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WEILIAN PROLONGATIONS OF ACTIONS OF SMOOTH CATEGORIES

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Abstract. First of all, we find some further properties of the characterization of fiber product preserving bundle functors on the category of all fibered manifolds in terms of an infinite sequence A of Weil algebras and a double sequence H of their homomorphisms from [5]. Then we introduce the concept of Weilian prolongation $W^A_H S$ of a smooth category S over $\mathbb{N}$ and of its action D. We deduce that the functor (A,H) transforms D-bundles into $W^A_H D$-bundles.

In [4] we clarified that every fiber product preserving bundle functor on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fiber preserving morphisms with local diffeomorphisms as base maps is of finite order and can be identified with a triple (A_m, H_m, t_m), where A_m is a Weil algebra, $H_m \colon G^r_m \to \mathrm{Aut}\ A_m$ is a group homomorphism of the r-th jet group in dimension m into the group of all algebra automorphisms of A_m and $t_m \colon \mathbb{D}^r_m \to A_m$ is an equivariant algebra homomorphism, $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m,\mathbb{R})$. Our next result from [5] can be formulated as follows. Write

$$A = (A_1, \dots, A_m, \dots) \tag{1}$$

for an infinite sequence of Weil algebras,

$$\mathrm{Hom}\ A = \big(\mathrm{Hom}\,(A_m, A_n)\big) \tag{2}$$

for the double sequence of the algebra homomorphisms and

$$L^r = (L^r_{m,n})\,, \quad L^r_{m,n} = J^r_0(\mathbb{R}^m,\mathbb{R}^n)_0 \tag{3}$$

for the skeleton of the category of r-jets. Then the fiber product preserving bundle functors F on the category $\mathcal{FM}$ of all fibered manifold morphisms of the base order r are in bijection with the pairs (A,H) of a sequence (1) and of a functor

$$H \colon L^r \to \mathrm{Hom}\ A\,, \quad H_{m,n} \colon L^r_{m,n} \to \mathrm{Hom}\,(A_m, A_n)\,. \tag{4}$$

In the first two sections of the present paper, we deduce certain new results concerning F and an arbitrary fiber product preserving bundle functor on $\mathcal{FM}_m$, that are to be used in the sequel. In Section 3 we consider a smooth category S over

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integers and we describe how the pair (A, H) of (1) and (4) induces a category $W_H^A S$ over $\mathbb{N}$. Then we consider an action D of S on a sequence $Z = (Z_1, \dots, Z_m, \dots)$ of manifolds and we deduce that $W_H^A S$ determines canonically an action $W_H^A D$ on the sequence

$$T^A Z = (T^{A_1} Z_1, \dots, T^{A_m} Z_m, \dots)\,.$$

In Section 5 we introduce the category of D-bundles and we prove that the functor $F = (A, H)$ transforms D-bundles into $W_H^A D$-bundles.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from the book [3].

1. The case of $\mathcal{FM}_m$ in trivializations. In general, every Weil algebra homomorphism $\mu\colon B \to C$ defines a natural transformation $\mu_M\colon T^B M \to T^C M$ of the corresponding Weil bundles over every manifold M, [2], [3], [6]. For a fibered manifold $p\colon Y \to M$, we define the vertical Weil bundle $V^B Y \subset T^B Y$ as the space of all B-velocities in the individual fibers of Y. Then μ_Y restricts and corestricts into a map

$$\mu_Y^V\colon V^B Y \to V^C Y\,,$$

that is a natural transformation of the bundle functors V^B and V^C on $\mathcal{FM}$.

