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TRANSLATIVE PACKING OF A CONVEX BODY BY SEQUENCES OF ITS HOMOTHETIC COPIES

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Abstract. Every sequence of positive or negative homothetic copies of a planar convex body C whose total area does not exceed 0.175 times the area of C can be translatively packed in C.

Let C be a planar convex body with area $|C|$. Moreover, let (C_i) be a finite or infinite sequence of homothetic copies of C. We say that (C_i) can be *translatively packed* in C if there exist translations σ_i such that $\sigma_i C_i$ are subsets of C and that they have pairwise disjoint interiors. Denote by $\phi(C)$ the greatest number such that every sequence of (positive or negative) homothetic copies of C whose total area does not exceed $\phi(C)|C|$ can be translatively packed in C. In [2] it is showed that $\phi(T) = \frac{2}{9} \approx 0.222$ for any triangle T. Moreover, $\phi(S) = 0.5$ for any square S (see [6]). By considerations presented in [7] or in Section 2.11 of [1] we have $\phi(C) \geq 0.125$. The aim of the paper is to prove that $\phi(C) \geq 0.175$ for any convex body C. It is very likely that $\phi(C) \geq \frac{2}{9}$ for any convex body C.

We say that a rectangle *is of type* $a \times h$ if one of its sides, of length a, is parallel to the first coordinate axis and the other side has length h. Moreover, let $[a_1, a_2] \times [b_1, b_2] = \{(x, y); a_1 \leq x \leq a_2, b_1 \leq y \leq b_2\}$.

The packing method presented in the proof of Theorem is similar to that from [3].

Lemma 1. *Let S be a rectangle of side lengths h_1 and h_2. Every sequence of squares of sides parallel to the sides of S and of side lengths not greater than λ can be translatively packed in S provided $\lambda \leq h_1$ and $\lambda \leq h_2$ and the total area of squares in the sequence does not exceed $\frac{1}{2}|S|$.*

Lemma 2. *Let S be a rectangle of side lengths h_1 and h_2. Every sequence of squares of sides parallel to the sides of S and of side lengths not greater than λ can be translatively packed in S provided $\lambda < h_1$ and $\lambda < h_2$ and the total area of squares in the sequence does not exceed $\lambda^2 + (h_1 - \lambda)(h_2 - \lambda)$.*

Lemma 3. *For each convex body C there exist homothetic rectangles P and R such that P is inscribed in C, R is circumscribed about C and that $\frac{1}{2}|R| \leq |C| \leq 2|P|$.*

Lemma 1 was proved by Moon and Moser in [6], Lemma 2 by Meir and Moser in [5] and Lemma 3 by Lassak in [4].

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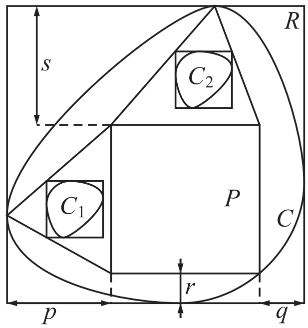

FIG. 1

Theorem. *Every (finite or infinite) sequence of positive or negative homothetic copies of a planar convex body C whose total area does not exceed $0.175|C|$ can be translatively packed in C.*

Proof. Let C be a planar convex body, let C_i be a homothetic copy of C with a ratio μ_i and let $\lambda_i = |\mu_i|$ for $i = 1, 2, \dots$. Moreover, assume that $\sum |C_i| \le 0.175|C|$. We can assume, without loss of generality, that $\lambda_1 \ge \lambda_2 \ge \dots \ge 0$. Obviously, $\lambda_1 \le \sqrt{0.175} < 0.42$. Let R be the rectangle described in Lemma 3. Moreover, let $P \subset C$ be a rectangle homothetic to R and of the area $|P| = \frac{1}{4}|R|$. Because of the affine invariant nature of the problem, we can assume that P and R are squares and that $R = [0,1] \times [0,1]$ (see Figure 1). Let p and r be numbers such that $P = [p, p + \frac{1}{2}] \times \big[r, r + \frac{1}{2}\big]$ and let $q = \frac{1}{2} - p$, $s = \frac{1}{2} - r$. We can assume that $s \ge p \ge q$ (see Figure 1).

