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ON DESZCZ SYMMETRIES OF WINTGEN IDEAL SUBMANIFOLDS

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Abstract. It was conjectured in [26] that, for all submanifolds M^n of all real space forms $\tilde{M}^{n+m}(c)$, the *Wintgen inequality* $\rho \leq H^2 - \rho^\perp + c$ is valid at all points of M, whereby ρ is the *normalised scalar curvature* of the Riemannian manifold M and H^2, respectively $\rho^\perp$, are the *squared mean curvature* and the *normalised scalar normal curvature* of the submanifold M in the ambient space $\tilde{M}$, and this conjecture was shown there to be true whenever codimension $m = 2$. For a given Riemannian manifold M, this inequality can be interpreted as follows: for all possible isometric immersions of M^n in space forms $\tilde{M}^{n+m}(c)$, the value of the *intrinsic* scalar curvature ρ of M puts a lower bound to all possible values of the *extrinsic* curvature $H^2 - \rho^\perp + c$ that M in any case can not avoid to "undergo" as a submanifold of $\tilde{M}$. And, from this point of view, then M is called a *Wintgen ideal submanifold* when it actually is able to achieve a realisation in $\tilde{M}$ such that this extrinsic curvature indeed everywhere assumes its theoretically smallest possible value as given by its normalised scalar curvature. For codimension $m = 2$ and dimension $n > 3$, we will show that the submanifolds M which realise such minimal extrinsic curvatures in $\tilde{M}$ do intrinsically enjoy some *curvature symmetries in the sense of Deszcz of their Riemann-Christoffel curvature tensor, of their Ricci curvature tensor and of their conformal curvature tensor of Weyl*, which properties will be described mainly following [20].

1. Deszcz symmetry

Let M^n be an $n-$dimensional *Riemannian manifold* with *metric* $(0,2)$ *tensor* g and *Levi-Civita connection* ∇. Let R denote the $(0,4)$ *Riemann-Christoffel curvature tensor* of M aswell as the *curvature operator* $R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, thus having

$$R(X,Y,Z,W) := g\big(R(X,Y)Z,W\big)\,, \tag{1}$$

whereby X, Y, etc. stand for arbitrary vector fields on M. By the action of the curvature operator working as a derivation on the curvature tensor R, the following

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$(0,6)$ tensor $R \cdot R$ is obtained:

$$\begin{aligned}(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) &:= (R(X,Y) \cdot R)(X_1, X_2, X_3, X_4)\\ &= -R(R(X,Y)X_1, X_2, X_3, X_4) - R(X_1, R(X,Y)X_2, X_3, X_4)\\ &\quad - R(X_1, X_2, R(X,Y)X_3, X_4) - R(X_1, X_2, X_3, R(X,Y)X_4)\,.\end{aligned} \tag{2}$$

We recall that Schouten, in the 1918 article in which he determined the (*pseudo-*)*parallel translation* independently of Levi-Civita, obtained the geometrical interpretation of the curvature tensor R as the second order measure of *the change of the direction of vector fields after their parallel transport around closed infinitesimal curves* on M [25]. Later on this actually became the most often used definition of the curvature tensor, cfr (1), whereby Schouten's geometrical view on R however almost equally often more or less got out of sight, mainly the formal aspects of Koszul's approach towards the connection ∇ being put in focus. The *locally flat spaces*, $R = 0$, thus are the Riemannian manifolds for which all directions remain preserved after parallel transport around all closed curves. The knowledge of the curvature tensor R is equivalent to the knowledge of the *sectional* or *Riemannian curvatures* $K(p,\pi)$, π being any 2-plane in the tangent space T_pM at any point p of M, as shown by Cartan; when π is spanned by orthonormal vectors u and v at p,

