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INVARIANCE OF g-NATURAL METRICS ON LINEAR FRAME BUNDLES

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Abstract. In this paper we prove that each g-natural metric on a linear frame bundle LM over a Riemannian manifold (M, g) is invariant with respect to a lifted map of a (local) isometry of the base manifold. Then we define g-natural metrics on the orthonormal frame bundle OM and we prove the same invariance result as above for OM. Hence we see that, over a space (M, g) of constant sectional curvature, the bundle OM with an arbitrary g-natural metric $\tilde{G}$ is locally homogeneous.

Introduction

There are well-known classical examples of "lifted metrics" on the linear frame bundle LM over a Riemannian manifold (M, g). Namely, these are the diagonal lift (which is also called the Sasaki-Mok metric), the horizontal lift and the vertical lift. As one can see, the classical constructions are examples of "natural transformations of second order". In [10], the present authors have fully classified all (possibly degenerate) naturally lifted metrics "of second order" on LM. They have proved that the complete family of such natural metrics (for a fixed base metric) is a module over real functions generated by some generalizations of known classical lifts. Our idea of naturality is closely related to that of A. Nijenhuis, D. B. A. Epstein, P. Stredder and others (see [6] for the full references). We have used for our purposes the concepts and methods developed by D. Krupka [12, 13] and D. Krupka and V. Mikolášová [15]. See also I. Kolář, P. W. Michor and J. Slovák [6], and D. Krupka and J. Janyška [14] for the concept of naturality in general. We shall use further the name "g-natural metrics" for our metrics on LM after the name proposed by M. T. K. Abbassi in [1] for the metrics naturally lifted to tangent bundles. The Riemannian geometry of linear frame bundles (for a special class of g-natural metrics) has been studied by Mok [16], Cordero and de León [2, 3], Cordero, Dodson and de León [4], and by the present authors in [11].

One of the properties of g-natural metrics to be expected is the "invariance property" saying that the lifts of (local) isometries are again (local) isometries. In

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fact, we have proved in [7] that this property holds for the g-natural metrics on the tangent bundles. After a short survey about our classification we present g-natural metrics on LM in the convenient setting. Then we prove that the invariance property really holds. Next we pass over to the orthonormal frame bundles.

We define g-natural metrics on orthonormal frame bundles OM as restrictions of g-natural metrics on LM to the submanifolds OM of LM, and we present an explicit formula. The Riemannian geometry of orthonormal frame bundles (for a special class of g-natural metrics) has been studied by Jensen [5], Mok [16], Cordero and de León [2, 3], Cordero, Dodson and de León [4], Zou [18], and by the present authors in [8] and [9]. We prove in this article that each g-natural metric on OM over (M, g) is invariant by a lifted map of a (local) isometry of (M, g), and hence each orthonormal frame bundle OM equipped with a g-natural metric $\tilde{G}$ over a space of constant sectional curvature is a locally homogeneous space.

1. Lifts of vectors

The linear frame bundle LM over a smooth manifold M consists of all pairs (x, u), where x is a point of M and u is a basis for the tangent space M_x of M at x. We denote by p the natural projection of LM to M defined by $p(x, u) = x$. If $(\mathcal{U};\ x^1, x^2, \dots, x^n)$ is a system of local coordinates in M, then a basis $u = (u_1, u_2, \dots, u_n)$ for M_x can be expressed in the unique way in the form

$$u_\lambda = \sum_{i=1}^{n} u^i_\lambda \Big(\frac{\partial}{\partial x^i}\Big)_x$$

for all indices $\lambda = 1, 2, \dots, n$, and hence $(p^{-1}(\mathcal{U});\ x^1, x^2, \dots, x^n, u^1_1, u^2_1, \dots, u^n_n)$ is a system of local coordinates in LM.

