ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 119–131

CONDITIONAL OSCILLATION OF HALF-LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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Abstract. We show that the half-linear differential equation

$$\left[r(t)\Phi(x')\right]' + \frac{s(t)}{t^p}\Phi(x) = 0 \tag{$*$}$$

with α-periodic positive functions r, s is conditionally oscillatory, i.e., there exists a constant $K > 0$ such that $(*)$ with $\frac{\gamma s(t)}{t^p}$ instead of $\frac{s(t)}{t^p}$ is oscillatory for $\gamma > K$ and nonoscillatory for $\gamma < K$.

1. Introduction

In this paper we study oscillatory properties of the half-linear equation

$$\left[r(t)\Phi(x')\right]' + s(t)\Phi(x) = 0\,, \quad \Phi(x) = x|x|^{p-2}\,, \tag{1.1}$$

where r and s are α-periodic ($\alpha > 0$) positive continuous functions and $p > 1$ is a real number conjugated with q, which means, that

$$\frac{1}{p} + \frac{1}{q} = 1\,.$$

Our research is motivated by the paper of K. M. Schmidt [2]. In that paper, the author studies oscillatory properties of the linear differential equation

$$\left[r(t)x'\right]' + \frac{\gamma s(t)}{t^2}x = 0\,, \quad t > 0 \tag{1.2}$$

where r, s are positive α-periodic functions and γ is a real parameter. The main result of [2] (after a minor reformulation) reads as follows.

Theorem 1.1. *Let*

$$K = \frac{1}{4}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{\mathrm{d}\tau}{r}\Big)^{-1}\Big(\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big)^{-1}\,,$$

then (1.2) *is oscillatory for $\gamma > K$ and nonoscillatory for $\gamma < K$.*

2000 *Mathematics Subject Classification:* Primary: 34C10.
Key words and phrases: oscillation theory, conditional oscillation, half-linear differential equations.
Research supported by the grant 201/07/0145 of the Czech Grant Agency.
Received November 3, 2007, revised March 2008. Editor O. Došlý.

The result presented in the previous theorem is interesting from the following point of view. It is known that the Euler equation

$$x'' + \frac{\gamma}{t^2} x = 0 \tag{1.3}$$

is conditionally oscillatory (i.e. there exists a constant γ_0 such that equation is oscillatory for $\gamma > \gamma_0$ and nonoscillatory for $\gamma < \gamma_0$) with the oscillation constant $\gamma_0 = \frac{1}{4}$. Theorem 1.1 shows that constant coefficients in (1.3) can be replaced by periodic functions and resulting equation remains conditionally oscillatory.

In our paper we show that a similar situation we have for half-linear equations. The Euler type half-linear differential equation

$$\big[\Phi(x')\big]' + \frac{\gamma}{t^p} \Phi(x) = 0\,, \tag{1.4}$$

is conditionally oscillatory (with $\gamma_0 = \big(\frac{p-1}{p}\big)^p$). The main result of our paper shows that also in half-linear case constant coefficients can be replaced by periodic ones, i.e., the equation

$$\big[r(t)\Phi(x')\big]' + \frac{\gamma s(t)}{t^p} \Phi(x) = 0$$

with periodic functions r, s remains conditionally oscillatory.

The basic difference between linear and half-linear differential equations is the fact that the solution space of half-linear equations is not additive (but remains homogeneous). The missing additivity (more or less) induces further differences as the absence of Wronskian-type identity, transform theory or reduction of order formula. Despite that, many results from linear equations may be extended to (1.1) (see e.g. [1]).

2. Preliminary results

We start with elements of oscillation theory of half-linear equation (1.1). It is known, see e.g. [1], that the linear Sturmian theory extends verbatim to half-linear equations. In particular, we have the following statements.

Proposition 2.1 (Sturmian separation theorem)**.** *Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of* (1.1)*. Then any other solution of this equation, which is not proportional to x, has exactly one zero in (t_1, t_2).*

Proposition 2.2 (Sturmian comparison theorem)**.** *Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of* (1.1) *and suppose, that*

$$S(t) \ge s(t)\,, \quad r(t) \ge R(t) > 0 \tag{2.1}$$

for $t \in [t_1, t_2]$. Then any solution of the equation

$$\big[R(t)\Phi(x')\big]' + S(t)\Phi(x) = 0 \tag{2.2}$$

has a zero in (t_1, t_2) or it is a multiple of the solution x. The last possibility is excluded if one of the inequalities in (2.1) *is strict on a set of positive measure.*

If (2.1) are satisfied in a given interval I, then (2.2) is said to be the *majorant equation* of (1.1) on I and (1.1) is said to be the *minorant equation* of (2.2) on I.