Observe that it is possible to place C_1 in $C \cap \big([0, t_1] \times [0,1]\big)$, where

$$t_1 = \lambda_1(1 + 2p)\,.$$

Indeed, it is possible to pack C_1 in $C \cap \big([t - \lambda_1, t] \times [0,1]\big)$, where $\frac{\frac{1}{2}}{\lambda_1} = \frac{p}{t - \lambda_1}$ (see Figure 2). Consequently, $t = \lambda_1(1 + 2p)$.

Consider four cases. In all cases we show that if $C_1, C_2, \dots$ cannot be translatively packed in C, then $\sum \lambda_i^2 > 0.175$, i.e. $\sum |C_i| = \sum \lambda_i^2 |C| > 0.175|C|$, which is again a contradiction.

Case 1, when $\lambda_1 \le \frac{p}{1+2p}$.

Obviously, it is possible to place C_1 in $C \cap \big([0,p] \times \big[r, \frac{1}{2} + r\big]\big)$. Since $\lambda_2 \le \lambda_1$ and $s \ge p$, it is possible to pack C_2 in $C \cap \big(\big[p, \frac{1}{2} + p\big] \times [1 - s, 1]\big)$ (see Figure 1).

By Lemma 2 we know that any sequence of squares of side lengths not greater than λ_3 whose total area does not exceed $\lambda_3^2 + (\frac{1}{2} - \lambda_3)^2$ can be translatively packed in $\frac{1}{2} \times \frac{1}{2}$. Each C_i is contained in a square R_i of sides parallel to the sides of R and with area $|R_i| = |C_i|/|C|$. Consequently, if the total area of $C_3, C_4, \dots$

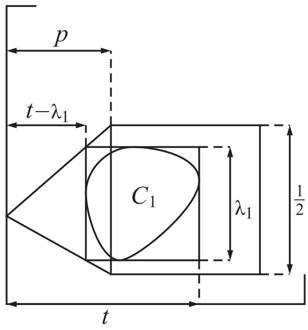

FIG. 2

does not exceed $\left[\lambda_3^2 + (\frac{1}{2} - \lambda_3)^2\right]|C|$, then the bodies can be translatively packed in $P = \frac{1}{2} \times \frac{1}{2}$.

This implies that if $C_1, C_2, \ldots$ cannot be translatively packed in C, then

$$\sum |C_i| = \sum \lambda_i^2 |C| > \lambda_1^2 |C| + \lambda_2^2 |C| + \left[\lambda_3^2 + \left(\frac{1}{2} - \lambda_3\right)^2\right] |C| .$$

Hence

$$\sum \lambda_i^2 > \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \left(\frac{1}{2} - \lambda_3\right)^2 \geq 3\lambda_3^2 + \left(\frac{1}{2} - \lambda_3\right)^2 = 4\lambda_3^2 - \lambda_3 + \frac{1}{4} \geq 0.1875 .$$

Case 2, when $\lambda_1 > \frac{p}{1+2p}$ *and* $\lambda_2 \leq \frac{p}{1+2p}$.

We place C_1 in $C \cap \left([0, t_1] \times [r, \frac{1}{2} + r]\right)$ (see Figure 2) and we place C_2 in $C \cap \left(\left[p, \frac{1}{2} + p\right] \times [1 - s, 1]\right)$. The remaining bodies $C_3, C_4, \ldots$ are packed in $\left[t_1, \frac{1}{2} + p\right] \times \left[r, \frac{1}{2} + r\right]$.