In [4], we deduced that every fiber product preserving bundle functor F_m on $\mathcal{FM}_m$ of the base order r is of the form $F_m = (A_m, H_m, t_m)$ specified in the introduction. The homomorphism H_m defines an action of G_m^r on $T^{A_m} Y$, $g \mapsto H_m(g)_Y$. So we can construct the associated bundle $P^r M[T^{A_m} Y, H_{m\,Y}]$. Then we have

$$F_m Y = \big\{\{u, X\} \in P^r M[T^{A_m} Y]; t_{m\,M} u = T^{A_m} p(X)\big\}\,,$$

where $t_{m\,M}\colon T_m^r M \to T^{A_m} M$ and we use the inclusion $P^r M \subset T_m^r M$. Let $\bar p\colon \bar Y \to \bar M$ be another $\mathcal{FM}_m$-object and $f\colon Y \to \bar Y$ be an $\mathcal{FM}_m$-morphism with the base map $\underline{f}\colon M \to \bar M$. Since $T^{A_m} f\colon T^{A_m} Y \to T^{A_m} \bar Y$ is a G_m^r-equivariant map, we can construct the induced morphism of associated bundles

$$P^r \underline{f}[T^{A_m} f]\colon P^r M[T^{A_m} Y] \to P^r \bar M[T^{A_m} \bar Y]\,. \tag{5}$$

Clearly, (5) maps $F_m Y$ into $F_m \bar Y$. This defines $F_m f$.

In the case of a product $p_1\colon M \times N \to M$, the condition $t_{m\,M} u = T^{A_m} p_1(X_1, X_2)$, $(X_1, X_2) \in T^{A_m} M \times T^{A_m} N$ yields $t_{m\,M} u = X_1$. Hence

$$F_m(M \times N) = P^r M[T^{A_m} N]\,.$$

Every $\mathcal{FM}_m$-morphism $f\colon M \times N \to \bar M \times \bar N$ is of the form $f = (\underline{f}, \widetilde{f})$, $\underline{f}\colon M \to \bar M$, $\widetilde{f}\colon M \times N \to \bar N$. Then

$$T^{A_m} f = (T^{A_m} \underline{f}, T^{A_m} \widetilde{f})\colon T^{A_m} M \times T^{A_m} N \to T^{A_m} \bar M \times T^{A_m} \bar N\,.$$

For $\{u, X\} \in P^r M[T^{A_m} N]$, we have to consider

$$\big\{u, (t_{m\,M} u, X)\big\} \in P^r M[T^{A_m} M \times T^{A_m} N]\,.$$

Then we obtain

$$P^r\underline{f}[T^{A_m}f]\big(\{u,(t_{m\,M}u,X)\}\big) = \\ \big\{P^r\underline{f}(u),(t_{m\,\bar M}P^r\underline{f}(u),T^{A_m}\widetilde{f}(t_{m\,M}u,X))\big\}\,.$$

This implies

Proposition 1. *For* $\{u,X\}\in P^rM[T^{A_m}N]$, *we have*

$$F_m(\underline{f},\widetilde{f})\big(\{u,X\}\big)=\big\{P^r\underline{f}(u),T^{A_m}\widetilde{f}(t_{m\,M}u,X)\big\}\,.$$

In the case $M=\mathbb{R}^m=\bar M$, we use the injection $\varrho_m\colon\mathbb{R}^m\to P^r\mathbb{R}^m$, $\varrho_m(x)=j^r_0\tau_x$, where $\tau_x\colon\mathbb{R}^m\to\mathbb{R}^m$ is the translation $y\mapsto y+x$. This defines an identification $F_m(\mathbb{R}^m\times N)\approx\mathbb{R}^m\times T^{A_m}N$, $\big\{\varrho_m(x),X\big\}\mapsto(x,X)$. Consider $f=(\underline{f},\widetilde{f})\colon\mathbb{R}^m\times N\to\mathbb{R}^m\times\bar N$ and write $\varphi\colon\mathbb{R}^m\to G^r_m$ for the map defined by $P^r\underline{f}\big(\varrho_m(x)\big)=\varrho_m\big(\underline{f}(x)\big)\varphi(x)$. Then we obtain the following formula for F_mf

$$\begin{aligned}F_mf(x,X)&=\big\{P^r\underline{f}\big(\varrho_m(x)\big),T^{A_m}\widetilde{f}(t_{m\,M}\varrho_m(x),X)\big\}\\ &=\big(\underline{f}(x),H_m(\varphi(x))_{\bar N}T^{A_m}\widetilde{f}(t_{m\,M}\varrho_m(x),X)\big)\,.\end{aligned}\tag{6}$$