Proposition 2.1 implies that (1.1) can be classified as *oscillatory* or *nonoscillatory*. Recall, that points $t_1, t_2 \in \mathbb{R}$ are said to be *conjugate* relative to equation (1.1), if there exists a nontrivial solution x of this equation, such that $x(t_1) = x(t_2) = 0$. Then, equation (1.1) is said to be *disconjugate* on an interval I, if this interval does not contain two points conjugate relative to equation (1.1). In the opposite case, equation (1.1) is said to be *conjugate on I*.

Now, let us recall the definition of oscillation and nonoscillation of equation (1.1) at zero and infinity.

Definition 1. Equation (1.1) is said to be *nonoscillatory at* 0, if there exists $\varepsilon > 0$ such that equation (1.1) is disconjugate on $[0, \varepsilon]$. In the opposite case, equation (1.1) is said to be *oscillatory at* 0.

Definition 2. Equation (1.1) is said to be *nonoscillatory at* ∞, if there exists $T_0 \in \mathbb{R}$ such that equation (1.1) is disconjugate on $[T_0, T]$ for every $T > T_0$. In the opposite case, equation (1.1) is said to be *oscillatory at* ∞.

If equation (1.1) is nonoscillatory at zero, then there exists a solution $v_{\max}$ of the Riccati equation

$$v' + s(t) + (p-1)r^{1-q}(t)|v|^q = 0 \tag{2.3}$$

associated to equation (1.1) such that $v_{\max}(t) > v(t)$ for t from a right neighbourhood of 0 for any other solution v of (2.3) which is defined in a right neighbourhood of 0. If equation (1.1) is nonoscillatory at infinity, then there exists a solution $v_{\min}$ of Riccati equation (2.3) such that $v_{\min}(t) < v(t)$ for any other solution for large t. We call $v_{\max}$ the *maximal solution* of (2.3) and $v_{\min}$ the *minimal solution* of (2.3).

Then, we define the principal solution of (1.1) at zero [infinity] as the nontrivial solution of the equation

$$x' = \Phi^{-1}\Big(\frac{v_{\max}(t)}{r(t)}\Big)\, x\,, \qquad \Big[x' = \Phi^{-1}\Big(\frac{v_{\min}(t)}{r(t)}\Big)\, x\Big]\,.$$

Now, let us briefly recall some basic facts concerning the half-linear Euler equation (1.4).

As mentioned in Introduction, equation (1.4) is conditionally oscillatory both at $t = 0$ and $t = \infty$ with the oscillation constant $\gamma_0 = \big(\frac{p-1}{p}\big)^p$ (see [1]).

Let $0 < \gamma < \gamma_0$, then (1.4) is not only nonoscillatory at 0 and ∞ but also disconjugated on $(0, \infty)$. Substituting $x(t) = t^\lambda$ into (1.4), we obtain an algebraic equation for λ

$$|\lambda|^p - \Phi(\lambda) + \frac{\gamma}{p-1} = 0\,.$$

and solving this equation, we find, that its roots $\lambda_2 < \lambda_1$ satisfy

$$0 < \lambda_2 < \frac{p-1}{p} < \lambda_1 < 1\,.$$

The principal solution of (1.4) at zero is t^{λ_1}, principal solution of (1.4) at infinity is t^{λ_2}, maximal and minimal solutions of the associated Riccati equation

$$w' + \frac{\gamma}{t^p} + (p-1)|w|^q = 0$$

are

$$w_{\max} = \Phi(\lambda_1)t^{1-p}, \quad w_{\min} = \Phi(\lambda_2)t^{1-p},$$

respectively.

Using the change of independent variable $t = \mathrm{e}^s$, $s \in \mathbb{R}$, we convert equation (1.4) into the equation with constant coefficients

$$\big[\Phi(y')\big]' - (p-1)\Phi(y') + \gamma\Phi(y) = 0\,. \tag{2.4}$$

The corresponding Riccati equation is

$$u' - (p-1)u + (p-1)|u|^q + \gamma = 0\,. \tag{2.5}$$

Denote

$$F(u) := \gamma - (p-1)u + (p-1)|u|^q\,.$$

Following lemmas and theorems will be useful in the next section of our paper.

