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# DERIVATIONS OF THE SUBALGEBRAS INTERMEDIATE THE GENERAL LINEAR LIE ALGEBRA AND THE DIAGONAL SUBALGEBRA OVER COMMUTATIVE RINGS

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Abstract. Let $R$ be an arbitrary commutative ring with identity, $\mathrm{gl}(n,R)$ the general linear Lie algebra over $R$, $d(n,R)$ the diagonal subalgebra of $\mathrm{gl}(n,R)$. In case 2 is a unit of $R$, all subalgebras of $\mathrm{gl}(n,R)$ containing $d(n,R)$ are determined and their derivations are given. In case 2 is not a unit partial results are given.

## 1. Introduction

Let $R$ be a commutative ring with identity, $R^*$ the subset of $R$ consisting of all invertible elements in $R$, $I(R)$ the set consisting of all ideals of $R$. Let $\mathrm{gl}(n,R)$ be the general linear Lie algebra consisting of all $n\times n$ matrices over $R$ and with the bracket operation: $[x,y]=xy-yx$. We denote by $d(n,R)$ (resp., $t(n,R)$) the subset of $\mathrm{gl}(n,R)$ consisting of all $n\times n$ diagonal (resp., upper triangular) matrices over $R$. Let $E$ be the identity matrix in $\mathrm{gl}(n,R)$, $RE$ the set $\{rE \mid r\in R\}$ consisting of all scalar matrices, and $E_{i,j}$ the matrix in $\mathrm{gl}(n,R)$ whose sole nonzero entry 1 is in the $(i,j)$ position. For $A\in \mathrm{gl}(n,R)$, we denote by $A'$ the transpose of $A$.

For $R$-modules $M$ and $K$, we denote by $\mathrm{Hom}_R(M,K)$ the set of all homomorphisms of $R$-modules from $M$ to $K$. $\mathrm{Hom}_R(M,M)$ is abbreviated to $\mathrm{Hom}_R(M)$. For $1\le i\le n$, $\chi_i\colon d(n,R)\to R$, defined by $\chi_i(\mathrm{diag}(d_1,d_2,\dots,d_n))=d_i$, is a standard homomorphism from $d(n,R)$ to $R$.

Recently, significant work has been done in studying automorphisms and derivations of matrix Lie algebras (or sometimes matrix algebras) and their subalgebras (see [1]–[7]). Derivations of the parabolic subalgebras of $\mathrm{gl}(n,R)$ were described in [7]. Derivations of the subalgebras of $t(n,R)$ containing $d(n,R)$ were determined in [6]. In this article, when 2 is a unit of $R$, all subalgebras of $\mathrm{gl}(n,R)$ containing $d(n,R)$ are determined and their derivations are given. In case 2 is not a unit partial results are given.

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## 2. The subalgebras of $\mathrm{gl}(n, R)$ containing $d(n, R)$

**Definition 2.1.** Let $\Phi = \{A_{i,j} \in I(R) \mid 1 \le i, j \le n\}$ be a subset of $I(R)$ consisting of $n^2$ ideals of $R$. We call $\Phi$ a *flag* of ideals of $R$, if

(1) $A_{i,i} = R$, $i = 1, 2, \ldots, n$.

(2) $A_{i,k}A_{k,j} \subseteq A_{i,j}$ for any $i$, $j$, $k$ $(1 \le i, j, k \le n)$.

**Example 2.2.** If $i \ne j$, let $A_{i,j}$ be 0, and let $A_{i,i} = R$ for $i = 1, 2, \ldots, n$. Then $\Phi = \{A_{i,j} \mid 1 \le i, j \le n\}$ is a flag of ideals of $R$.

**Example 2.3.** If all $A_{i,j}$ are taken to be $R$, then $\Phi = \{A_{i,j} \mid 1 \le i, j \le n\}$ is a flag of ideals of $R$.

**Theorem 2.4.** *If $\Phi = \{A_{i,j} \mid 1 \le i, j \le n\}$ is a flag of ideals of $R$, then $L_\Phi = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}E_{i,j}$ is a subalgebra of $\mathrm{gl}(n, R)$ containing $d(n, R)$.*

**Proof.** Suppose that $\Phi = \{A_{i,j} \mid 1 \le i, j \le n\}$ is a flag of ideals of $R$ and $L_\Phi = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}E_{i,j}$. Let

$$x = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}E_{i,j} \in L_\Phi\,, \qquad y = \sum_{i=1}^n \sum_{j=1}^n b_{i,j}E_{i,j} \in L_\Phi\,,$$

where $a_{i,j}, b_{i,j} \in A_{i,j}$. It is obvious that $rx + sy \in L_\Phi$ for any $r, s \in R$. Notice that

$$[x, y] = \sum_{i=1}^n \sum_{j=1}^n c_{i,j}E_{i,j}\,, \quad \text{where} \quad c_{i,j} = \sum_{k=1}^n (a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\,.$$