$$K(p,\pi) = R(u,v,v,u)]\,. \tag{3}$$

The simplest non-flat Riemannian manifolds M are *the spaces of constant curvature* $K = c$, i.e. the spaces whose function K is *isotropic* (meaning that at each point p the Gauss curvature $K(p,\pi)$ at p of the local surface formed by the geodesics of M which pass through p and whose tangent vector at p lies in π has the same value for all choices of planes π at p, thus K becoming a real function on M, which by the Lemma of Schur, for $n > 2$, then necessarily has to be constant). These *real space forms* M of curvature c are characterised by their curvature tensor R being given by

$$R = \frac{c}{2} g \wedge g\,, \tag{4}$$

whereby the Nomizu-Kulkarni square $g \wedge g$ of the metric tensor g may well be considered as the simplest $(0,4)$ tensor on M with the same algebraic symmetry properties as R. The real space forms can be obtained from the locally flat spaces by *projective transformations* and their class is closed under such transformations. A main interest of Riemann, Helmholtz, Lie, ... in the spaces of constant curvature was related to the fact that these real space forms are precisely the Riemannian manifolds which satisfy *the axiom of free mobility.*

As a generalisation of the spaces of constant curvature Cartan introduced the *locally symmetric spaces*, i.e. the spaces with *parallel curvature tensor* R, $\nabla R = 0$, or, equivalently, as shown by Levi, the spaces for which up to the first order *the sectional curvatures for all planes* π *remain preserved after parallel transport of* π *along arbitrary curves in* M. As Cartan proved, these *Cartan symmetric spaces* are characterised to be the Riemannian manifolds for which *the geodesic reflections in* *all points are local isometries.* In the spirit of Schouten, recently one of the authors and Haesen obtained the geometrical interpretation of the tensor $R\cdot R$ as the second order measure of *the change of the sectional curvatures for planes* π *at points* p *after the parallel transport of* π *around infinitesimal co-ordinate parallelograms in* M *cornered at* p [20]. Thus, the *semi-symmetric* or *Szabó symmetric spaces* [28, 29], $R\cdot R = 0$, are the Riemannian manifolds for which *all sectional curvatures remain preserved after parallel transport around all infinitesimal co-ordinate parallelograms.* The Cartan symmetric spaces clearly are a non-trivial subclass of the class of the Szabó symmetric spaces. Now, proceeding similarly as when going before from the localy flat spaces to the spaces of constant curvature, one of the authors and Haesen proceeded from the semi-symmetric spaces to the *pseudo-symmetric spaces in the sense of Deszcz*, or, put otherwise, to the *Deszcz symmetric spaces.* So, starting from the Szabó symmetric spaces, $R\cdot R = 0$, they looked for the simplest spaces for which the $(0,6)$ tensor $R\cdot R$ is not identically zero, and these turn out to be the spaces for which $R\cdot R$ and the $(0,6)$ *Tachibana tensor* $Q(g,R) = -\wedge_g\cdot R$ are proportional, say $R\cdot R = L_R(Q(g,R))$ for some real valued function L_R on M. Throughout the paper, the pseudo-symmetry function L_R will also more simply be denoted by L. Hereby we recall that the endomorphism $X\wedge_g Y$ being defined by $(X\wedge_g Y)Z := g(Y,Z)X - g(X,Z)Y$, this Tachibana tensor is given by

$$
\begin{aligned}
Q(g,R)(X_1,X_2,X_3,X_4;X,Y) &= -\big((X\wedge_g Y)\cdot R\big)(X_1,X_2,X_3,X_4)\\
&= R\big((X\wedge_g X)X_1,X_2,X_3,X_4\big) + R\big(X_1,(X\wedge_g Y)X_2,X_3,X_4\big)\\
&\quad + R\big(X_1,X_2,(X\wedge_g Y)X_3,X_4\big) + R\big(X_1,X_2,X_3,(X\wedge_g Y)X_4\big)\,.
\end{aligned}
\tag{5}
$$