Lemma 2.1. *Consider the Riccati equation*

$$w' + \frac{\gamma}{t^p} + (p-1)|w|^p = 0\,, \quad \gamma < \Big(\frac{p-1}{p}\Big)^p \tag{2.6}$$

associated with the nonoscillatory Euler half-linear equation (1.4)*. If $w(T) \geq 1$ for some $T > 0$, then there exists $\tau \in \big(T\mathrm{e}^{-\int_1^\infty \frac{\mathrm{d}u}{F(u)}}, T\big)$ such that $w(\tau+) = \infty$.*

Proof. We convert equation (1.4) into equation (2.4) with associated Riccati equation (2.5). Suppose, by contradiction, that there exists a solution u of (2.5) extensible to $-\infty$ which satisfies $u(S) \geq 1$, where $S = \log T$, and integrate equation (2.5) on the interval $[s, S]$, where $S \in \mathbb{R}$ is fixed. Any solution, different from maximal and minimal ones (for which is $F(u) = 0$), is implicitly given by the formula

$$-\int_{u(s)}^{u(S)} \frac{\mathrm{d}u}{F(u)} = \int_{u(S)}^{u(s)} \frac{\mathrm{d}u}{F(u)} = S - s\,.$$

Hence

$$\int_1^\infty \frac{\mathrm{d}u}{F(u)} > S - s = \log T - \log t = -\log\frac{t}{T}\,,$$

i.e., $t > T\mathrm{e}^{-\int_1^\infty \frac{\mathrm{d}u}{F(u)}}$ which implies the existence of $\tau \in \big(T\mathrm{e}^{-\int_1^\infty \frac{\mathrm{d}u}{F(u)}}, T\big)$ such that $w(\tau+) = \infty$. □

Lemma 2.2. *Consider Riccati equation* (2.6) *associated with the nonoscillatory half-linear Euler equation* (1.4)*. If $v(T) \leq 0$ for some $T > 0$, then there exists $\tau \in \big(T, T\mathrm{e}^{\int_{-\infty}^0 \frac{\mathrm{d}u}{F(u)}}\big)$ such that $v(\tau-) = -\infty$.*

Proof. Similarly as in the Proof of Lemma 2.1, we use conversion to equations (2.4) and (2.5). Suppose the existence of a solution u of (2.5) extensible to ∞ that satisfies $u(S) \leq 0$ and integrate equation (2.5) on the interval $[S, s]$, where $S \in \mathbb{R}$ is fixed. Any solution, different from maximal and minimal ones, is implicitly

$$\int_{u(s)}^{u(S)} \frac{\mathrm{d}u}{F(u)} = s - S\,. \tag{2.7}$$

Again, this contradicts the existence of such a solution u, because the left hand side of equation (2.7) is bounded and the right hand side tends to infinity as $s \to \infty$. $\square$

We finish this section with formulating a couple of lemmas and theorems without proofs (see e.g. [1]).

Lemma 2.3. *Consider a pair of equations*

$$v' + C(t) + (p-1)|v|^q = 0\,, \tag{2.8}$$

$$w' + c(t) + (p-1)|w|^q = 0\,, \tag{2.9}$$

where $C(t) \geq c(t) > 0$ for $t \in (a,b)$. If $\tau, T \in (a,b), \tau < T$, and a solution w of (2.9) *exists on $(\tau, T]$ and satisfies $w(\tau+) = \infty$, then there exists $\widetilde{\tau} \in [\tau, T)$ such that the solution v of* (2.8) *given by the initial condition $v(T) = w(T)$ satisfies $v(\widetilde{\tau}+) = \infty$.*

Lemma 2.4. *Consider a pair of equations* (2.8), (2.9). *If $\tau, T \in (a,b), T < \tau$, and a solution w of* (2.9) *exists on $[T, \tau)$ and satisfies $w(\tau-) = -\infty$, then there exists $\widetilde{\tau} \in (T, \tau]$ such that the solution v of* (2.8) *given by the initial condition $v(T) = w(T)$ satisfies $v(\widetilde{\tau}-) = -\infty$.*

Following theorems compare solutions of a pair of Riccati equations associated with nonoscillatory half-linear differential equations.