By Lemma 2 we deduce that if (C_i) cannot be translatively packed in C, then the sum of λ_i^2 is greater than

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \left(\frac{1}{2} + p - t_1 - \lambda_3\right)\left(\frac{1}{2} - \lambda_3\right).$$

Consequently,

$$\sum \lambda_i^2 > \lambda_1^2 + 2\lambda_3^2 + \left[\frac{1}{2} + p(1 - 2\lambda_1) - \lambda_1 - \lambda_3\right]\left(\frac{1}{2} - \lambda_3\right).$$

Since $\lambda_1 < \frac{1}{2}$ and $p \geq \frac{1}{4}$, we have $\sum \lambda_i^2 \geq f_1(\lambda_1, \lambda_3)$, where

$$f_1(\lambda_1, \lambda_3) = \lambda_1^2 + 2\lambda_3^2 + \left(\frac{3}{4} - \frac{3}{2}\lambda_1 - \lambda_3\right)\left(\frac{1}{2} - \lambda_3\right).$$

By using the standard method of finding the absolute minimum of the function of two variables it is easy to check that $f_1(\lambda_1, \lambda_3) \geq f_1\left(\frac{7}{26}, \frac{11}{78}\right) > 0.185$.

Case 3, when $\lambda_2 > \frac{p}{1+2p}$ *and* $p > 0.41$.

We place C_1 in $C \cap \big([0, t_1] \times \big[r, \frac{1}{2} + r\big]\big)$. The remaining copies $C_2, C_3, \dots$ are packed in $\big[t_1, \frac{1}{2} + p\big] \times \big[r, \frac{1}{2} + r\big]$. If (C_i) cannot be translatively packed in C, then

$$\sum \lambda_i^2 > \lambda_1^2 + \lambda_2^2 + (\frac{1}{2} + p - \lambda_1 - 2\lambda_1 p - \lambda_2)(\frac{1}{2} - \lambda_2).$$

By taking 0.41 instead of p we obtain that

$$\sum \lambda_i^2 > \lambda_1^2 + \lambda_2^2 + (0.91 - 1.82\lambda_1 - \lambda_2)(0.5 - \lambda_2).$$

A standard computation shows that this value is greater than 0.175.

Case 4, when $\lambda_2 > \frac{p}{1+2p}$ *and* $p \le 0.41$.

First of all, we show that $t_1 + t_2 + \lambda_3 \le 1$, where $t_2 = \lambda_2(1+2q)$. By $p + \frac{1}{2} + q = 1$ we have $t_2 = \lambda_2(2 - 2p)$. If $\lambda_3 > 1 - t_1 - t_2$, then

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 > \lambda_1^2 + \lambda_2^2 + \big[1 - \lambda_1(1 + 2p) - \lambda_2(2 - 2p)\big]^2 .$$

By $\lambda_1 \ge \lambda_2$ and $p < 0.41$ we have

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 > \lambda_1^2 + \lambda_2^2 + \big(1 - 1.82\lambda_1 - 1.18\lambda_2\big)^2 .$$

It is easy to check that this value is greater than 0.175, which is a contradiction.

We place C_1 in $C \cap \big([0, t_1] \times [r, \frac{1}{2} + r]\big)$ and we place C_2 in $C \cap \big([1 - t_2, 1] \times [r, \frac{1}{2} + r]\big)$. The remaining bodies $C_3, C_4, \dots$ are packed in $[t_1, 1 - t_2] \times \big[r, \frac{1}{2} + r\big]$. By Lemma 1 we deduce that if (C_i) cannot be translatively packed in C, then

$$\sum \lambda_i^2 > \lambda_1^2 + \lambda_2^2 + \frac{1}{2} \cdot \frac{1}{2}\big[1 - \lambda_1(1 + 2p) - \lambda_2(2 - 2p)\big] .$$

By taking 0.41 instead of p we obtain that

$$\sum \lambda_i^2 > \lambda_1^2 - 0.455\lambda_1 + \lambda_2^2 - 0.295\lambda_2 + 0.25 .$$

A standard computation shows that this value is greater than 0.175. □

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