$Q(g,R)$ may well be the simplest $(0,6)$ tensor on a Riemannian manifold M which shares the algebraic curvature symmetries of the $(0,6)$ tensor $R\cdot R$. A classical result states that the identical vanishing of this Tachibana tensor, $Q(g,R)=0$, characterises the real space forms. Moreover, also similarly as before, results of Mikesh and Venzi and of Defever and Deszcz learn that *when a Riemannian manifold* M *admits a projective mapping onto a Szabó symmetric space, then* M *itself must be a Deszcz symmetric space* and that *the class of Deszcz symmetric spaces is closed under projective transformations.*

Planes π and $\bar\pi$ spanned by vectors u, v, and x, y, respectively, at a same point p of M, are said to be *curvature-dependent* if $Q(g,R)(u,v,v,u;x,y)\neq 0$, (which is independent of the choices of bases for π and $\bar\pi$). For such planes, *the double sectional curvature* or *the sectional curvature of Deszcz* $L(p,\pi,\bar\pi)$ is defined as the real number given by

$$
L(p,\pi,\bar\pi) := \frac{(R\cdot R)(u,v,v,u;x,y)}{Q(g,R)(u,v,v,u;x,y)}\,,
\tag{6}
$$

(which is independent of the choice of bases). And likewise as in Cartan's result on the equivalence of information contained in the $(0,4)$ tensor R and in the sectional curvatures $K(p,\pi)$, there is the equivalence of information contained in the $(0,6)$ tensor $R\cdot R$ and in the sectional curvatures $L(p,\pi,\bar\pi)$ of Deszcz. This sectional curvature of Deszcz, $L(p,\pi,\bar\pi)$, is a scalar valued *Riemannian invariant* depending on any two curvature-dependent 2-planes π and $\bar{\pi}$ at any point p of M. And whereas $(R \cdot R)(u, v, v, u; x, y)$ measures the second order change of a sectional curvature $K(p, \pi)$ when π is transported parallely around an infinitesimal co-ordinate parallelogram cornered at p with sides directed by x and y, $Q(g, R)(u, v, v, u; x, y)$ in some sense calibrates this change by the change of $K(p, \pi)$ under the performance of a kind of rotation of π at p defined in connection with infinitesimal rotations of projections of $\pi = u \wedge v$ onto $\bar{\pi} = x \wedge y$. We restrict ourselves at this place just to mention further that, like the sectional curvatures $K(p, \pi)$ can nicely be interpreted geometrically in terms of the parallelogramoïds of Levi-Civita, which was likely the main goal of the 1917 article in which he defined his parallel transport [23], also the sectional curvatures $L(p, \pi, \bar{\pi})$ of Deszcz can be well interpreted geometrically in terms of these parallelogramoïds. And the Riemannian curvatures of Deszcz are *isotropic*, i.e. at any point p of M the numbers $L(p, \pi, \bar{\pi})$ do not depend on the planes π and $\bar{\pi}$, but can only depend on the points p of M, if and only if the Riemannian manifold M is *Deszcz symmetric*, $R \cdot R = L(Q(g, R))$ for some function $L\colon M \to R$. In the present situation however, there is no lemma of Schur, which then would further force this function L to be constant; therefore, Kowalski and Sekizawa called the Deszcz symmetric spaces for which the double sectional curvature L is indeed a constant, independent of the planes π and $\bar{\pi}$ aswell as of the points p of M, the *pseudo-symmetric spaces of constant type* L [22]. By way of examples in this respect we'd like to mention here that the 3*-dimensional Thurston geometries* [30], which in a kind of axiomatic way originated as natural extensions of the spaces of constant curvature with their typical free mobility, all do have constant *Deszcz Riemann curvature* $L = 0, +1, -1$: $L = 0$ for $E^3(K = c = 0)$, $S^3(K = c > 0)$, $H^3(K = c < 0)$, $S^2 \times E^1$ and $H^2 \times E^1$, $L = 1$ for $\widetilde{SL(2, R)}$ and for the 3−dimensional Heisenberg group H_3, and $L = -1$ for the Lie group SOL. For further information on pseudo-symmetry in the sense of Deszcz also concerning the physical space-times, see e.g. [2, 11, 19, 18, 31].