Theorem 2.1. *Consider a pair of half-linear differential equations*

$$\big[r(t)\Phi(x')\big]' + c(t)\Phi(x) = 0\,, \tag{2.10}$$

$$\big[R(t)\Phi(y')\big]' + C(t)\Phi(y) = 0 \tag{2.11}$$

and suppose that (2.11) *is a Sturmian majorant of* (2.10) *for large t, i.e., there exists $T \in \mathbb{R}$ such that $0 < R(t) \leq r(t)$, $c(t) \leq C(t)$ for $t \in [T, \infty)$. Suppose that the majorant equation* (2.11) *is nonoscillatory and denote $v_{\min}, w_{\min}$ minimal solutions of*

$$v' + c(t) + (p-1)r^{1-q}(t)|v|^q = 0\,, \tag{2.12}$$

$$w' + C(t) + (p-1)R^{1-q}(t)|w|^q = 0\,, \tag{2.13}$$

respectively. Then $v_{\min}(t) \leq w_{\min}(t)$ for large t.

Theorem 2.2. *Consider a pair of half-linear differential equations* (2.10), (2.11) *and suppose that* (2.11) *is a Sturmian majorant of* (2.10) *for t from a right neighbourhood of* 0, *i.e., there exists $\varepsilon \in \mathbb{R}$ such that $0 < R(t) \leq r(t)$, $c(t) \leq$*

$C(t)$ for $t \in (0,\varepsilon]$. Suppose that the majorant equation (2.11) *is nonoscillatory and denote $v_{\max}$, $w_{\max}$ maximal solutions of* (2.12), (2.13), *respectively. Then $v_{\max}(t) \geq w_{\max}(t)$ for t from a right neighbourhood of* 0.

3. Conditional oscillation of equations with periodic coefficients

The main result of our paper reads as follows.

Theorem 3.1. *Consider the equation*

$$\big[r(t)\Phi(x')\big]' + \gamma\frac{s(t)}{t^p}\Phi(x) = 0\,, \tag{3.1}$$

where r and s are α-periodic ($\alpha > 0$) positive continuous functions, and $\gamma \in \mathbb{R}$. Let

$$K := q^{-p}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{\mathrm{d}\tau}{r^{q-1}}\Big)^{1-p}\Big(\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big)^{-1}\,. \tag{3.2}$$

Then equation (3.1) *is oscillatory if $\gamma > K$ and nonoscillatory if $\gamma < K$.*

Proof. Let $\gamma > K$. Suppose, by contradiction, that (3.1) is nonoscillatory. It means that the associated Riccati equation (2.3) has a solution, which exists on some interval $[T,\infty)$. Because r and s are α-periodic, positive and continuous, the equation

$$\big[r_{\max}\Phi(x')\big]' + \gamma\frac{s_{\min}}{t^p}\Phi(x) = 0\,,$$

where

$$r_{\max} = \max\big\{r(t), t \geq 0\big\}\,,$$

$$s_{\min} = \min\big\{s(t), t \geq 0\big\}\,.$$

is a minorant of (3.1), hence it is also nonoscillatory.

Denote $\mu := \frac{s_{\min}}{r_{\max}}$. Solving the Euler-type equation

$$\big[\Phi(x')\big]' + \gamma\frac{\mu}{t^p}\Phi(x) = 0\,, \tag{3.3}$$

with $\mu\gamma \leq \big(\frac{p-1}{p}\big)^p$ we find, that the principal solutions at zero and infinity are t^{λ_1}, t^{λ_2}, respectively, where $0 < \lambda_2 < \lambda_1 < 1$ are roots of the equation

$$|\lambda|^p - \Phi(\lambda) + \gamma\frac{\mu}{p-1} = 0\,,$$

see Section 2.

