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ASYMPTOTIC PROPERTIES OF TRINOMIAL DELAY DIFFERENTIAL EQUATIONS

Jozef Džurina and Renáta Kotorová

Abstract. The aim of this paper is to study asymptotic properties of the solutions of the third order delay differential equation

$$(*)\qquad \left(\frac{1}{r(t)}\, y'(t)\right)'' - p(t)\, y'(t) + g(t)\, y\big(\tau(t)\big) = 0\,.$$

Using suitable comparison theorem we study properties of Eq. $(*)$ with help of the oscillation of the second order differential equation.

We consider the third order delay differential equation

$$(1)\qquad \left(\frac{1}{r(t)}\, y'(t)\right)'' - p(t)\, y'(t) + g(t)\, y\big(\tau(t)\big) = 0$$

and the corresponding second order differential equation

$$(2)\qquad v''(t) = p(t)\, r(t)\, v(t)\,.$$

We always assume that

(i) $r(t)$, $p(t)$ and $g(t) \in C\big([t_0,\infty)\big)$, $p(t) \geqslant 0$, $r(t) > 0$, $g(t) > 0$.

(ii) $\tau(t) \in C^1\big([t_0,\infty)\big)$, $\tau(t) \leqslant t$, $\tau'(t) > 0$ and $\tau(t) \to \infty$ as $t \to \infty$.

Differential equations of the third order has been subject of insensitive studying in the literature (see enclosed references).

Lazer in [16] has shown that particular case of (1), namely differential equation without delay

$$(3)\qquad y'''(t) - p(t)\, y'(t) + g(t)\, y(t) = 0$$

has the following structure of the nonoscillatory solution:

Lemma A. *Let $y(t)$ be a nonoscillatory solution of* (3)*. Then there exists $t_1 \geqslant t_0$ such that either*

$$(4)\qquad y(t)\, y'(t) < 0$$

or

$$(5)\qquad y(t)\, y'(t) \geqslant 0$$

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for $t \geqslant t_1$ *and more over if* $y(t)$ *satisfies* (4) *then also*

$$(-1)^i y(t)\, y^{(i)}(t) > 0\,, \qquad 0 \leqslant i \leqslant 3,\ t \geqslant t_1\,. \tag{6}$$

It is known that (3) always has a solution satisfying (6). For simplifying formulation of our theorems we say that (3) has property (P_0) if every nonoscillatory solution of (3) satisfies (6).

Lazer presented a sufficient condition for property (P_0) of (3) (see [16]). This result has been improved by several authors (see e.g. [6], [9], [11], [18] and [19]).

We define corresponding property (P_0) of (1). For this reason we introduce the following notation. We denote

$$D_0 y = y\,, \quad D_1 y = \frac{1}{r}(D_0 y)'\,, \quad D_2 y = (D_1 y)'\,, \quad D_3 y = (D_2 y)'\,.$$

We say that (1) has property (P_0) if every nonoscillatory solution $y(t)$ of (1) satisfies

$$(-1)^i y(t)\, D_i y(t) \geqslant 0\,, \quad 0 \leqslant i \leqslant 3\,. \tag{7}$$

We present a general technique, based on suitable comparison theorem, that enables us to study property (P_0) of (1) with help of asymptotic properties of the second order differential equation.

Lemma 1. *The operator* $Ly \equiv \left(\frac{1}{r(t)}\, y'(t)\right)'' - p(t) y'(t)$ *can be written as*

$$Ly \equiv \frac{1}{v}\Big(v^2\Big(\frac{1}{v}\,\frac{y'}{r}\Big)'\Big)'\,,$$

where $v(t)$ *is a positive solution of* (2).

Proof. Straightforward computation shows that

$$Ly \equiv \frac{1}{v}\Big(v^2\Big[-\frac{v'}{v^2}\,\frac{y'}{r} + \frac{1}{v}\Big(\frac{y'}{r}\Big)'\Big]\Big)' = \Big(\frac{y'}{r}\Big)'' - \frac{v''}{v}\,\frac{y'}{r} = \Big(\frac{1}{r(t)} y'(t)\Big)'' - p(t)\, y'(t)\,.$$

□

Corollary 1. *If* $v(t)$ *is a positive solution of* (2)*, then equation* (1) *can be rewritten as*

$$\Big(v^2\Big(\frac{1}{vr}\, y'\Big)'\Big)' + v(t)\, g(t)\, y\big(\tau(t)\big) = 0\,. \tag{8}$$

In the sequel we shall study properties of (1) with help of its binomial representation (8). For this reason it is useful for (8) to be in the canonical form because properties of canonical equations are well known. We remind that equation (8) is in the canonical form if $\int^\infty v^{-2}(t)\, dt = \int^\infty v(t) r(t)\, dt = \infty$.