Consider the case t_m is the zero homomorphism $\mathcal{O}$. Then we have $t_{m\,M}u=j^{A_m}\hat x$, $u\in P^r_xM$, where $\hat x$ is the constant map of $\mathbb{R}^k$ into x, $k=$ the width of A_m. This implies directly

$$F_mY=P^rM[V^{A_m}Y,H^V_{m\,Y}]\,.$$

Then F_mf can be written as

$$F_mf=P^r\underline{f}[V^{A_m}f]\,.$$

In the case of $M=\mathbb{R}^m=\bar M$, (6) yields

$$F_mf(x,X)=\big(\underline{f}(x),H_m(\varphi(x))_{\bar N}(V^{A_m}f(x,X))\big)\,.\tag{7}$$

2. The case of $\mathcal{FM}$. Consider a fiber product preserving bundle functor F on $\mathcal{FM}$ and write F_m for its restriction to $\mathcal{FM}_m$. By [5], F is determined by the sequences A and H from (1) and (4) as follows. If we restrict $H_{m,m}$ to the open subset $G^r_m\subset L^r_{m,m}$, we obtain a group homomorphism $H_m\colon G^r_m\to\operatorname{Aut}A_m$. Then $F_m=(A_m,H_m,\mathcal{O})$. Hence $F_mY=P^rM[V^{A_m}Y]$. Further, let $f\colon Y\to\bar Y$ be an $\mathcal{FM}$-morphism over $\underline{f}\colon M\to\bar M$, $\dim\bar M=n$. For every $u\in P^r_xM$ and $v\in P^r_{\underline{f}(x)}\bar M$, we can write $j^r_x\underline{f}$ in the form $j^r_x\underline{f}=\{u,v,g\}$, $g=v^{-1}\circ j^r_x\underline{f}\circ u\in L^r_{m,n}$. Then our construction of $Ff\colon FY\to F\bar Y$ of [5] can be expressed in the form

$$Ff\big(\{u,X\}\big)=\big\{v,H(g)^V_{\bar Y}(V^{A_m}f(X))\big\}\,,\quad X\in V^{A_m}_xY\,.\tag{8}$$

In the case $Y=\mathbb{R}^m\times N$ and $\bar Y=\mathbb{R}^n\times\bar N$, we consider $\varrho_m\colon\mathbb{R}^m\to P^r\mathbb{R}^m$ and $\varrho_n\colon\mathbb{R}^n\to P^r\mathbb{R}^n$. Then $j^r\underline{f}$ defines a map $\varphi\colon\mathbb{R}^m\to L^r_{m,n}$ by

$$\varphi(x)=\varrho_n\big(\underline{f}(x)\big)^{-1}\circ j^r_x\underline{f}\circ\varrho_m(x)\,.$$

In the corresponding identifications

$$F(\mathbb{R}^m\times N)=\mathbb{R}^m\times T^{A_m}N\,,\quad F(\mathbb{R}^n\times\bar N)=\mathbb{R}^n\times T^{A_n}\bar N\,,$$

(8) is of the form

$$Ff(x,X) = \big(\underline{f}(x), H(\varphi(x))_{\bar N}(V^{A_m} f(x,X))\big)\,, \quad X \in T^{A_m}N\,. \tag{9}$$

Clearly, (7) is a special case of (9).

3. The category $W^A_H S$. Investigating the prolongation of principal bundles with respect to the functor $F_m = (A_m, H_m, t_m)$ on $\mathcal{FM}_m$, M. Doupovec and the author used essentially the fact that every Lie group G induces the semidirect group product

$$W^{A_m}_{H_m} G = G^r_m \rtimes T^{A_m} G$$

with the group composition

$$(g_2, C_2)(g_1, C_1) = (g_2 \circ g_1, H_m(g_1^{-1})_G(C_2) \bullet C_1)\,,$$

where $\bullet$ denotes the induced group composition in $T^{A_m}G$. Replacing A_m by the sequence A and H_m by the double sequence H, we extend this construction to a smooth category S over $\mathbb{N}$.