By assumption (2) on $\Phi$, we know that $(a_{i,k}b_{k,j} - b_{i,k}a_{k,j}) \in A_{i,j}$, forcing $c_{i,j} \in A_{i,j}$ and $[x, y] \in L_\Phi$. Hence $L_\Phi$ is a subalgebra of $\mathrm{gl}(n, R)$. Assumption (1) on $\Phi$ shows that $L_\Phi$ contains $d(n, R)$. $\square$

The following result shows that these $L_\Phi$ nearly exhaust all subalgebras of $\mathrm{gl}(n, R)$ containing $d(n, R)$.

**Theorem 2.5.** *If $L$ is a subalgebra of $\mathrm{gl}(n, R)$ containing $d(n, R)$, then there exists a flag $\Phi = \{A_{i,j} \mid 1 \le i, j \le n\}$ of ideals of $R$ such that*

$$2L \subseteq L_\Phi \subseteq L\,.$$

**Proof.** Let $L$ be a subalgebra of $\mathrm{gl}(n, R)$ containing $d(n, R)$. For $\forall i, j$ $(1 \le i, j \le n)$, define

$$A_{i,j} = \{a_{i,j} \in R \mid a_{i,j}E_{i,j} \in L\}\,,$$

and set

$$\Phi = \{A_{i,j} \mid 1 \le i, j \le n\}\,,$$

$$L_\Phi = \sum_{i=1}^n \sum_{j=1}^n A_{i,j}E_{i,j}\,.$$

In the following, we will prove that $\Phi$ is a flag of ideals of $R$, and $2L \subseteq L_\Phi \subseteq L$. It's obvious that all $A_{i,j}$ are ideals of $R$ and $A_{i,i} = R$ for $i = 1, 2, \cdots, n$. If $i \ne j$

and $a_{i,k} \in A_{i,k}$, $a_{k,j} \in A_{k,j}$, then by $[a_{i,k}E_{i,k}, a_{k,j}E_{k,j}] = a_{i,k}a_{k,j}E_{i,j} \in L$, we see that $a_{i,k}a_{k,j} \in A_{i,j}$, forcing $A_{i,k}A_{k,j} \subseteq A_{i,j}$. If $i = j$, since $A_{i,i} = R$, we also have that $A_{i,k}A_{k,j} \subseteq A_{i,j}$. Thus $\Phi$ is a flag of ideals of $R$. It is easy to see that $L_\Phi \subseteq L$. On the other hand, for $x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}E_{i,j} \in L$, if $k \neq l$, then by

$$\big[E_{k,k}, [E_{l,l}, -x]\big] = a_{k,l}E_{k,l} + a_{l,k}E_{l,k} \in L\,,$$

$$[E_{k,k}, a_{k,l}E_{k,l} + a_{l,k}E_{l,k}] = a_{k,l}E_{k,l} - a_{l,k}E_{l,k} \in L\,,$$

we see that $2a_{k,l}E_{k,l} \in L$, $2a_{l,k}E_{l,k} \in L$. This shows that $2a_{k,l} \in A_{k,l}$, $2a_{l,k} \in A_{l,k}$, forcing $2x \in L_\Phi$. So $2L \subseteq L_\Phi$. □

**Corollary 2.6.** *Assume that $2 \in R^*$, then $L$ is a subalgebra of $\mathrm{gl}(n,R)$ containing $d(n,R)$ if and only if there exists a flag $\Phi = \{A_{i,j} \mid 1 \leq i, j \leq n\}$ of ideals of $R$ such that $L = L_\Phi$.*

**Remark 2.7.** Without the assumption $2 \in R^*$, Corollary 2.6 does not hold. The following is an example. Let $R$ be $Z/2Z$ ($Z$ is the ring of all integer numbers), then $R$ has only two ideals: 0 and $R$. Set $L = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \mid a, b, c \in Z/2Z \right\}$. Then $L$ is a subalgebra of $\mathrm{gl}(2, Z/2Z)$ containing $d(2, Z/2Z)$, but $L \neq L_\Phi$ for any flag $\Phi = \{A_{i,j} \mid 1 \leq i, j \leq 2\}$ of ideals of $R$.