Denote the maximal solution near $t = 0$ of the Riccati equation associated to equation (3.3) by

$$v_{\max}(t) := t^{1-p}\Phi(\lambda_1)\,,$$

and the minimal solution for large t by

$$v_{\min}(t) := t^{1-p}\Phi(\lambda_2)\,.$$

Introducing the function $w = \frac{r\Phi(x')}{\Phi(x)}$, we may transform equation (3.1) to the Riccati equation

$$w' + \gamma \frac{s(t)}{t^p} + (p-1) r^{1-q}(t) |w|^q = 0$$

with the maximal solution (at $t = 0$) $w_{\max}$ and the minimal solution (at $t = \infty$) $w_{\min}$ and denote

$$\zeta(t) := -w t^{p-1}\,, \quad \xi(t) := \frac{1}{\alpha} \int_t^{t+\alpha} \zeta(\tau)\, \mathrm{d}\tau\,. \tag{3.4}$$

First, suppose that there exists $t_n \to \infty$ such that $\zeta(t_n) \le -1$, i.e.,

$$w(t_n) = -t_n^{1-p} \zeta(t_n) \ge t_n^{1-p} > \Phi(\lambda_1) t_n^{1-p} = v_{\max}(t_n) \ge w_{\max}(t_n)\,.$$

Consider the solution of (3.3) given by the initial condition $v(t_n) = t_n^{1-p}$, i.e.,

$$v(t_n) - v_{\max}(t_n) = [1 - \Phi(\lambda_1)] t_n^{1-p}\,.$$

Then, by Lemma 2.1, there exists $\tau_n \to \infty, \tau_n < t_n$, such that $v(\tau_n +) = \infty$. But this means, by Lemma 2.3, that $w(\widetilde{\tau}_n +) = \infty$ for some $\tau_n \le \widetilde{\tau}_n < t_n$, which is a contradiction.

Next, suppose that there exists a sequence $\hat{t}_n \to \infty$ such that $\zeta(\hat{t}_n) \ge 0$, i.e.,

$$w(\hat{t}_n) \le 0 < v_{\min}(\hat{t}_n) = \Phi(\lambda_2) \hat{t}_n^{1-p} \le w_{\min}(\hat{t}_n)\,.$$

This means (from Lemma 2.2 and Lemma 2.4), that there exists $\widehat{\tau}_n > \hat{t}_n$ such that $w(\widehat{\tau}_n -) = \infty$, which contradicts the fact, that $w(t)$ exists on $[T, \infty)$.

Hence, there exists $T_0 > T$ such that

$$v_{\min} = \Phi(\lambda_2) t^{1-p} \le w \le \Phi(\lambda_1) t^{1-p} = v_{\max}$$

for $t \ge T_0$. Multiplying the last inequality by $-t^{p-1}$, we obtain

$$0 > -\Phi(\lambda_2) \ge \zeta(t) \ge -\Phi(\lambda_1) > -1\,.$$

Let us denote

$$A := (p-1) \Big(\frac{p}{\alpha} \int_0^\alpha \frac{1}{r^{q-1}(\tau)}\, \mathrm{d}\tau \Big)^{-\frac{1}{q}}\,, \quad B := \big|\xi(t)\big| \Big(\frac{p}{\alpha} \int_0^\alpha \frac{1}{r^{q-1}(\tau)}\, \mathrm{d}\tau \Big)^{\frac{1}{q}}\,.$$

We have

$$\begin{aligned} \zeta'(t) &= \big[-w(t) t^{p-1} \big]' = -\big[w'(t) t^{p-1} + (p-1) w(t) t^{p-2} \big] \\ &= \frac{1}{t} \Big[(p-1)\zeta(t) + s(t)\gamma + (p-1) \frac{|\zeta(t)|^q}{r^{q-1}(t)} \Big]\,. \end{aligned}$$

Next, for $t \geq T_0$

$$\begin{aligned}\int_t^{t+\alpha} \left|\zeta'(\tau)\right| \mathrm{d}\tau &\leq \frac{1}{t} \int_t^{t+\alpha} \left|(p-1)\zeta(\tau) + \gamma s(\tau) + (p-1)\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\right| \mathrm{d}\tau \\ &\leq \frac{1}{t} \int_t^{t+\alpha} \Big[(p-1) + \gamma s(\tau) + \frac{p-1}{r^{q-1}(\tau)}\Big] \mathrm{d}\tau = \frac{C}{t}\,,\end{aligned} \tag{3.5}$$

where

$$C := \int_t^{t+\alpha} \Big[(p-1) + \gamma s(\tau) + \frac{p-1}{r^{q-1}(\tau)}\Big] \mathrm{d}\tau\,.$$

Hence, for every $t > T_0$ and $\tau_1, \tau_2 \in [t, t+\alpha]$ we have

$$\left|\zeta(\tau_1) - \zeta(\tau_2)\right| \leq \int_t^{t+\alpha} \left|\zeta'(\tau)\right| \mathrm{d}\tau \leq \frac{C}{t}\,.$$

Due to the continuity of the function ζ, there exists $\tau_0 \in [t, t+\alpha]$ such that

$$\xi(t) = \zeta(\tau_0) \quad \Rightarrow \quad |\zeta(\tau) - \xi(t)| \leq \frac{C}{t}\,,$$

where $\tau \in [t, t+\alpha]$.