A similar study concerning the geometrical meaning of *Ricci pseudo-symmetry in the sense of Deszcz*, $R \cdot S = L_S\big(Q(g, S)\big) = L_S(\wedge_g \cdot S)$, of Riemannian manifolds M whose Ricci $(0, 2)$ tensor is denoted by S and whereby L_S denotes a real valued function on M, was carried out by one of the authors and Jahanara, Haesen and Sentürk in [21]; in this respect, see also [10, 12].

As shown in [13], a 3-dimensional Riemannian manifold M is Deszcz symmetric if and only if it is *quasi-Einstein*, i.e. if its Ricci tensor has an eigenvalue of multiplicity ≥ 2. We recall that, for arbitrary dimensions $n \geq 3$, the *Einstein spaces* M are characterised by the *isotropy of their Ricci curvatures*, i.e. by the fact that their Ricci curvatures $\mathrm{Ric}(p, d)$ at every point $p \in M$ are independent of the tangent direction d, which then in a Schur-like way further implies that the scalar valued curvature function Ric is actually moreover independent also of p, i.e. is a *constant* function. In [21], the invariance of the Ricci curvatures $\mathrm{Ric}(p, d)$ under the parallel transport of the direction d around co-ordinate parallelograms tangent to planes $\bar{\pi}$ and cornered at p was studied, and related to the *Ricci semi-symmetry* of M, $R \cdot S = 0$. And from here, for the Ricci tensor S and the associated scalar valued curvatures $\mathrm{Ric}(p,d)$, in analogy with the above discussion for the Riemann-Christoffel tensor R and the associated scalar valued curvatures $K(p,\pi)$, the *Ricci curvatures* $\mathrm{Ric}(p,d,\bar{\pi})$ *of Deszcz* were introduced of which the *isotropy*, i.e. their independance at all points $p \in M$ of the directions d as well as of the planes $\bar{\pi}$, was shown to be the geometrical meaning of the Ricci pseudo-symmetry in the sense of Deszcz. The class of Riemannian manifolds with pseudo-symmetric tensor S is considerably larger than the class of those with pseudo-symmetric tensor R (which obviously it contains as a subclass).

At this stage we finally would like to add that a Riemannian manifold M is said to have a *pseudo-symmetric conformal Weyl tensor* C if, in the same notations as before,

$$C \cdot C = L_C\big(Q(g,C)\big) = L_C(\wedge_g \cdot C)\,. \tag{7}$$

The classes of the Deszcz symmetric Riemannian manifolds and of the Riemannian manifolds with pseudo-symmetric Weyl tensor are distinct. In this respect, we just briefly mention hereafter some related observations on *hypersurfaces* M^n *in Euclidean spaces* E^{n+1} which may help to get a better view on the appearance of the latter intrinsic curvature property later on, to be looked back at then, of course, taking into account the Gauss equation of the 2-codimensional submanifolds of not necessarily flat real space forms in comparison with the one of Euclidean hyprsurfaces.

A hypersurface M^n of dimension $n > 2$ in a Euclidean space is *semi symmetric* (including, as particular cases the hypersurfaces which are of constant curvature or more generally which are locally symmetric) if and only if zero is a principal curvature of multiplicity $\geq n-2$ or if M^n has at most two distinct principal curvatures of which one is zero in case there really are two. Such a hypersurface M^n is *properly pseudo-symmetric*, i.e. is Deszcz symmetric but not Szabó symmetric, if and only if M^n has exactly two distinct principal curvatures, none of which is zero. And a Euclidean hypersurface M^n of dimension $n > 3$ has a *pseudo-symmetric Weyl tensor* C if and only if it is a 2-*quasi umbilical* hypersurface, i.e. if it has a principal curvature with multiplicity $\geq n-2$. The property of a Riemannian manifold to have a pseudo-symmetric Weyl tensor C is *invariant under the conformal* deformations of the metric tensor g. For further information on this intrinsic curvature condition, see e.g. [1, 13, 14].