## 3. Construction of certain derivations of $L_\Phi$

Let $L_\Phi = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}E_{i,j}$ be a fixed subalgebra of $\mathrm{gl}(n,R)$ containing $d(n,R)$, with $\Phi = \{A_{i,j} \in I(R) \mid 1 \leq i, j \leq n\}$ a flag of ideals of $R$. We denote by Der $L_\Phi$ the set consisting of all derivations of $L_\Phi$. We now construct certain derivations of $L_\Phi$ for building the derivation algebra *Der* $L_\Phi$ of $L_\Phi$. For $A_{i,j} \in \Phi$, let $B_{i,j}$ denote the annihilator of $A_{i,j}$ in $R$, i.e., $B_{i,j} = \{r \in R \mid rA_{ij} = 0\}$.

### (A) **Inner derivations**

Let $x \in L_\Phi$, then $\mathrm{ad}\ x\colon L_\Phi \to L_\Phi$, $y \mapsto [x,y]$, is a derivation of $L_\Phi$, called the *inner derivation* of $L_\Phi$ induced by $x$. Let $\mathrm{ad}\ L_\Phi$ denote the set consisting of all $\mathrm{ad}\ x$, $x \in L_\Phi$, which forms an ideal of Der $L_\Phi$.

### (B) **Transpose derivations**

**Definition 3.3.** Let $\Pi = \{\pi_{i,j} \in \mathrm{Hom}_R(A_{i,j}, A_{j,i}) \mid 1 \leq i, j \leq n\}$ be a set consisting of $n^2$ homomorphisms of $R$-modules. We call $\Pi$ *suitable for transpose derivations*, if the following conditions are satisfied for all $i, j$ $(1 \leq i, j \leq n)$:
(1) $\pi_{i,i} = 0$;
(2) $\pi_{i,j}(A_{i,k}A_{k,j}) = 0$ for all $k$ which satisfies $k \neq i$ and $k \neq j$;
(3) $\pi_{i,j}(A_{i,j}) \subseteq B_{k,j}$ and $\pi_{i,j}(A_{i,j}) \subseteq B_{i,k}$ for all $k$ which satisfies $k \neq i$ and $k \neq j$;
(4) $2\pi_{i,j}(A_{i,j}) = 0$.

**Remark.** In case 2 is a unit, (4) means that $\pi_{i,j}$ are necessarily zero maps.

Using the homomorphism $\Pi = \{\pi_{i,j} \in \mathrm{Hom}_R(A_{i,j}, A_{j,i}) \mid 1 \leq i, j \leq n\}$ which is suitable for transpose derivations, we define $\phi_\Pi\colon L_\Phi \to L_\Phi$ by sending any $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}E_{i,j} \in L_\Phi$ to $\sum_{i=1}^{n}\sum_{j=1}^{n} \pi_{i,j}(a_{i,j})E_{j,i}$.

**Lemma 3.4.** *The map $\phi_\Pi$ as defined above, is a derivation of $L_\Phi$.*

**Proof.** Let

$$x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}E_{i,j} \in L_\Phi\,, \qquad a_{i,j} \in A_{i,j}\,,$$

$$y = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}E_{i,j} \in L_\Phi\,, \qquad b_{i,j} \in A_{i,j}\,.$$

Obviously, $\phi_\Pi(rx+sy) = r\phi_\Pi(x) + s\phi_\Pi(y)$ for $\forall r, s \in R$. Write

$$[x,y] = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}E_{i,j}\,, \quad \text{where} \quad c_{i,j} = \sum_{k=1}^{n}(a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\,.$$

Because $\Pi$ is suitable for transpose derivations, we have that

$$\begin{aligned}\phi_\Pi\big([x,y]\big) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{i,j}(c_{i,j})E_{j,i} = \sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{i,j}\Big(\sum_{k=1}^{n}(a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\Big)E_{j,i}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\big[(a_{i,i}-a_{j,j})\pi_{i,j}(b_{i,j}) + (b_{j,j}-b_{i,i})\pi_{i,j}(a_{i,j})\big]E_{j,i}\\ &\quad\text{(by assumption (2)).}\end{aligned}$$

On the other hand,

$$\begin{aligned}\big[\phi_\Pi(x), y\big] + \big[x, \phi_\Pi(y)\big] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\Big[\sum_{k=1}^{n}\big(\pi_{k,j}(a_{k,j})b_{k,i} - b_{j,k}\pi_{i,k}(a_{i,k})\\ &\qquad - \pi_{k,j}(b_{k,j})a_{k,i} + a_{j,k}\pi_{i,k}(b_{i,k})\big)\Big]E_{j,i}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\big[(a_{j,j}-a_{i,i})\pi_{i,j}(b_{i,j}) + (b_{i,i}-b_{j,j})\pi_{i,j}(a_{i,j})\big]E_{j,i}\\ &\quad\text{(by assumption (3)).}\end{aligned}$$

By assumption (4) on $\Pi$, we see that $\phi_\Pi\big([x,y]\big) = \big[\phi_\Pi(x), y\big] + \big[x, \phi_\Pi(y)\big]$. Hence $\phi_\Pi$ is a derivation of $L_\Phi$. □

$\phi_\Pi$ is called a *transpose derivation* of $L_\Phi$.