Now we present some useful properties of solutions of (2) (see Corollary 6.4 in [10]).

Lemma 2. *Eq.* (2) *possesses the following couple of solutions*

$$v(t) > 0\,, \quad v'(t) \le 0 \quad \text{and} \quad v''(t) \ge 0 \tag{9}$$

and

$$v(t) > 0\,, \quad v'(t) > 0 \quad \text{and} \quad v''(t) \ge 0\,, \tag{10}$$

for all t large enough.

We say that $v(t)$ is the solution of degree 0 of (2) when it satisfies (9), on the other hand $v(t)$ is said to be the solution of degree 2 of (2) if it satisfies (10).

The following lemma permits to obtain a solution of degree 0 of (2) if the corresponding solution of degree 2 is known.

Lemma 3. *If $v_2(t)$ is a solution of (2) of degree 2 then*

$$v_1(t) = v_2(t) \int_t^\infty v_2^{-2}(s)\, ds$$

is also solution of (2) and $v_1(t)$ is of degree 0.

Proof. It is easy to see that since $v_2(t)$ is of degree 2 then $\int^\infty v_2^{-2}(s)\, ds < \infty$ so $v_1(t)$ is well defined. Simple computation shows

$$v_1''(t) = v_2''(t) \int_t^\infty v_2^{-2}(s)\, ds = p(t)\, r(t)\, v_2(t) \int_t^\infty v_2^{-2}(s)\, ds = p(t)\, r(t)\, v_1(t)$$

and so $v_1(t)$ is another solution of (2). On the other hand,

$$v_1'(t) = v_2'(t) \int_t^\infty v_2^{-2}(s)\, ds - \frac{1}{v_2(t)}\,.$$

Since $v_2(\infty) = \infty$, we have

$$\frac{1}{v_2(t)} = \int_t^\infty v_2'(s) v_2^{-2}(s)\, ds \geq v_2'(t) \int_t^\infty v_2^{-2}(s)\, ds\,.$$

Thus we conclude that $v_1'(t) < 0$. So $v_1(t)$ is of degree 0. □

Lemma 4. *If $v(t)$ is a solution of degree 0 of (2) then $\int^\infty v^{-2}(t)\, dt = \infty$.*

Proof. It is easy to see that $v(t)$ satisfies $c > v(t)$, eventually which implies assertion of the lemma. □

A solution $v_1(t)$ of degree 0 is the key solution of (2) because if it satisfies

$$\int^\infty v_1(t)\, r(t)\, dt = \infty \tag{11}$$

then Eq. (1) can be represented in the *canonical* form of (8).

Remark 1. It is useful to notice that if $v(t)$ is a solution of degree 0 of (2) then condition (11) implies

$$\int^\infty r(s)\, ds = \infty\,. \tag{11_1}$$

We present sufficient condition for every solution of degree 0 of (2) to satisfy (11). Let us denote $\widetilde{P}_r(t) = \int_t^\infty p(s)\, r(s)\, ds$ (we suppose that $\int^\infty p(s)\, r(s)\, ds < \infty$).

Lemma 5. *Assume that*

$$\int_{t_0}^\infty r(t)\, \mathrm{e}^{-\int_{t_0}^t \widetilde{P}_r(s)\, ds}\, dt = \infty\,. \tag{12}$$

Then every solution of degree 0 of (2) satisfies (11).

