearly $F' = \alpha\, F_1$ and $G' = \alpha\, G_1$. By the first fundamental theorem and Lemmas 2.5, 2.6 we obtain

$$\begin{aligned}
T(r,F) &= T(r,\frac{1}{F}) + O(1) = N(r,0;F) + m(r,\frac{1}{F}) + O(1)\\
&\le N(r,0;F) + m(r,\frac{F'}{F}) + m(r,0;F') + O(1)\\
&= T(r,F') + N(r,0;F) - N(r,0;F') + S(r,F)\\
&\le T(r,F_1) + (n+1)N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) - nN(r,0;f)\\
&\quad - N(r,\beta_1;f) - N(r,\beta_2;f) - N(r,0;f') + S(r)\\
&= T(r,F_1) + N(r,0;f) + N(r,\gamma_1;f) + N(r,\gamma_2;f) - N(r,\beta_1;f)\\
&\quad - N(r,\beta_2;f) - N(r,0;f') + S(r)\,. \qquad \square
\end{aligned}$$

3. Proof of the theorem

Proof of Theorem 1.1. Let F, G be defined as in Lemma 2.9 and F_1 and G_1 be defined as in Lemma 2.6. Then it follows that F' and G' share "$(\alpha;2)$" and hence F_1 and G_1 share "$(1,2)$". Suppose $H \not\equiv 0$. Then by Lemmas 2.4, 2.6 and (2.6) we get

$$\begin{aligned}
T(r,F_1) &\le N_2(r,0;F_1) + N_2(r,\infty;F_1) + N_2(r,0;G_1)\\
&\quad + N_2(r,\infty;G_1) + S(r,f) + S(r,g)\\
&\le 2\overline{N}(r,0;f) + N(r,\beta_1;f) + N(r,\beta_2;f) + 2\overline{N}(r,0;g)\\
&\quad + N(r,\beta_1;g) + N(r,\beta_2;g) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g)\\
&\quad + N(r,0;f') + \overline{N}(r,0;g') + S(r)\,.
\end{aligned} \tag{3.1}$$

Now from Lemmas 2.5, 2.12 and 2.13 we can obtain from (3.1) for $\varepsilon(>0)$

$$\begin{aligned}
(n+3)T(r,f) &\le 2\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) + 3T(r,f) + 2\overline{N}(r,0;g)\\
&\quad + 2\overline{N}(r,\infty;g) + 2T(r,g) + N(r,0;g') + S(r)\\
&\le 5T(r,f) + 5T(r,g) + 2\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + S(r)\\
&\le (15 - 2\Theta(\infty;f) - 3\Theta(\infty;g) + 2\varepsilon)\ T(r) + S(r)\,.
\end{aligned} \tag{3.2}$$

In a similar manner we can obtain

$$(n+3)T(r,g) \le (15 - 3\Theta(\infty;f) - 2\Theta(\infty;g) + 2\varepsilon)\,T(r) + S(r)\,. \tag{3.3}$$

From (3.2) and (3.3) we get

$$\big[n - 12 + 2\Theta(\infty;f) + 2\Theta(\infty;g) + \min\{\Theta(\infty;f);\Theta(\infty;g)\} - 2\varepsilon\big]T(r) \le S(r)\,. \tag{3.4}$$

Since ε (> 0) is arbitrary, (3.4) implies a contradiction. Hence $H \equiv 0$.
Since for $\varepsilon > 0$ we have

$$\begin{aligned}\overline{N}(r,0;f') &\leq T(r,f') - m\Big(r,\frac{1}{f'}\Big)\\ &\leq m(r,f) + N(r,\infty;f) + \overline{N}(r,\infty;f) - m\Big(r,\frac{1}{f'}\Big) + S(r,f)\\ &\leq (2-\Theta(\infty;f)+\varepsilon)T(r,f) - m\Big(r,\frac{1}{f'}\Big) + S(r,f)\,.\end{aligned}$$

We note that

$$\begin{aligned}&\overline{N}(r,0;F_1) + \overline{N}(r,\infty;F_1) + \overline{N}(r,0;G_1) + \overline{N}(r,\infty;G_1)\\ &\quad\leq \overline{N}(r,0;f) + \overline{N}(r,\beta_1;f) + \overline{N}(r,\beta_2;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;f')\\ &\qquad + \overline{N}(r,0;g) + \overline{N}(r,\beta_1;g) + \overline{N}(r,\beta_2;g) + \overline{N}(r,\infty;g) + \overline{N}(r,0;g')\\ &\quad\leq \ (12 - 2\Theta(\infty;f) - 2\Theta(\infty;g) + 2\varepsilon)\,T(r)\\ &\qquad - m(r,0;f') - m(r,0;g') + S(r)\,. \end{aligned}\tag{3.5}$$

Also using Lemma 2.5 we get

$$\begin{aligned}T(r,F') + m\Big(r,\frac{1}{f'}\Big) &= m\big(r,f^n(af^2+bf+c)f'\big) + m\Big(r,\frac{1}{f'}\Big)\\ &\quad + N(r,\infty;f^n(af^2+bf+c)f') \geq m\left(r,f^n(af^2+bf+c)\right)\\ &\quad + N(r,\infty;f^n\left(af^2+bf+c\right)) = T(r,f^n\left(af^2+bf+c\right))\\ &= (n+2)T(r,f) + O(1)\,.\end{aligned}\tag{3.6}$$

Similarly

$$T(r,G') + m\Big(r,\frac{1}{g'}\Big) \geq (n+2)T(r,g) + O(1)\,. \tag{3.7}$$

From (3.6) and (3.7) we get

$$\max\left\{T(r,F_1),T(r,G_1)\right\} \geq (n+2)T(r) - m\Big(r,\frac{1}{f'}\Big) - m\Big(r,\frac{1}{g'}\Big) + O(1)\,. \tag{3.8}$$

By (3.5) and (3.8) applying Lemma 2.7 we get either $F_1 \equiv G_1$ or $F_1G_1 \equiv 1$.

Now from Lemma 2.8 it follows that $F_1G_1 \not\equiv 1$. Again $F_1 \equiv G_1$ implies $F' \equiv G'$. So from Lemmas 2.9 and 2.11 the theorem follows. □

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