## (C) **Ring derivations**

**Definition 3.5.** Let $\Sigma = \{\sigma_{i,j} \in \mathrm{Hom}_R(A_{i,j}),\, \sigma \in \mathrm{Hom}_R\big(d(n,R)\big) \mid 1 \le i,\, j \le n\}$ be a set consisting of $n^2+1$ endomorphisms of $R$-modules. We call $\Sigma$ *suitable for ring derivations* if the following conditions are satisfied for $\forall\, i, j$ $(1 \le i, j \le n)$:

(1) $\chi_i\big(\sigma(D)\big) - \chi_j\big(\sigma(D)\big) \subseteq (B_{i,j} \cap B_{j,i})$ for $\forall\, D \in d(n,R)$;

(2) $\sigma\big(a_{i,j}a_{j,i}(E_{i,i}-E_{j,j})\big) = \big(\sigma_{i,j}(a_{i,j})a_{j,i} + a_{i,j}\sigma_{j,i}(a_{j,i})\big)(E_{i,i}-E_{j,j})$, $\forall\, a_{i,j} \in A_{i,j}, \forall\, a_{j,i} \in A_{j,i}$;

(3) $\sigma_{i,i} = 0$, $i = 1, 2, \ldots n$

(4) When $i \neq j$, $\sigma_{i,j}(a_{i,k}a_{k,j}) = \sigma_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(a_{k,j})$ for $\forall\ k$ $(1 \leq k \leq n)$, $\forall\ a_{i,k} \in A_{i,k}$ and $\forall\ a_{k,j} \in A_{k,j}$.

Using $\Sigma = \{\sigma_{i,j} \in \mathrm{Hom}_R(A_{i,j}),\ \sigma \in \mathrm{Hom}_R(d(n,R)) \mid 1 \leq i,\ j \leq n\}$ which is suitable for ring derivations, we define $\phi_\Sigma \colon L_\Phi \to L_\Phi$ by sending any $\sum_{i=1}^n \sum_{j=1}^n a_{i,j} E_{i,j} \in L_\Phi$ to $\sum_{1\leq i\neq j\leq n} \sigma_{i,j}(a_{i,j})E_{i,j} + \sigma\big(\sum_{k=1}^n a_{k,k}E_{k,k}\big)$.

**Lemma 3.6.** *The map $\phi_\Sigma$, as defined above, is a derivation of $L_\Phi$.*

**Proof.** Let $x = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}E_{i,j} \in L_\Phi$, $y = \sum_{i=1}^n \sum_{j=1}^n b_{i,j}E_{i,j} \in L_\Phi$, where $a_{i,j}, b_{i,j}$ lie in $A_{i,j}$. It is obvious that $\phi_\Sigma(rx+sy) = r\phi_\Sigma(x) + s\phi_\Sigma(y)$ for any $r, s \in R$. We know $[x,y] = \sum_{i=1}^n \sum_{j=1}^n c_{i,j}E_{i,j}$, where $c_{i,j} = \sum_{k=1}^n (a_{i,k}b_{k,j} - b_{i,k}a_{k,j})$. Because $\Sigma$ is suitable for ring derivations, we have that