Proof. Let $v(t)$ satisfy (9). Integrating (2) from t to ∞, one gets

$$v'(\infty) - v'(t) = \int_t^\infty p(s)\, r(s)\, v(s)\, ds\,,$$

where $v'(\infty) = \lim_{t\to\infty} v'(t)$. We claim that $\lim_{t\to\infty} v'(t) = 0$. If not then $\lim_{t\to\infty} v'(t) = \ell$, $\ell < 0$. Then $v'(t) \le \ell$. Integrating from t_1 to t, we have $v(t) \le v(t_1) + \ell(t - t_1) \to -\infty$ as $t \to \infty$. This is a contradiction. Thus we conclude that

$$-v'(t) = \int_t^\infty p(s)\, r(s)\, v(s)\, ds \le v(t) \int_t^\infty p(s)\, r(s)\, ds = v(t)\, \widetilde{P}_r(t)\,.$$

Then integrating from t_1 to t, we have

$$v(t) \ge v(t_1)\, \mathrm{e}^{-\int_{t_1}^t \widetilde{P}_r(s)\, ds}\,.$$

Multiplying by $r(t)$ and integrating from t to ∞ we get in view of (12) the desired property. ☐

To simplify our notation, we set

$$L_0 y = y\,, \quad L_1 y = \frac{1}{vr}(L_0 y)'\,, \quad L_2 y = v^2 (L_1 y)'\,, \quad L_3 y = (L_2 y)'\,.$$

We present structure of nonoscillatory solutions $y(t)$ of (8) provided that (8) is in canonical form (see e.g. [8] or [13]).

Lemma 6. *Let* (8) *be in canonical form then every positive solution of* (8) *satisfies either*

$$L_0 y(t) > 0\,, \quad L_1 y(t) < 0\,, \quad L_2 y(t) > 0\,, \quad L_3 y(t) < 0\,, \tag{13}$$

or

$$L_0 y(t) > 0\,, \quad L_1 y(t) > 0\,, \quad L_2 y(t) > 0\,, \quad L_3 y(t) < 0 \tag{14}$$

for large t.

Following Kiguradze we say that Eq. (8) has property (A) if every positive solution $y(t)$ of (8) satisfies (13).

In the sequel we shall assume that $v(t)$ a solution of degree 0 of (2) satisfies (11). Then (8) is a canonical representation of equation (1). Therefore any positive nonosillatory solution $y(t)$ of (1) is also a solution of (8) and it satisfies either

$$L_0 y > 0\,, \quad L_1 y > 0$$

or

$$L_0 y > 0\,, \quad L_2 y < 0\,.$$

It means that either

$$y > 0\,, \quad y' > 0\,,$$

or

$$y > 0\,, \quad y' < 0\,.$$

So (1) has the same structure of nonoscillatory solutions like (3).

Lemma 7. *If $v(t)$ is solution of degree* 0 *of* (2) *satisfying* (11). *Then a positive solution of* (8) *satisfies* (13) *if and only if it satisfies* (7).

Proof. $\Rightarrow$: Assume that $y(t)$ satisfies (13). Therefore $L_0 y = y > 0$, $L_1 y = \frac{1}{rv}\, y' < 0$, eventually. From this $D_0 y > 0$ and $D_1 y = \frac{1}{r}\, y' < 0$ and $y' < 0$. It follows from (1) that $D_3 y = \left(\frac{1}{r} y'\right)'' < 0$. So $D_2 y$ is of constant sign. If we admit $D_2 y < 0$ then $D_1 y = \frac{1}{r} y'$ is decreasing so $\frac{1}{r}\, y'(t) \leqslant \frac{1}{r(t_1)}\, y'(t_1) = -c_1 < 0$. Integrating from t_1 to t and taking Remark 1 into account, we see $y(t) < y(t_1) - c_1 \int_{t_1}^{t} r(s)\, ds \to -\infty$ as $t \to \infty$. A contradiction with the positivity of $y(t)$. So we conclude $D_2 y > 0$ and y satisfies (7).

$\Leftarrow$: Now, $D_0 y > 0$ and $D_1 y < 0$ implies $L_0 y > 0$ and $L_1 y < 0$. It follows for (8) that $L_3 y < 0$. Then $L_2 y(t)$ is decreasing. If we admit $L_2 y(t) < 0$ for $t \geq t_1$, then $L_1 y(t) \leqslant -\ell < 0$ and integrating from t_1 to t one gets $y(t) < y(t_2) - \ell \int_{t_1}^{t} r(s)\, v(s)\, ds \to -\infty$ as $t \to \infty$. $\square$

Lemma 7 can be now reformulated as:

Theorem 1. *If $v(t)$ is solution of degree* 0 *of* (2) *satisfying* (11). *Then Eq.* (1) *has property* (P_0) *if and only if* (8) *has property* (A).

Now we are prepared to provide main results.