Definition 1. A smooth category S over $\mathbb{N}$ is a category over $\mathbb{N}$ such that each set $S_{m,n}$ is a smooth manifold and every composition map

$$\varkappa_{m,n,p}\colon S_{n,p} \times S_{m,n} \to S_{m,p}$$

is a smooth map.

Having in mind the description (9) of $F = (A, H)$, we define

$$(W^A_H S)_{m,n} = L^r_{m,n} \times T^{A_n} S_{m,n}\,. \tag{10}$$

For every $(g_1, C_1) \in L^r_{m,n} \times T^{A_n} S_{m,n}$ and $(g_2, C_2) \in L^r_{n,p} \times T^{A_p} S_{n,p}$, we define their composition by

$$(g_2, C_2)(g_1, C_1) = \big(g_2 \circ g_1, C_2 \bullet H(g_2)_{S_{m,n}}(C_1)\big)\,, \tag{11}$$

where $\bullet$ denotes the induced map

$$T^{A_p}\varkappa_{m,n,p}\colon T^{A_p} S_{n,p} \times T^{A_p} S_{m,n} \to T^{A_p} S_{m,p}\,.$$

Proposition 2. *$W^A_H S$ is a smooth category over $\mathbb{N}$, that will be called the Weilian (A,H)-prolongation of S.*

Proof. It suffices to verify explicitly the associativity of (11), for the remaining steps of the proof are trivial. By the associativity in S and the functoriality of H, we obtain (with omitting the subscripts of H)

$$(g_3, C_3)\big(g_2 \circ g_1, C_2 \bullet H(g_2)(C_1)\big) = \big(g_3 \circ g_2 \circ g_1, C_3 \bullet H(g_3)(C_2) \bullet H(g_3 \circ g_2)(C_1)\big)\,.$$

□

Further, if $\bar S = (\bar S_{m,n})$ is another smooth category over $\mathbb{N}$ and $\varphi\colon S \to \bar S$ is a smooth functor, i.e. all maps $\varphi_{m,n}\colon S_{m,n} \to \bar S_{m,n}$ are smooth, then the rule

$$(W^A_H \varphi)_{m,n} = \mathrm{id}_{L^r_{m,n}} \times T^{A_n} \varphi_{m,n}$$

defines a smooth functor $W^A_H \varphi\colon W^A_H S \to W^A_H \bar S$.

4. The action $W^A_H D$. Consider a sequence $Z = (Z_1, \dots, Z_m, \dots)$ of manifolds.

Definition 2. An action D of S on Z is a double sequence $D_{m,n} \colon S_{m,n} \times Z_m \to Z_n$ of smooth maps such that $D_{m,m}(e_m, y) = y$, $e_m =$ the unit of $D_{m,m}$, $y \in Z_m$, and

$$D_{n,p}\big(s_2, D_{m,n}(s_1, y)\big) = D_{m,p}\big(\varkappa_{m,n,p}(s_2, s_1), y\big), \quad s_1 \in S_{m,n},\ s_2 \in S_{n,p}.$$

In [1], we deduced that every action of a Lie group G on a manifold Q induces an action of $W^{A_m}_{H_m} G$ on $T^{A_m} Q$. Analogously we introduce the action $W^A_H D$ of $W^A_H S$ on the sequence

$$T^A Z = (T^{A_1} Z_1, \dots, T^{A_m} Z_m, \dots).$$

For $(g, C) \in L^r_{m,n} \times T^{A_n} S_{m,n}$ and $B \in T^{A_m} Z_m$, we define

$$(W^A_H D)_{m,n}\big((g, C), B\big) = C{\bullet}H(g)_{Z_m}(B), \tag{12}$$

where $\bullet$ denotes the map $T^{A_n} D_{m,n} \colon T^{A_n} S_{m,n} \times T^{A_n} Z_m \to T^{A_n} Z_n$.