Let g be a Riemannian metric on the manifold M and ∇ its Levi-Civita connection. Then the tangent space $(LM)_{(x,u)}$ of LM at $(x, u) \in LM$ splits into the horizontal and vertical subspace $H_{(x,u)}$ and $V_{(x,u)}$ with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}\,.$$

If a point $(x, u) \in LM$ and a vector $X \in M_x$ are given, then there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_*(X^h) = X$. We call X^h the *horizontal lift* of X to TM at (x, u). We define naturally n different vertical lifts of $X \in M_x$. If ω is a one-form on M, then $\iota_\mu\omega$, $\mu = 1, 2, \dots, n$, are functions on LM defined by $(\iota_\mu\omega)(x, u) = \omega(u_\mu)$ for all $(x, u) = (x, u_1, u_2, \dots, u_n) \in LM$. The *vertical lifts* $X^{v,\lambda}$, $\lambda = 1, 2, \dots, n$, of $X \in M_x$ to LM at (x, u) are the n vectors such that $X^{v,\lambda}(\iota_\mu\omega) = \omega(X)\delta^\lambda_\mu$, $\lambda, \mu = 1, 2, \dots, n$, hold for all one-forms ω on M, where δ^λ_μ denotes the Kronecker's delta. The n vertical lifts are always uniquely determined, and they are linearly independent if $X \neq 0$. They are expressed in a local coordinate system as

$$X^{v,\lambda}_{(x,u)} = \sum_{i=1}^{n} \xi^i \Big(\frac{\partial}{\partial u^i_\lambda}\Big)_{(x,u)}$$

for all indices $\lambda = 1, 2, \dots, n$ (*cf.* [3]).

In an obvious way we can define horizontal and vertical lifts of vector fields on M. These are uniquely defined vector fields on TM. The *canonical vertical vector fields* on LM are vector fields U^μ_λ, $\lambda, \mu = 1, 2, \ldots, n$, defined, in terms of local coordinates, by $U^\mu_\lambda = \sum_{i=1}^n u^i_\lambda \partial/\partial u^i_\mu$. Here U^μ_λ's do not depend on the choice of local coordinates and they are defined globally on LM.

For a vector $u_\lambda = \sum_{i=1}^n u^i_\lambda (\partial/\partial x^i)_x \in M_x$, $\lambda = 1, 2, \ldots, n$, we see that

$$(u_\lambda)^h_{(x,u)} = \sum_{i=1}^n u^i_\lambda \Big(\frac{\partial}{\partial x^i}\Big)^h_{(x,u)} \tag{1.1}$$

and

$$(u_\lambda)^{v,\mu}_{(x,u)} = \sum_{i=1}^n u^i_\lambda \Big(\frac{\partial}{\partial x^i}\Big)^{v,\mu}_{(x,u)} = \sum_{i=1}^n u^i_\lambda \Big(\frac{\partial}{\partial u^i_\mu}\Big)_{(x,u)} = U^\mu_\lambda(x,u). \tag{1.2}$$

2. g-natural metrics

We say that a bundle morphism of the form $\zeta\colon LM \oplus TM \oplus TM \longrightarrow M \times \mathbb{R}$ is an *L-metric* on M if it is linear in the second and the third argument (and smooth in the first argument). We also say that ζ is symmetric or skew-symmetric if it is symmetric or skew-symmetric with respect to the second and third argument, respectively. Any Riemannian metric g on M is a symmetric L-metric which is independent on u. In our special case, letting g be a given Riemannian metric on M, we speak about *natural L-metrics derived from* g which are L-metrics ζ, for a fixed $u \in LM$, whose components $\zeta(u)_{ij} = \zeta(u, \partial/\partial x^i, \partial/\partial x^j)$ with respect to a system of local coordinates $(x^1, x^2, \ldots, x^n)$ in M are solutions of the system of differential equations

$$2\sum_{a=1}^n g_{ap}\frac{\partial \zeta_{ij}}{\partial g_{aq}} - \sum_{\alpha=1}^n u^q_\alpha \frac{\partial \zeta_{ij}}{\partial u^p_\alpha} = \zeta_{ip}\delta^q_j + \zeta_{pj}\delta^q_i\,, \quad i,j,p,q = 1,2,\ldots,n\,.$$

We obtain

Theorem 2.1 ([10])**.** *Let (M, g) be an n-dimensional Riemannian manifold. Then all natural L-metrics ζ on M derived from g are given by*

$$\zeta(u; X, Y) = \sum_{\alpha,\beta=1}^n \varphi_{\alpha\beta}(w_{\rho\sigma})\omega^\alpha(X)\omega^\beta(Y)\,, \tag{2.1}$$

where $\{\omega^1, \omega^2, \ldots, \omega^n\}$ are the dual frame to a linear frame $u = \{u_1, u_2, \ldots, u_n\}$ and $\varphi_{\alpha\beta}$, $\alpha, \beta = 1, 2, \ldots, n$, are arbitrary smooth functions of $n(n+1)/2$ variables $w_{\rho\sigma} = g(u_\rho, u_\sigma)$, $1 \le \rho \le \sigma \le n$.