2. Wintgen ideal submanifolds

For surfaces M^2 in E^3, the *Euler inequality* $K \leq H^2$, whereby K is the intrinsic *Gauss curvature* of M^2 and H^2 is the extrinsic *squared mean curvature* of M^2 in E^3, at once follows from the fact that $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ whereby k_1 and k_2 denote the *principal curvatures* of M^2 in E^3. And, obviously, $K = H^2$ everywhere on M^2 if and only if the surface M^2 is *totally umbilical* in E^3, i.e. $k_1 = k_2$ at all points of M^2, or still, by a *theorem of Meusnier*, if and only if M^2 is a part of a *plane* E^2 or of a *round sphere* S^2 in E^3. In the late seventies, Wintgen proved that the *Gauss curvature* K and the *squared mean curvature* H^2 and the *normal curvature* $K^\perp$ of any surface M^2 in E^4 always satisfy the inequality

$K \leq H^2 - K^\perp$, and that actually *the equality holds if and only if the curvature ellipse of* M^2 *in* E^4 *is a circle* [32]; (see e.g. also [7, 8] for results on the studies in global differential geometry of Smale, Lashof, Chen, Willmore, a.o. concerning the Euler characteristic of the normal bundle, the number of self-intersections and the total mean curvature of submanifolds). This inequality between the most important intrinsic and extrinsic scalar valued curvatures of surfaces M^2 in E^4 was shown to hold more generally for all surfaces M^2 in arbitrary dimensional space forms $\tilde{M}^{2+m}(c)$, inclusive the above characterisation of the equality case, by Rouxel [24] and by Guadalupe and Rodriguez [17]. After these extensions, in 1981 and in 1983 respectively, in 1999 De Smet and Dillen and Vrancken and one of the authors proved the Wintgen inequality $\rho \leq H^2 - \rho^\perp$ *for all submanifolds* M^n *of codimension* 2 *in all real space forms* $\tilde{M}^{n+2}(c)$ and characterised the equality as follows in terms of the shape operators.

Theorem ([26]). *For every submanifold* M^n *of arbitrary dimension* n *and codimension* 2 *in a real space form* $\tilde{M}^{n+2}(c)$ *of curvature* c*, at every point* p *of* M^n*:*

$$\rho \leq H^2 - \rho^\perp + c\,, \tag{8}$$

and equality holds if and only if there exist orthonormal bases $\{e_1, e_2, e_3, \ldots, e_n\}$ *and* $\{\xi_1, \xi_2\}$ *of respectively the tangent space* T_pM *and the normal space* $T_p^\perp M$ *such that the corresponding Weingarten maps are given by*

$$A_{\xi_1} = \begin{pmatrix} \lambda & \mu & 0 & \ldots & 0 \\ \mu & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda \end{pmatrix}, \quad A_{\xi_2} = \begin{pmatrix} \mu & 0 & 0 & \ldots & 0 \\ 0 & -\mu & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \tag{9}$$

for some $\lambda, \mu \in R$.

Next, we recall a few basic formula's and definitions to be a bit more precise (see also [5]). First, the *normalised scalar curvature* ρ of the Riemannian manifold M^n is defined to be

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} R(e_i, e_j, e_j, e_i)\,, \tag{10}$$

$i, j, \ldots \in \{1, 2, \ldots, n\}$. Denoting the Riemannian metric and the corresponding connection on $\tilde{M}$ by $\tilde{g}$ and $\tilde{\nabla}$, the formula's of Gauss and Weingarten, respectively, read like