Proposition 3. *$W^A_H D$ is an action of $W^A_H S$ on $T^A Z$.*

Proof. For another $(\bar g, \bar C) \in (W^A_H D)_{n,p}$, the fact that D is an action and H is a functor yields directly

$$\bar C{\bullet}H(\bar g)\big(C{\bullet}H(g)(B)\big) = \big(\bar C{\bullet}H(\bar g)(C)\big){\bullet}H(\bar g \circ g)(B).$$

□

Let $\bar D$ be another action of S on $\bar Z = (\bar Z_1, \dots, \bar Z_m, \dots)$. An action morphism $\psi \colon D \to \bar D$ is a sequence $(\psi_m \colon Z_m \to \bar Z_m)$ of smooth maps such that

$$\bar D_{m,n}\big(s, \psi_m(y)\big) = \psi_n\big(D_{m,n}(s, y)\big), \quad s \in S_{m,n},\ y \in Z_m.$$

Then it is easy to see that

$$T^A \psi = (T^{A_m} \psi_m \colon T^{A_m} Z_m \to T^{A_m} \bar Z_m)$$

is a morphism of the actions $W^A_H D$ and $W^A_H \bar D$.

5. (A, H) transforms D-bundles into $W^A_H D$-bundles. The elementary D-bundles are the products $M \times Z_m$. The elementary morphisms of D-bundles are the pairs $f_0 \colon M \to \bar M$ and $f_1 \colon M \to S_{m,n}$, that are interpreted as $\mathcal{FM}$-morphisms

$$f = (f_0, f_1) \colon M \times Z_m \to \bar M \times Z_n \quad f(x, y) = \big(f_0(x), f_1(x)(y)\big), \tag{13}$$

where $f_1(x)(y) = D_{m,n}\big(f_1(x), y\big)$. Globally, the category $D\mathcal{B}$ of D-bundles is defined by the standard "gluing together" procedure. So the structure of D-bundle on a fibered manifold $p \colon Y \to M$ is determined by an open covering (U_α) of M and a family of local trivializations

$$\psi_\alpha \colon p^{-1}(U_\alpha) \to U_\alpha \times Z_m$$

such that all transition functions are the elementary $D\mathcal{B}$- morphisms. If $\bar p \colon \bar Y \to \bar M$ is another D-bundle, an $\mathcal{FM}$-morphism $f \colon Y \to \bar Y$ is said to be a $D\mathcal{B}$-morphism, if it is expressed by elementary $D\mathcal{B}$-morphisms in the generating trivializations of the $D\mathcal{B}$-structures on Y and $\bar Y$. We say that (13) is an admissible local expression of a $D\mathcal{B}$-morphism.

We recall that we have a canonical injection $i^B_M\colon M \to T^B M$ for every Weil bundle $T^B M$, $i^B_M(x) = j^B \hat{x}$, $x \in M$.

Consider a fiber product preserving bundle functor $F = (A, H)$ on $\mathcal{FM}$. By Section 1, we have $F(\mathbb{R}^m \times Z_m) = \mathbb{R}^m \times T^{A_m} Z_m$. Consider an elementary $D\mathcal{B}$-morphism $f = (f_0, f_1)\colon \mathbb{R}^m \times Z_m \to \mathbb{R}^n \times Z_n$ and write $f_{1x}\colon Z_m \to Z_n$ for the map $y \mapsto f_1(x)(y)$. By (9) and (13), we obtain

$$Ff(x,X) = \big(f_0(x), H(\varphi(x))\big)_{Z_n}\big((T^{A_m} f_{1x}(X))\big)\,, \quad X \in T^{A_m} Z_m\,. \tag{14}$$

But we have $T^{A_m} f_{1x}(X) = T^{A_m} D_{m,n}(j^{A_m} \widehat{f_1(x)}, X)$ and

$$H\big(\varphi(x)\big)_{Z_n}\big(T^{A_m} D_{m,n}(j^{A_m}\widehat{f_1(x)}, X)\big) = T^{A_n} D_{m,n}\big(j^{A_n}\widehat{f_1(x)}, H(\varphi(x))_{Z_m}(X)\big)\,.$$

By (12) and (13), (14) is an elementary morphism of $W^A_H D$-bundles. Since our constructions are of functorial character, we have deduced

Proposition 4. *The functor (A, H) transforms D-bundles into $W^A_H D$-bundles.*

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