Remark. Let $(\theta^1, \theta^2, \ldots, \theta^n)$ be the frame of (M, g) metrically equivalent to a basis $u = (u_1, u_2, \ldots, u_n)$ for M_x, which is given by $\theta^\alpha(u_\lambda) = g_x(u_\alpha, u_\lambda)$ for all indices $\alpha, \lambda = 1, 2, \ldots, n$. Then, each ω^α of the frame $(\omega^1, \omega^2, \ldots, \omega^n)$ dual to u is a linear combination of θ^α's whose factors are smooth functions of variables $w_{\rho\sigma} = g(u_\rho, u_\sigma)$, $\rho, \sigma = 1, 2, \ldots, n$. Because the factors $\varphi_{\alpha\beta}(w_{\rho\sigma})$ in (2.1) are functions of the variables $w_{\rho\sigma}$, if we take $(\omega^1, \omega^2, \ldots, \omega^n)$ in Theorem 2.1 as the frame metrically equivalent to u, then all natural L-metrics ζ on M derived from g are still written in the form (2.1).

For a given Riemannian metric g on M, we define the *classical lifts of L-metrics from M to LM with respect to g* as symmetric $(0,2)$-tensor fields on LM which are constructed as follows:

(a) Let ζ be a symmetric L-metric and $(\zeta^{\lambda\mu})$, $1 \leq \lambda \leq \mu \leq n$, a family of arbitrary L-metrics. The *diagonal lift* $\xi^{d,g}$ of the family $\xi = (\zeta, \zeta^{\lambda\mu})$ with respect to g is defined by

$$\xi^{d,g}_{(x,u)}(X^h, Y^h) = \zeta_x(u; X, Y)\,,$$

$$\zeta^{d,g}_{(x,u)}(X^h, Y^{v,\mu}) = 0\,, \qquad \mu = 1, 2, \dots, n\,,$$

$$\xi^{d,g}_{(x,u)}(X^{v,\lambda}, Y^{v,\mu}) = \zeta^{\lambda\mu}_x(u; X, Y)\,, \qquad 1 \leq \lambda \leq \mu \leq n\,,$$

for all $X, Y \in M_x$.

(b) The *horizontal lift* $\xi^{h,g}$ of an n-tuple $\xi = (\zeta^\mu)$ of L-metrics with respect to g is defined by

$$\xi^{h,g}_{(x,u)}(X^h, Y^h) = 0\,,$$

$$\xi^{h,g}_{(x,u)}(X^h, Y^{v,\mu}) = \zeta^\mu_x(u; Y, X)\,, \qquad \mu = 1, 2, \dots, n\,,$$

$$\xi^{h,g}_{(x,u)}(X^{v,\lambda}, Y^{v,\mu}) = 0\,, \qquad \lambda, \mu = 1, 2, \dots, n\,,$$

for all $X, Y \in M_x$.

If we take $\zeta = g$ and $\zeta^{\lambda\mu} = g\,\delta^{\lambda\mu}$ in (a), and $\zeta^\mu = g$ in (b), then $\xi^{d,g}$ and $\xi^{h,g}$ are just the diagonal lift g^d and the horizontal lift g^h, respectively. Also, if we take $\zeta = g$ and $\zeta^{\lambda\mu} = 0$ in (a), then $\xi^{d,g}$ is the vertical lift g^v.

Thus we have all metrics on LM which come from a second order natural transformation of a given Riemannian metric on M:

Theorem 2.2 ([10]). *Let g be a Riemannian metric on an n-dimensional smooth manifold M, $n \geq 2$, and let G be a* (*possibly degenerate*) *pseudo-Riemannian metric on the linear frame bundle LM which comes from a second order natural transformation of g. Then there are families $\xi_1 = (\zeta, \zeta^{\lambda\mu})$ and $\xi_2 = (\zeta^\nu)$ of natural L-metrics derived from g, where $1 \leq \lambda \leq \mu \leq n$, $\nu = 1, 2, \dots, n$ and ζ is symmetric, such that*

$$G = {\xi_1}^{d,g} + {\xi_2}^{h,g}\,.$$

Moreover, all natural L-metrics derived from g are given by Theorem 2.1.