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{11}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla_X^\perp \xi\,, \tag{12}$$

whereby $\xi, \eta, \ldots$ denote normal vector fields on M in $\tilde{M}$, h is the *second fundamental form*, A_ξ is the *shape operator* or the *Weingarten map* of M with respect to the normal vector field ξ and $\nabla^\perp$ is the *normal connection* of M in $\tilde{M}$. The *mean curvature vector field* $\overrightarrow{H}$ of M in $\tilde{M}$ is defined as $\overrightarrow{H} = \frac{1}{n} trace\ h$ and its length $H = \|\overrightarrow{H}\|$ is the *mean curvature* of M in $\tilde{M}$. By the *equation of Ricci*, the *normal curvature tensor* $R^\perp$ of M in $\tilde{M}$ is given as follows:

$$R^\perp(X, Y; \xi, \eta) := \tilde{g}\big(R^\perp(X, Y)\xi, \eta\big) = g\big([A_\xi, A_\eta]X, Y\big)\,, \tag{13}$$

whereby $R^\perp(X, Y) := \nabla^\perp_X \nabla^\perp_Y - \nabla^\perp_Y \nabla^\perp_X - \nabla^\perp_{[X,Y]}$. The *normalised scalar normal curvature* $\rho^\perp$ of M in $\tilde{M}$ is then defined to be

$$\rho^\perp = \frac{2}{n(n-1)} \Big\{ \sum_{i<j} \big[R^\perp(e_i, e_j; \xi_1, \xi_2)\big]^2 \Big\}^{\frac{1}{2}}\,. \tag{14}$$

We remark that $\rho^\perp = 0$ if and only if *the normal connection is flat*, which, as follows from (13) and as already observed by Cartan [4], is equivalent to the simultaneous diagonalisability of all shape oprators A_ξ.

Finally, we would like to recall that, as shown by Chen, the extrinsic normal curvature tensor $R^\perp$ and the "mixing" $(\rho - H^2)\, g$ of intrinsic and extrinsic quantities of submanifolds are both *conformal invariants* (see e.g. [7, 6]).

The Wintgen inequality (8) *was conjectured to hold for all submanifolds* M^n *in all space forms* $\tilde{M}^{n+m}(c)$, in the paper [26] of "DDVV", (and some people recently mention this as "the DDVV conjecture" although probably *"conjecture on Wintgen's inequality"* may well be more appropriate; and moreover, there might exist papers in differential geometry by several other DDVV's as well ...). In any case, for pertinent comments and recent contributions to this conjecture, for descriptions and classification results on the submanifolds satisfying the equality in the Wintgen inequality, i.e. on *"Wintgen ideal submanifolds"*, we refer to the works of Bryant, Choi, Dillen, Fastenakels, Lu, Suceavă and Van der Veken in the bibliography. In particular, *Choi and Lu recently proved the conjecture on Wintgen's inequality to be true for all* 3*−dimensional submanifolds* M^3 *in arbitrary dimensional space forms* $\tilde{M}^{3+m}(c)$ [9]. In view of this result (of which we learned just after finishing the work for the present paper) and in view of the fact that some symmetry properties of Wintgen ideal submanifolds turn out to be somewhat special for dimension 3 as compared to dimensions $n > 3$, in the following we will only consider submanifolds M^n of dimension $n > 3$ and of codimension 2, (intending of dealing with the situation of dimension $n = 3$ and codimension $m \geq 2$ at a later occasion).

3. About three Deszcz symmetries of Wintgen ideal submanifolds

Theorem 1. *A Wintgen ideal submanifold* M^n *of dimension* $n > 3$ *and with codimension* 2 *in a real space form* $\tilde{M}^{n+2}(c)$ *of curvature* c *is Deszcz symmetric, i.e. satisfies* $R \cdot R = L(Q(g, R))$*, if and only if* M^n *is totally umbilical in* $\tilde{M}^{n+2}(c)$*, in which case* $L = 0$*, or if* M^n *is minimal in* $\tilde{M}^{n+2}(c)$*, in which case* $L = c$.