Now, we estimate the value of the function ξ'.

$$\begin{aligned}\xi'(t) &= \frac{1}{\alpha}\big[\zeta(t+\alpha) - \zeta(t)\big] = \frac{1}{\alpha} \int_t^{t+\alpha} \zeta'(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{\tau}\Big[(p-1)\zeta(\tau) + s(\tau)\gamma + (p-1)\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\Big] \mathrm{d}\tau \\ &\geq \frac{1}{t+\alpha}\Big[(p-1)\xi(t) + \frac{\gamma}{\alpha}\int_0^{\alpha} s(\tau)\,\mathrm{d}\tau + \frac{p-1}{\alpha}\int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big] \\ &\quad + \frac{(p-1)\alpha}{t(t+\alpha)}\xi(t) \\ &= \frac{1}{t+\alpha}\Big[(p-1)\xi(t) + \frac{A^p}{p} + \frac{B^q}{q} + \frac{\gamma}{\alpha}\int_0^{\alpha} s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} + \frac{(p-1)\alpha}{t}\xi(t) \\ &\quad + \frac{p-1}{\alpha}\int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau - \frac{B^q}{q}\Big]\,.\end{aligned}$$

Denote

$$
\begin{aligned}
&X := (p-1)\xi(t) + \frac{A^p}{p} + \frac{B^q}{q}\,,\\
(3.6)\qquad &Y := \frac{\gamma}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} + \frac{(p-1)\alpha}{t}\xi(t)\,,\\
&Z := \frac{p-1}{\alpha}\int_t^{t+\alpha} \frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau - \frac{B^q}{q}\,.
\end{aligned}
$$

Next, we estimate quantities appearing in (3.6). It follows from Young's inequality, that $\frac{A^p}{p} + \frac{B^q}{q} - AB \geq 0$, so (using $\xi \leq 0$)

$$X = \frac{A^p}{p} + \frac{B^q}{q} + (p-1)\xi = \frac{A^p}{p} + \frac{B^q}{q} - (p-1)|\xi| = \frac{A^p}{p} + \frac{B^q}{q} - AB \geq 0\,.$$

As for the term Y, we denote

$$K_\gamma := Y = \frac{\gamma}{\alpha}\int_0^\alpha s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} + \frac{(p-1)\alpha}{t}\xi(t)$$

and show, that $K_\gamma \geq 0$.

$$
\begin{aligned}
K_\gamma &= \frac{\gamma}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau - \frac{(p-1)^p}{p}\Big(\frac{p}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}} + \frac{(p-1)\alpha}{t}\xi\\
&= \frac{\gamma}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau - q^{-p}\frac{\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau}{\Big(\frac{1}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{\frac{p}{q}}\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau} + \frac{(p-1)\alpha}{t}\frac{1}{\alpha}\int_t^{t+\alpha}\zeta\,\mathrm{d}\tau\\
&\geq \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big[\gamma - q^{-p}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}}\Big(\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big)^{-1}\Big] - \frac{p-1}{t}\\
&= \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau(\gamma - K) - \frac{p-1}{t} > 0\,,
\end{aligned}
$$

for $t \geq T_1$, because $\gamma > K$.

Finally, to estimate the last expression in (3.6), let us introduce the function

$$F(x,y) := \begin{cases} \frac{|x|^q - |y|^q}{|x|-|y|}\,, & x \neq y, [x,y] \in M\,,\\ q\Phi^{-1}(|x|)\,, & x = y\,,\end{cases}$$

where $M := [-1,0] \times [-1,0]$.