The family of all natural metrics G on LM over an n-dimensional Riemannian manifold (M, g) depends on $n(n^3 + 3n^2 + n + 1)/2$ arbitrary functions of $n(n+1)/2$ variables.

For any point $(x, u) \in LM$, if we take the frame $(\omega^1, \omega^2, \dots, \omega^n)$ in Theorem 2.1 as the frame metrically equivalent to $u = (u_1, u_2, \dots, u_n)$, then the metrics G in

Theorem 2.2 is expressed in the following form:

$$\begin{aligned} G_{(x,u)}(X^h, Y^h) &= \sum_{\alpha,\beta=1}^{n} \varphi_{\alpha\beta}(w_{\rho\sigma}) g_x(u_\alpha, X) g_x(u_\beta, Y)\,, \\ G_{(x,u)}(X^h, Y^{v,\mu}) &= \sum_{\alpha,\beta=1}^{n} \varphi_{\alpha\,\beta}^{\;\mu}(w_{\rho\sigma}) g_x(u_\alpha, X) g_x(u_\beta, Y)\,, \\ G_{(x,u)}(X^{v,\lambda}, Y^{v,\mu}) &= \sum_{\alpha,\beta=1}^{n} \varphi_{\alpha\beta}^{\lambda\mu}(w_{\rho\sigma}) g_x(u_\alpha, X) g_x(u_\beta, Y) \end{aligned} \tag{2.2}$$

for all $X, Y \in M_x$, where $\varphi_{\alpha\beta}$, $\varphi_{\alpha\,\beta}^{\;\mu}$ and $\varphi_{\alpha\beta}^{\lambda\mu}$, $\alpha, \beta, \lambda, \mu = 1, 2, \ldots, n$, are arbitrary smooth functions of $n(n+1)/2$ variables $w_{\rho\sigma} = g(u_\rho, u_\sigma)$, $1 \le \rho \le \sigma \le n$.

In the following we shall call G a *g-natural metric* on LM.

3. Invariance of g-natural metrics

Let ϕ be a (local) transformation of a manifold M. Then we define a transformation Φ of LM by

$$\Phi(x, u) = (\phi x, \phi_{*x} u_1, \phi_{*x} u_2, \ldots, \phi_{*x} u_n) \tag{3.1}$$

for all $(x, u) = (x, u_1, u_2, \ldots, u_n) \in LM$.

Proposition 3.1. *Let ϕ be a (local) affine transformation of a manifold M with an affine connection ∇ and let Φ be the lift of ϕ to LM defined as above. Then we have*

$$\Phi_*(X^h) = (\phi_* X)^h, \qquad \Phi_*(X^{v,\lambda}) = (\phi_* X)^{v,\lambda} \tag{3.2}$$

for all $X \in \mathfrak{X}(M)$ and all indices $\lambda = 1, 2, \ldots, n$. In particular, for the canonical vertical vector fields, we have

$$\Phi_*(U_\lambda^\mu) = U_\lambda^\mu \tag{3.3}$$

for all indices $\lambda, \mu = 1, 2, \ldots, n$.

Proof. We use the formula $p \circ \Phi = \phi \circ p$. For all vector fields X and functions f on M we calculate at $(x, u) \in LM$:

$$\begin{aligned} \big(p_{*\Phi(x,u)}\big(\Phi_{*(x,u)}(X^h_{(x,u)})\big)\big) f &= X^h_{(x,u)}(f \circ p \circ \Phi) = X^h_{(x,u)}(f \circ \phi \circ p) \\ &= \big(p_{*(x,u)}(X^h_{(x,u)})\big)(f \circ \phi) = X_x(f \circ \phi) = (\phi_{*x} X_x) f\,. \end{aligned}$$