Proof. By the Gauss equation

$$R(X,Y,Z,W) = c\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} + \tilde{g}\big(h(Y,Z),h(X,W)\big) - \tilde{g}\big(h(X,Z),h(Y,W)\big) \tag{15}$$

and since

$$h(X,Y) = g\big(A_{\xi_1}(X),Y\big)\xi_1 + g\big(A_{\xi_2}(X),Y\big)\xi_2\,, \tag{16}$$

the Riemann-Christoffel curvature tensor R of a submanifold M^n in $\tilde{M}^{n+2}(c)$ for which $\rho = H^2 - \rho^\perp + c$ is explicitly determined by the characteristic shape operators A_{ξ_1} and A_{ξ_2} as given in (9) for some orthonormal normal vectors ξ_1 and ξ_2 with respect to some orthonormal tangent vectors $e_1, e_2, \ldots, e_n$. Hence $R\cdot R$ and $Q(g,R)$ are readily computable. And, on the other hand, also from the expressions of A_{ξ_1} and A_{ξ_2} given in (9), it is clear that M^n is *totally umbilical* and respectively *minimal* in $\tilde{M}^{n+2}(c)$ if and only if $\mu = 0$ and respectively $\lambda = 0$.

Then, to begin with, the comparison of the components

$$(R\cdot R)\,(e_1,e_4,e_3,e_4;e_1,e_3) = (\lambda\mu)^2 \tag{17}$$

and

$$\big(Q(g,R)\big)(e_1,e_4,e_3,e_4;e_1,e_3) = 0 \tag{18}$$

learns that if M^n is a *pseudo-symmetric space*, then certainly $\lambda\mu$ has to vanish.

If $\mu = 0$, then M^n is a *totally umbilical* submanifold in $\tilde{M}^{n+2}(c)$, (and the converse obviously holds as well). Hence it is itself a real space form [5], (of curvature $c+\lambda^2$), and thus, in particular, it is semi-symmetric, i.e. pseudo-symmetric with $L = 0$. So, the part of the statement in the theorem related to the totally umbilical submanifolds is done with.

If $\lambda = 0$, then M^n is a *minimal* submanifold of $\tilde{M}^{n+2}(c)$, (and the converse obviously holds as well). And then, all the components of the $(0,6)$ tensors $R\cdot R$ and $Q(g,R)$, with respect to the tangent basis $\{e_1,e_2,\ldots,e_n\}$, are either "together" zero, or, when non-zero, they are, up possibly to a common change of sign coming from the algebraic identities satisfied by the tensors $R\cdot R$ and $Q(g,R)$, found to be given "in pairs" with the same values as in formulae (17) and (18), or else, with the same values as, for instance, given in the following formulae:

$$(R\cdot R)\,(e_1,e_2,e_2,e_3;e_1,e_3) = 2\mu^2 c \tag{19}$$

and

$$\big(Q(g,R)\big)\,(e_1,e_2,e_2,e_3;e_1,e_3) = 2\mu^2\,. \tag{20}$$

So, the part of the statement in the theorem related to the minimal submanifolds is also done with. When the ambient space is Euclidean, these submanifolds are semi-symmetric, $L = 0$, and, otherwise, they are properly pseudo-symmetric of constant type $L = c \neq 0$. $\square$

Theorem 2. *A Wintgen ideal submanifold M^n of dimension $n > 3$ and with codimension 2 in a real space form $\tilde{M}^{n+2}(c)$ of curvature c is Deszcz Ricci-symmetric, i.e. satisfies $R \cdot S = L_S\big(Q(g,S)\big)$ for some function $L_S \colon M \to R$, if and only if it is Deszcz symmetric.*