Theorem 2. *Let $v(t)$ be a positive solution of degree* 0 *of* (2) *satisfying* (11). *If the differential inequality*

$$\left(v^2 z'(t)\right)' + \left[v\big(\tau(t)\big)\, r\big(\tau(t)\big)\, \tau'(t) \int_t^\infty v(s)\, g(s)\, ds\right] z\big(\tau(t)\big) \leqslant 0 \tag{15}$$

has no positive solution then (1) *has property* (P_0).

Proof. Assume that (1) has not property (P_0). It follows from Theorem 1 that (8) has not property (A). Therefore a solution $y(t)$ exists that satisfies (14).

Integrating (8) from t to ∞, we have

$$L_2 y(t) = c + \int_t^\infty v(s)\, g(s)\, y\big(\tau(s)\big)\, ds \tag{16}$$

where $c = \lim\limits_{t\to\infty} L_2 y(t)$. Since

$$y(t) = y(t_1) + \int_{t_1}^{t} v(x)\, r(x)\, L_1 y(x)\, dx\,,$$

then it follows from (16) that

$$L_2 y(t) \geqslant \int_t^\infty v(s)\, g(s) \Big(\int_{t_1}^{\tau(s)} v(x)\, r(x)\, L_1 y(x)\, dx\Big)\, ds\,.$$

We may assume that $\tau(t) > t_1$, so

$$L_2 y(t) \geqslant \int_t^\infty v(s)\, g(s) \int_{\tau(t)}^{\tau(s)} v(x)\, r(x) L_1 y(x) dx\, ds\,.$$

Changing the order of integration leads to

$$L_2 y(t) \geqslant \int_{\tau(t)}^{\infty} v(x)\, r(x)\, L_1 y(x) \int_{\tau^{-1}(x)}^{\infty} v(s)\, g(s) ds\, dx = \int_{\tau(t)}^{\infty} L_1 y(x)\, G(x)\, dx\,,$$

where

$$G(x) = v(x)\, r(x) \int_{\tau^{-1}(x)}^{\infty} v(s)\, g(s)\, ds\,,$$

and $\tau^{-1}(t)$ is the inverse function to $\tau(t)$. Integrating from t_1 to t, we get

$$L_1 y(t) \geqslant L_1 y(t_1) + \int_{t_1}^{t} \frac{1}{v^2(s)} \int_{\tau(s)}^{\infty} L_1 y(x)\, G(x)\, dx\, ds\,. \tag{17}$$

Let us denote the right hand side of (17) by $z(t)$. Then $z(t) > 0$ and

$$0 = \big(v^2 z'(t)\big)' + L_1 y\big(\tau(t)\big)\, G\big(\tau(t)\big)\, \tau'(t) \geqslant \big(v^2 z'(t)\big)' + G\big(\tau(t)\big)\, \tau'(t)\, z\big(\tau(t)\big)$$

so $z(t)$ is a positive solution of (15). A contradiction. □

Using the fact that differential inequality (15) has no positive solution if and only if the corresponding differential equation is oscillatory (see [8] or [13]) the previous theorem can be reformulated as:

Corollary 2. *Let $v(t)$ be a positive solution of degree 0 of (2) satisfying (11). If the second order differential equation*

$$\big(v^2 z'(t)\big)' + \Big[v\big(\tau(t)\big)\, r\big(\tau(t)\big)\, \tau'(t) \int_{t}^{\infty} v(s)\, g(s)\, ds\Big]\, z\big(\tau(t)\big) = 0 \tag{18}$$

is oscillatory then Eq. (1) has property (P_0).

Theorem 3. *Let $v(t)$ be a positive solution of degree 0 of (2) satisfying (11). If*

$$\liminf_{t\to\infty} \Big(\int_{t_1}^{\tau(t)} \frac{1}{v^2(s)}\, ds\Big) \times \Big(\int_{t}^{\infty} v\big(\tau(x)\big)\, r\big(\tau(x)\big)\tau'(x) \int_{x}^{\infty} v(s)\, g(s)\, ds\, dx\Big) > \frac{1}{4} \tag{19}$$

then Eq. (1) has property (P_0).