Since Φ preserves the horizontal distribution, we have

$$\Phi_{*(x,u)}(X^h_{(x,u)}) = (\phi_{*x} X_x)^h_{\Phi(x,u)}\,.$$

Next, using the formula $\iota_\nu \omega \circ \Phi = \iota_\nu(\phi^* \omega)$ for all one-forms ω on M and all indices $\nu = 1, 2, \ldots, n$, we calculate at $(x, u) \in LM$ for all $X \in \mathfrak{X}(M)$:

$$\begin{aligned} \big(\Phi_{*(x,u)}(X^{v,\mu}_{(x,u)})\big)(\iota_\nu \omega) &= (X^{v,\mu}_{(x,u)})\big((\iota_\nu \omega) \circ \Phi\big) = (X^{v,\mu}_{(x,u)})\big(\iota_\nu(\phi^* \omega)\big) \\ &= (\phi^* \omega)_x (X_x) \delta_\nu^\mu = \omega_{\phi(x)}(\phi_{*x} X_x) \delta_\nu^\mu = (\phi_{*x} X_x)^{v,\mu}(\iota_\nu \omega)\,. \end{aligned}$$

Since Φ preserves the vertical distribution, we have $\Phi_{*(x,u)}(X^{v,\mu}_{(x,u)}) = (\phi_{*x}X)^{v,\mu}_{\Phi(x,u)}$. In particular, we have $\Phi_{*(x,u)}({u_\lambda}^{v,\mu}) = (\phi_{*x}u_\lambda)^{v,\mu}$, that is, by (1.2), we have (3.3). □

Theorem 3.2. *Let ϕ be a (local) isometry of a Riemannian manifold (M,g). Then every g-natural metric G on the linear frame bundle LM over (M,g) is invariant by the lift Φ of ϕ. In other words, Φ is a (local) isometry of (LM,G) whose projection on (M,g) is ϕ.*

Proof. For any basis $u = (u_1, u_2, \dots, u_n)$ for M_x, $x \in M$, we assume that the frame $(\omega^1, \omega^2, \dots, \omega^n)$ in Theorem 2.2 is metrically equivalent to u. We abbreviate $(\phi_{*x}u_1, \phi_{*x}u_2, \dots, \phi_{*x}u_n)$ as $\phi_{*x}u$. Let X and Y be vectors from M_x. Then, by Proposition 3.1 and the first formula of (2.2), we have at any point $(x,u) \in LM$ that

$$\begin{aligned}(\Phi^*G)_{(x,u)}\big(X^h_{(x,u)},\, Y^h_{(x,u)}\big) &= G_{\Phi(x,u)}(\Phi_{*(x,u)}X^h_{(x,u)},\, \Phi_{*(x,u)}Y^h_{(x,u)})\\ &= G_{(\phi x,\phi_{*x}u)}((\phi_{*x}X)^h_{(\phi x,\phi_{*x}u)},\, (\phi_{*x}Y)^h_{(\phi x,\phi_{*x}u)})\\ &= \sum_{\alpha,\beta=1}^{n} \varphi_{\alpha\beta}\big(g_{\phi x}(\phi_{*x}u_\rho,\, \phi_{*x}u_\sigma)\big)g_{\phi x}(\phi_{*x}u_\alpha,\, \phi_{*x}X)g_{\phi x}(\phi_{*x}u_\beta,\, \phi_{*x}Y)\,.\end{aligned}$$

Now, since ϕ is a (local) isometry of (M,g), the right-hand side of this formula is just $G_{(x,u)}\big(X^h_{(x,u)}, Y^h_{(x,u)}\big)$. Hence we have

$$(\Phi^*G)_{(x,u)}\big(X^h_{(x,u)}, Y^h_{(x,u)}\big) = G_{(x,u)}(X^h_{(x,u)},\, Y^h_{(x,u)})$$

for all $X, Y \in M_x$.

The rest of the assertion is proved by similar calculations. □

Let us remark that the natural projection p of (LM,G) onto (M,g) is *not* a Riemannian submersion, in general.

4. The orthonormal frame bundle

The orthonormal frame bundle

$$OM = \{(x,u) \in LM \mid g_x(u_\lambda, u_\mu) = \delta_{\lambda\mu}, \lambda, \mu = 1,2,\dots,n\}$$

over a Riemannian manifold (M,g) is an $n(n+1)/2$-dimensional subbundle of LM with the structure group $O(n)$.