Proof. We need to show that amongst the Wintgen ideal submanifolds M^n in $\tilde{M}^{n+2}(c)$ with $n > 3$, the Deszcz Ricci symmetric ones are totally umbilical or minimal. The Ricci tensor of such M^n in $\tilde{M}^{n+2}$: $S(X,Y) = \sum_{i=1}^n R(E_i, X, Y, E_i)$, whereby $\{E_1, E_2, \dots, E_n\}$ is any orthonormal local tangent frame on M^n, is readily obtained from the Gauss equation (15) together with (16). And, comparing the non-trivial components of the $(0,4)$ tensors $R \cdot S$ and $Q(g,S)$, we find that M^n is Deszcz Ricci symmetric if and only if the following system of three equations holds, (obtained, for instance, by considering the components $\{e_1, e_1; e_1, e_2\}$, $\{e_3, e_2; e_1, e_3\}$ and $\{e_3, e_1; e_1, e_3\}$, respectively):

$$\lambda\mu \left\{L_S - c - \lambda^2 + 2\mu^2\right\} = 0\,, \tag{21}$$

$$\lambda\mu \left\{(n-2)(L_S - c - \lambda^2) + 2\mu^2\right\} = 0\,, \tag{22}$$

$$\mu^2 \left\{2(L_S - c - \lambda^2) + (n-2)\lambda^2\right\} = 0\,. \tag{23}$$

Using that, essentially, $n > 3$, this obviously leads to a contradiction unless $\lambda\mu = 0$. □

Theorem 3. *Every Wintgen ideal submanifold M^n of dimension $n > 3$ and with codimension 2 in a real space form $\tilde{M}^{n+2}(c)$ of curvature c is a Riemannian manifold with pseudo-symmetric conformal Weyl tensor.*

Proof. The $(0,4)$ *conformal Weyl tensor* C of a Riemannian manifold M^n is given by

$$\begin{aligned} C(X_1, X_2, X_3, X_4) &:= R(X_1, X_2, X_3, X_4) \\ &\quad - \frac{1}{n-2}\big\{S(X_1, X_3)g(X_2, X_4) + S(X_2, X_4)g(X_1, X_3) \\ &\quad - S(X_1, X_4)g(X_2, X_3) - S(X_2, X_3)g(X_1, X_4)\big\} \\ &\quad + \frac{\tau}{(n-1)(n-2)}\big\{g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_2, X_3)\big\}\,, \end{aligned} \tag{24}$$

whereby τ denotes the *scalar curvature* of M, i.e. $\tau := \mathit{trace}\, S$. With the same definitions related to C as given explicitly before for R, in a straightforward way, it is found from (9) that for all submanifolds M^n in $\tilde{M}^{n+2}(c)$ for which $\rho = H^2 - \rho^\perp + c$:

$$C \cdot C = L_C\big(Q(g,C)\big) \tag{25}$$

whereby

$$L_C = \frac{2(n-3)}{(n-1)(n-2)}\ \mu^2. \tag{26}$$

□

Proposition 1. *A Wintgen ideal submanifold M^n of dimension $n > 3$ and with codimension 2 in a real space form $\tilde{M}^{n+2}(c)$ of curvature c is minimal if and only if the pseudo-symmetry function of its conformal Weyl tensor C is given by*

$$L_C = \frac{n-3}{(n-1)(n-2)}\,(c - K_{\inf})\,. \tag{27}$$

Proof. The *sectional curvatures* K_{ij}, in the tangent frame $\{e_1, e_2, \dots, e_n\}$ of the Theorem of Section 2, of a Wintgen ideal submanifold M^n in a real space form $\tilde{M}^{n+2}(c)$, by (9), are given by

$$K_{12} = \lambda^2 - 2\mu^2 + c\,, \tag{28}$$

and, for $(i,j) \neq (1,2)$, by

$$K_{ij} = \lambda^2 + c\,. \tag{29}$$

From these values of the particular sectional curvatures for the plane sections spanned by the vectors of this frame, one sees that the *minimal value* $K_{\inf}$ *of the sectional curvature function* $K(p,\pi)$ for the tangent 2−planes π at p on such submanifolds M^n is attained on the plane spanned by e_1 and e_2. And then from (26) and (28) at once follows the proposition. □

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