Proof. Condition (19) guaranties oscillation of (18) (see e.g. [8]). □

Example 1. Let us consider delay differential equation

$$\Big(\frac{1}{t}\, y'(t)\Big)'' - \frac{2}{t^3}\, y'(t) + g(t)\, y(\alpha t) = 0\,, \quad 0 < \alpha < 1\,. \tag{E$_1$}$$

Now $v(t) = \frac{1}{t}$ is a solution of degree 0 of equation $v''(t) = \frac{2}{t^2}\, v(t)$. It follows from Theorem 3 that (E_1) satisfies (P_0) if

$$\liminf_{t\to\infty} \alpha^4 t^3 \int_{t}^{\infty} g(s)\Big(1 - \frac{t}{s}\Big)\, ds > \frac{3}{4}\,. \tag{E$_2$}$$

For the partial case of (E_1), namely for

$$\Big(\frac{1}{t}\, y'(t)\Big)'' - \frac{2}{t^3}\, y'(t) + \frac{b}{t^4}\, y(\alpha t) = 0\,, \quad b > 0 \tag{E$_3$}$$

condition (E_2) reduces to $b\alpha^4 > 9$.

Employing additional condition presented in the next theorem our results here concerning property (P_0) can be formulated in stronger form

Theorem 4. *Let all assumptions of Theorem 3 hold. If moreover*

$$\int_{t_0}^{\infty} v(s_3)\, r(s_3) \int_{s_3}^{\infty} v^{-2}(s_2) \int_{s_2}^{\infty} v(s_1)\, g(s_1)\, ds_1\, ds_2\, ds_3 = \infty \tag{20}$$

then every nonoscillatory solution of (1) *tends to zero as* $t \to \infty$.

Proof. Let $y(t) > 0$ be a solution of (1). Theorem 3 insures that $y(t)$ satisfies (7), so $y'(t) < 0$ and there exists $\lim\limits_{t\to\infty} y(t) = \ell \geq 0$. If we admit $\ell > 0$ then $y\big(\tau(t)\big) > \ell$. Integrating (8) twice from t to ∞ and then from t_1 to t, we have

$$\begin{aligned} y(t) = y(t_1) - &\int_{t_1}^{t} v(s_3) r(s_3) \int_{s_3}^{\infty} v^{-2}(s_2) \int_{s_2}^{\infty} v(s_1)\, g(s_1)\, y\big(\tau(s_1)\big)\, ds_1\, ds_2\, ds_3 \\ &\leq y(t_1) - \ell \int_{t_1}^{t} v(s_3)\, r(s_3) \int_{s_3}^{\infty} v^{-2}(s_2) \int_{s_2}^{\infty} v(s_1)\, g(s_1)\, ds_1\, ds_2\, ds_3 \to -\infty \end{aligned}$$

as $t \to \infty$. Consequently, $\lim\limits_{t\to\infty} y(t) = 0$. $\square$

Remark. If nonoscillatory solution $y(t)$ of (1) tends to zero as $t \to \infty$ then moreover $\lim\limits_{t\to\infty} D_i\, y(t) = 0$ for $i = 0, 1, 2$.

Example 2. We consider once more differential equation (E_3). Since (20) holds for (E_2) then Theorem 4 implies that every nonoscillatory solution of (E_2) tends to zero as $t \to \infty$ provided that $b\alpha^4 > 9$. One such nonoscillatory solution is $y(t) = t^{\beta}$, where β is a negative solution of $\beta(\beta - 1)(\beta - 4) + b\alpha^{\beta} = 0$.

Employing additional conditions to coefficients of (2), we can essentially simplify our results. At first we say that $\widetilde{v}(t)$ is an asymptotic expression of a function $v(t)$ as $t \to \infty$ if $\lim\limits_{t\to\infty} \frac{v(t)}{\widetilde{v}(t)} = 1$. We will denote this fact by $v \sim \widetilde{v}$. It is useful to observe that for any $\lambda \in (0, 1)$, we have

$$\lambda \widetilde{v}(t) \leq v(t) \leq \frac{1}{\lambda}\, \widetilde{v}(t)\,, \tag{21}$$

eventually.

Lemma 8. *Assume that*

$$\int^{\infty} s\, r(s)\, p(s)\, ds < \infty\,. \tag{22}$$

Then there exists a solution $v(t)$ *of degree* 0 *of* (2) *such that* $v(t) \sim 1$.

Proof. Let $t_1 \geq t_0$ is such that $\int_{t_1}^{\infty} s\, r(s)\, p(s)\, ds < 1$. In the proof of Lemma 5 we have shown that if $v_1(t)$ is a solution of degree 0 of (2) then $\lim\limits_{t\to\infty} v_1'(t) = 0$.