The vector fields $T_{\lambda\mu}$ defined on LM by

$$T_{\lambda\mu} = U^\mu_\lambda - U^\lambda_\mu \tag{4.1}$$

for all indices $\lambda, \mu = 1,2,\dots,n$ are tangent to OM at every point $(x,u) \in OM$. Here $T_{\lambda\mu}$ is skew-symmetric with respect to all indices $\lambda, \mu = 1,2,\dots,n$. In particular $T_{\lambda\lambda} = 0$ for all $\lambda = 1,2,\dots,n$. At every point $(x,u) \in OM$, a collection $\{T_{\lambda\mu}(x,u) \mid 1 \le \lambda < \mu \le n\}$ forms a basis for the vertical space $V_{(x,u)} \cap (OM)_{(x,u)}$.

Definition 4.1. Consider a smooth Riemannian manifold (M, g). A *g-natural metric* on the orthonormal frame bundle OM over (M, g) is restriction $\tilde{G}$ of any g-natural metric G given by (2.2) on the linear frame bundle LM to the submanifold $OM \subset LM$.

We say that a natural L-metric on M derived from g is *of constant type* if all factors $\varphi_{\alpha\beta}(w_{\rho\sigma})$ in (2.1) are constants for all indices $\alpha, \beta = 1, 2, \ldots, n$.

Theorem 4.2. *Let g be a Riemannian metric on an n-dimensional smooth manifold M, $n \geq 2$, and let $\tilde{G}$ be a g-natural metrics on the orthonormal frame bundle OM. Then there are families $\xi_1 = (\zeta, \zeta^{\lambda\mu})$ and $\xi_2 = (\zeta^\nu)$ of natural L-metrics of constant type derived from g, where $1 \leq \lambda \leq \mu \leq n$, $\nu = 1, 2, \ldots, n$ and ζ is symmetric, such that*

$$\tilde{G} = {\xi_1}^{d,g} + {\xi_2}^{h,g}\,.$$

Equivalently, we have

Corollary 4.3. *The g-natural metrics $\tilde{G}$ in Theorem 4.2 can be expressed at any point $(x, u) \in OM$ in the following form:*

$$\begin{aligned} \tilde{G}_{(x,u)}(X^h_{(x,u)},\, Y^h_{(x,u)}) &= \sum_{\alpha,\beta=1}^{n} c_{\alpha\beta}\, g_x(u_\alpha, X) g_x(u_\beta, Y)\,, \\ \tilde{G}_{(x,u)}\big(X^h_{(x,u)},\, T_{\lambda\mu}(x,u)\big) &= \sum_{\alpha=1}^{n} (c_{\alpha}{}^{\mu}_{\lambda} - c_{\alpha}{}^{\lambda}_{\mu}) g_x(u_\alpha, X)\,, \\ \tilde{G}_{(x,u)}\big(T_{\lambda\mu}(x,u),\, T_{\nu\omega}(x,u)\big) &= c^{\mu\omega}_{\lambda\nu} - c^{\mu\nu}_{\lambda\omega} - c^{\lambda\omega}_{\mu\nu} + c^{\lambda\nu}_{\mu\omega} \end{aligned} \tag{4.2}$$

for all $X, Y \in M_x$ and all indices $\lambda, \mu, \nu, \omega = 1, 2, \ldots, n$, where $c_{\alpha\beta}$, $c_{\alpha}{}^{\mu}_{\beta}$ and $c^{\lambda\mu}_{\alpha\beta}$, $\alpha, \beta, \lambda, \mu = 1, 2, \ldots, n$, are constants.