Then, we have

$$Z = \frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta|^q}{r^{q-1}} \,\mathrm{d}\tau - \frac{B^q}{q} = \frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\zeta|^q}{r^{q-1}} \,\mathrm{d}\tau - \frac{|\xi|^q}{q} \frac{q(p-1)}{\alpha} \int_t^{t+\alpha} \frac{1}{r^{q-1}} \,\mathrm{d}\tau$$
$$= -\frac{p-1}{\alpha} \int_t^{t+\alpha} \frac{|\xi|^q - |\zeta|^q}{r^{q-1}} \,\mathrm{d}\tau \geq -\frac{p-1}{\alpha} \int_t^{t+\alpha} |\xi - \zeta| \frac{|\xi|^q - |\zeta|^q}{|\xi| - |\zeta|} \frac{1}{r^{q-1}} \,\mathrm{d}\tau$$
$$\geq -\frac{(p-1)CD}{\alpha t} \int_0^{\alpha} \frac{1}{r^{q-1}} \,\mathrm{d}\tau \,,$$

where we have used (3.5) and $D := \max_M F(\xi, \zeta) < \infty$.

Altogether for $t \geq T := \max\{T_0, T_1, T_2\}$, where

$$T_2 := \frac{2CD(p-1)}{\alpha K_\gamma} \int_0^{\alpha} \frac{1}{r^{q-1}(\tau)} \,\mathrm{d}\tau \,,$$

we obtain

$$\xi'(t) \geq \frac{1}{t+\alpha} \Big[K_\gamma - \frac{CD(p-1)}{\alpha t} \int_0^{\alpha} \frac{1}{r^{q-1}(\tau)} \,\mathrm{d}\tau \Big]$$
$$\geq \frac{1}{t+\alpha} \Big(K_\gamma - \frac{K_\gamma}{2} \Big) = \frac{K_\gamma}{2(t+\alpha)} \,,$$

which means, that

$$\xi(t) \geq \xi(T) + \frac{K_\gamma}{2} \log \frac{t+\alpha}{T+\alpha} \to \infty \quad \text{as} \quad t \to \infty \,,$$

which is a contradiction. Thus, equation (3.1) is oscillatory for $\gamma > K$.

In the next part of the proof, we show, that (3.1) is nonoscillatory for $\gamma < K$. Denote $\mu := \frac{s_{\max}}{r_{\min}}$. Equation (3.3) is now a majorant equation of equation (3.1). We show that the majorant equation (3.3) is nonoscillatory, which implies, that equation (3.1) is also nonoscillatory.

Denote

$$\xi_0 := -\Big[\frac{p}{\alpha(p-1)} \int_0^{\alpha} \frac{1}{r^{q-1}(\tau)} \,\mathrm{d}\tau \Big]^{1-p} .$$

We will show that there exists T such that $\xi(t)$ defined by (3.4) in the previous part of the proof satisfies $\xi(t) \leq \xi_0$, $(t \geq T)$. By contradiction, assume that

$$t_0 := \sup\{t \geq T, \xi(\tau) \leq \xi_0, \tau \in [T, t]\} < \infty \,.$$

Then $\xi'(t_0) \geq 0$ and $\xi(t_0) = \xi_0$. We estimate the value of $\xi'(t_0)$. We obtain

$$
\begin{aligned}
\xi'(t_0) &= \frac{1}{\alpha}\int\limits_{t_0}^{t_0+\alpha} \frac{1}{\tau}\Big[(p-1)\zeta(\tau) + \gamma s(\tau) + (p-1)\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\Big]\,\mathrm{d}\tau \\
&\leq \frac{1}{t_0}\Big[(p-1)\xi(t_0) + \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s(\tau)\,\mathrm{d}\tau + \frac{p-1}{\alpha}\int\limits_{t_0}^{t_0+\alpha}\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big] \\
&\quad - \frac{(p-1)\alpha}{t_0(t_0+\alpha)}\xi(t_0) \\
&= \frac{1}{t_0}\Big[(p-1)\xi(t_0) + \frac{A^p}{p} + \frac{B^q}{q} + \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} - \frac{(p-1)\alpha}{t_0+\alpha}\xi(t_0) \\
&\quad + \frac{p-1}{\alpha}\int\limits_{t_0}^{t_0+\alpha}\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau - \frac{B^q}{q}\Big]\,.
\end{aligned}
$$