Moreover $\lim_{t\to\infty} v_1(t) = \ell \geq 0$. At first we need to show that $\ell > 0$. We assume the contrary. Then integrating (2) twice from $t \geq t_1$ to ∞, one gets

$$\begin{aligned} v_1(t) &= \int_t^\infty \int_x^\infty p(s)\, r(s)\, v_1(s)\, ds\, dx = \int_t^\infty (s-t)\, p(s)\, r(s)\, v_1(s)\, ds \\ &\leq \int_t^\infty s\, p(s)\, r(s)\, v_1(s)\, ds \leq v_1(t) \int_t^\infty s\, r(s)\, p(s)\, ds < v_1(t)\,. \end{aligned}$$

A contradiction. Then $v_1 \sim \ell \neq 0$ and so $v = \frac{1}{\ell}\, v_1$ is required solution. $\square$

Remark. Assertion of the Lemma 8 can be found also in [4].

Roughly speaking, Lemma 8 says that if $p(t)$ is "*small enough*" then we have an estimate of required solution of (2).

Theorem 5. *Assume that* $\int^\infty sr(s)p(s)\, ds < \infty$ *and* $\int^\infty r(s)\, ds = \infty$. *If*

$$\liminf_{t\to\infty} \tau(t) \Big(\int_t^\infty r\big(\tau(x)\big)\, \tau'(x) \int_x^\infty g(s)\, ds\, dx \Big) > \frac{1}{4} \tag{23}$$

then (1) *has property* (P_0). *If moreover*

$$\int_{t_0}^\infty r(s_3) \int_{s_3}^\infty \int_{s_2}^\infty g(s_1)\, ds_1\, ds_2\, ds_3 = \infty \tag{24}$$

then every nonoscillatory solution of (1) *tends to zero as* $t \to \infty$.

Proof. It is clear from (23) that there exists $\lambda \in (0,1)$ such that

$$\liminf_{t\to\infty} \lambda^2 \tau(t) \Big(\int_t^\infty \lambda r\big(\tau(x)\big)\, \tau'(x) \int_x^\infty \lambda g(s)\, ds\, dx \Big) > \frac{1}{4}\,. \tag{25}$$

Since $v(t) \sim \widetilde{v}(t) \equiv 1$ then (11) reduces to $\int^\infty r(s)\, ds = \infty$ and moreover (21) takes the form

$$\lambda \leq v(t) \leq \frac{1}{\lambda}\,, \tag{26}$$

eventually. Applying (26) to (25), it is easy to see that (19) is satisfied and Theorem 3 guarantees has property (P_0) of (1).

On the other hand, in view of (24), we have

$$\int_{t_0}^\infty \lambda r(s_3) \int_{s_3}^\infty \lambda^2 \int_{s_2}^\infty \lambda g(s_1)\, ds_1\, ds_2\, ds_3 = \infty\,. \tag{27}$$

Applying (26) to (27), we get (20) and the second part of the assertion follows from Theorem 4. $\square$

Theorem 5 implies that if $p(t)$ is "*small enough*" then we can replace required solution of (2) by constant 1.

Example 3. Let us consider delay differential equation

$$\Big(\frac{1}{t}\, y'(t)\Big)'' - \frac{c}{t^4}\, y'(t) + \frac{b}{t^4}\, y(\alpha t) = 0\,, \quad c > 0\,, \quad b > 0\,, \quad 0 < \alpha < 1\,. \tag{E$_4$}$$

Theorem 5 guarantees property (P_0) of (E_4) provided that $b\alpha^3 > \frac{3}{4}$. Moreover, (24) is satisfied and so every nonoscillatory solution of (E_4) tends to zero as $t \to \infty$.

Remark. Theorem 3 permits to deduce property (P_0) of (1) from the absence of positive solutions of the corresponding second order differential equation. This comparison theorem generalizes earlier ones of Džurina [8], Parhi and Padhi [18].

Theorems 3, 4 and 5 present easily verifiable criteria for desired property of (1) and are applicable for a wide class of equations. Propositions of Theorem 5 includes only coefficients $\tau(t)$, $p(t)$ and $g(t)$, no solving of (2) is needed and results of this type are not known for property (P_0) of (1).

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Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University of Košice Letná 9, 042 00 Košice
E-mail: `jozef.dzurina@tuke.sk` `renata.kotorova@tuke.sk`