Proof. Let $(x, u) = (x, u_1, u_2, \ldots, u_n)$ be any point of OM. Then we have $w_{\rho\sigma} = g_x(u_\rho, u_\sigma) = \delta_{\rho\sigma}$ for all indices $\rho, \sigma = 1, 2, \ldots, n$, where $\delta_{\rho\sigma}$'s are the Kronecker's deltas. Hence all factors $\varphi_{\alpha\beta}$, $\varphi_{\alpha}{}^{\mu}_{\beta}$ and $\varphi^{\lambda\mu}_{\alpha\beta}$, $\alpha, \beta, \lambda, \mu = 1, 2, \ldots, n$, are arbitrary constants. By (4.1) and (1.2), we have (4.2). $\square$

Let ϕ be a (local) transformation. We denote by $\tilde{\Phi}$ restriction of Φ defined by (3.1) to OM. Then, using the fact that $\tilde{\Phi}_{*(x,u)}\boldsymbol{X} = \Phi_{*(x,u)}\boldsymbol{X}$ for all vectors $\boldsymbol{X} \in (OM)_{(x,u)}$, from (3.2), (3.1) and (4.1) we have

$$\begin{aligned} \tilde{\Phi}_{*(x,u)}(X^h_{(x,u)}) &= (\phi_{*x}X)^h_{\tilde{\Phi}(x,u)}\,, \\ \tilde{\Phi}_{*(x,u)}\big(T_{\lambda\mu}(x,u)\big) &= T_{\lambda\mu}\big(\tilde{\Phi}(x,u)\big) \end{aligned} \tag{4.3}$$

at $(x, u) \in OM$ for all vectors $X \in M_x$ and all indices $\lambda, \mu = 1, 2, \ldots, n$.

Theorem 4.4. *Let ϕ be a (local) isometry of a Riemannian manifold (M, g). Then every g-natural metric $\tilde{G}$ on the orthonormal frame bundle OM over (M, g) is invariant by the lift $\tilde{\Phi}$ of ϕ. In other words, $\tilde{\Phi}$ is a (local) isometry of $(OM, \tilde{G})$ whose projection on (M, g) is ϕ.*

Proof. We use Corollary 4.3. Let (x,u) be any point of OM. By the first formula of (4.3), we have

$$\begin{aligned}(\tilde{\Phi}^*\tilde{G})_{(x,u)}(X^h_{(x,u)},\, Y^h_{(x,u)}) &= \tilde{G}_{\tilde{\Phi}(x,u)}\big(\tilde{\Phi}_{*(x,u)}(X^h_{(x,u)}),\, \tilde{\Phi}_{*(x,u)}(Y^h_{(x,u)})\big)\\ &= \sum_{\alpha,\beta=1}^{n} c_{\alpha\beta}\, g_{\phi x}(\phi_{*x}u_\alpha,\, \phi_{*x}X)\, g_{\phi x}(\phi_{*x}u_\beta,\, \phi_{*x}Y)\\ &= \tilde{G}_{(x,u)}(X^h_{(x,u)}, Y^h_{(x,u)})\end{aligned}$$

for all $X, Y \in M_x$. Next, by (4.3), we have

$$\begin{aligned}(\tilde{\Phi}^*\tilde{G})_{(x,u)}(X^h_{(x,u)},\, T_{\lambda\mu}(x,u)) &= \tilde{G}_{\tilde{\Phi}(x,u)}\big(\tilde{\Phi}_{*(x,u)}(X^h_{(x,u)}),\, \tilde{\Phi}_{*(x,u)}(T_{\lambda\mu}(x,u))\big)\\ &= \sum_{\alpha=1}^{n}(c_{\alpha}{}^{\mu}_{\lambda} - c_{\alpha}{}^{\lambda}_{\mu})\, g_{\phi x}(\phi_{*x}u_\alpha,\, \phi_{*x}X)\\ &= \tilde{G}_{(x,u)}\big(X^h_{(x,u)},\, T_{\lambda\mu}(x,u)\big)\end{aligned}$$

for all $X \in M_x$ and all indices $\lambda, \mu = 1, 2, \dots, n$. Finally, by the second formula of (4.3), we have easily that

$$(\tilde{\Phi}^*\tilde{G})_{(x,u)}\big(T_{\lambda\mu}(x,u),\, T_{\nu\omega}(x,u)\big) = \tilde{G}_{(x,u)}\big(T_{\lambda\mu}(x,u),\, T_{\nu\omega}(x,u)\big)$$

for all $\lambda, \mu, \nu, \omega = 1, 2, \dots, n$. □

Corollary 4.5. *If (M,g) is a space of constant sectional curvature, then its orthonormal frame bundle $(M, \tilde{G})$ with a g-natural metric is always locally homogeneous.*

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