Again, we denote

$$
\begin{aligned}
X &:= (p-1)\xi(t_0) + \frac{A^p}{p} + \frac{B^q}{q}\,, \\
Y &:= \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s(\tau)\,\mathrm{d}\tau - \frac{A^p}{p} - \frac{(p-1)\alpha}{t+\alpha}\xi(t_0)\,, \\
Z &:= \frac{p-1}{\alpha}\int\limits_{t_0}^{t_0+\alpha}\frac{|\zeta(\tau)|^q}{r^{q-1}(\tau)}\,\mathrm{d}\tau - \frac{B^q}{q}\,,
\end{aligned}
\tag{3.7}
$$

with A, B given by

$$
A := (p-1)\Big(\frac{p}{\alpha}\int\limits_0^{\alpha}\frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big)^{-\frac{1}{q}}, \quad B := |\xi(t_0)|\Big(\frac{p}{\alpha}\int\limits_0^{\alpha}\frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big)^{\frac{1}{q}},
$$

and we estimate quantities appearing in (3.7).

Since $A^p = B^q$, we have

$$
\frac{A^p}{p} + \frac{B^q}{q} = A^p\Big(\frac{1}{p} + \frac{1}{q}\Big) = A^{1+\frac{p}{q}} = A(B^q)^{\frac{1}{q}} = AB = -(p-1)\xi\,,
$$

which means, that $X = 0$ in (3.7).

Next, we denote

$$
-K_\gamma := Y = \frac{\gamma}{\alpha}\int\limits_0^{\alpha} s\,\mathrm{d}\tau - \frac{A^p}{p} - \frac{(p-1)\alpha}{t_0+\alpha}\xi\,.
$$

Then

$$\begin{aligned}
-K_\gamma &= \frac{\gamma}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau - \frac{(p-1)^p}{p}\Big(\frac{p}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}} - \frac{(p-1)\alpha}{t_0+\alpha}\xi \\
&\le \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big[\gamma - q^{-p}\Big(\frac{1}{\alpha}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\Big)^{-\frac{p}{q}}\Big(\frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau\Big)^{-1}\Big] + \frac{p-1}{t_0+\alpha} \\
&= \frac{1}{\alpha}\int_0^\alpha s\,\mathrm{d}\tau(\gamma-K) + \frac{p-1}{t_0+\alpha} < 0\,,
\end{aligned}$$

because $\gamma < K$, i.e., $K_\gamma > 0$.

Finally, similarly as in the previous computation, we have

$$\begin{aligned}
Z &= \frac{p-1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{|\zeta|^q}{r^{q-1}}\,\mathrm{d}\tau - \frac{B^q}{q} \\
&= \frac{p-1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{|\zeta|^q}{r^{q-1}}\,\mathrm{d}\tau - \frac{|\xi|^q}{q}\,\frac{q(p-1)}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{1}{r^{q-1}}\,\mathrm{d}\tau \\
&= \frac{p-1}{\alpha}\int_{t_0}^{t_0+\alpha} \frac{|\zeta|^q - |\xi|^q}{r^{q-1}}\,\mathrm{d}\tau \le \frac{CD(p-1)}{\alpha t_0}\int_0^\alpha \frac{1}{r^{q-1}}\,\mathrm{d}\tau\,.
\end{aligned}$$

Altogether for $t_0 \ge T := \max\{T_0, T_1, T_2\}$, where T_0, T_1, T_2 are defined earlier, we obtain

$$\begin{aligned}
\xi'(t_0) &\le \frac{1}{t_0}\Big[-K_\gamma + \frac{CD(p-1)}{\alpha t_0}\int_0^\alpha \frac{1}{r^{q-1}(\tau)}\,\mathrm{d}\tau\Big] \\
&\le \frac{1}{t_0}\Big(-K_\gamma + \frac{K_\gamma}{2}\Big) = -\frac{K_\gamma}{2t_0} < 0\,,
\end{aligned}$$

which is a contradiction.

□

Remark 1. It is still an open problem to decide whether equation (3.1) is oscillatory or not in the case, $\gamma = K$, with K given by (3.2).

Remark 2. For $r(t) \equiv 1 \equiv s(t)$, equation (3.1) reduces to Euler equation (1.4) and our oscillation constant K defined by (3.2) reduces to the well known constant $\gamma_0 = \big(\frac{p-1}{p}\big)^p$.

References

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