$$\begin{aligned}
\phi_\Sigma\big([x,y]\big) &= \sum_{1\leq i\neq j\leq n} \Big[\sum_{k=1}^n \big(\sigma_{i,j}(a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\big)\Big] E_{i,j} \\
&\quad + \sigma\Big[\sum_{i=1}^n \sum_{k=1}^n (a_{i,k}b_{k,i} - b_{i,k}a_{k,i})E_{i,i}\Big] \\
&= \sum_{1\leq i\neq j\leq n} \Big[\sum_{k=1}^n \big(\sigma_{i,j}(a_{i,k}b_{k,j} - b_{i,k}a_{k,j})\big)\Big] E_{i,j} \\
&\quad + \sigma\Big(\sum_{i=1}^n \sum_{k=1}^n a_{i,k}b_{k,i}(E_{i,i} - E_{k,k})\Big) \\
\text{(note that } &\sum_{i=1}^n \sum_{k=1}^n (a_{i,k}b_{k,i} - b_{i,k}a_{k,i})E_{i,i} = \sum_{i=1}^n \sum_{k=1}^n a_{i,k}b_{k,i}(E_{i,i} - E_{k,k})) \\
&= \sum_{1\leq i\neq j\leq n} \Big[\sum_{k=1}^n \big(\sigma_{i,k}(a_{i,k})b_{k,j} + a_{i,k}\sigma_{k,j}(b_{k,j}) \\
&\quad - \sigma_{i,k}(b_{i,k})a_{k,j} - b_{i,k}\sigma_{k,j}(a_{k,j})\big)\Big] E_{i,j} \\
&\quad + \sum_{i=1}^n \sum_{k=1}^n \big[\sigma_{i,k}(a_{i,k})b_{k,i} + a_{i,k}\sigma_{k,i}(b_{k,i})\big](E_{i,i} - E_{k,k}), \\
&\quad \text{(by assumption (2) and (4)).}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\big[\phi_\Sigma(x), y\big] + \big[x, \phi_\Sigma(y)\big] &= \Big[\sum_{1\leq i\neq j\leq n} \sigma_{i,j}(a_{i,j})E_{i,j} + \sigma\Big(\sum_{i=1}^n a_{i,i}E_{i,i}\Big), y\Big] \\
&\quad + \Big[x, \sum_{1\leq i\neq j\leq n} \sigma_{i,j}(b_{i,j})E_{i,j} + \sigma\Big(\sum_{i=1}^n b_{i,i}E_{i,i}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\sum_{1\le i\ne j\le n} \sigma_{i,j}(a_{i,j})E_{i,j}, y\Big] + \Big[x, \sum_{1\le i\ne j\le n} \sigma_{i,j}(b_{i,j})E_{i,j}\Big] \\
&\qquad \text{(by assumption (1))} \\
&= \Big[\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{i,j}(a_{i,j})E_{i,j}, y\Big] + \Big[x, \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{i,j}(b_{i,j})E_{i,j}\Big] \\
&\qquad \text{(by assumption (3))} \\
&= \sum_{1\le i\ne j\le n} \Big[\sum_{k=1}^{n} \sigma_{i,k}(a_{i,k})b_{k,j} - b_{i,k}\sigma_{k,j}(a_{k,j}) \\
&\quad - \sigma_{i,k}(b_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(b_{k,j})\Big]E_{i,j} \\
&\quad + \sum_{i=1}^{n} \Big[\sum_{k=1}^{n} \sigma_{i,k}(a_{i,k})b_{k,i} + b_{k,i}\sigma_{i,k}(a_{i,k}) \\
&\quad - \sigma_{i,k}(b_{i,k})a_{k,i} - a_{k,i}\sigma_{i,k}(b_{i,k})\Big]E_{i,i} \\
&= \sum_{1\le i\ne j\le n} \Big[\sum_{k=1}^{n} \sigma_{i,k}(a_{i,k})b_{k,j} - b_{i,k}\sigma_{k,j}(a_{k,j}) \\
&\quad - \sigma_{i,k}(b_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(b_{k,j})\Big]E_{i,j} \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n} \big[\sigma_{i,k}(a_{i,k})b_{k,i} + b_{k,i}\,\sigma_{i,k}(a_{i,k})\big](E_{i,i} - E_{k,k})\,.
\end{aligned}
$$

We see that

$$\big[\phi_\Sigma(x), y\big] + \big[x, \phi_\Sigma(y)\big] = \phi_\Sigma\big([x,y]\big)\,.$$

Hence $\phi_\Sigma$ is a derivation of $L_\Phi$. □

$\phi_\Sigma$ is called a *ring derivation* of $L_\Phi$.

## 4. The derivation algebra of $L_\Phi$

If $n > 1$, for each fixed $k$ $(1 \le k \le n-1)$, we assume that $n = kq + p$ with $q$ and $p$ two non-negative integers and $p \le k-1$. Let $D_k = \operatorname{diag}\big(E_k, 2E_k, \ldots, qE_k, (q+1)E_p\big) \in d(n,R)$, $k = 1, 2, \ldots, n-1$ (where $E_k$ denotes the $k \times k$ identity matrix). Let $\Phi = \big\{A_{i,j} \in I(R) \mid 1 \le i < j \le n\big\}$ be a flag of ideals of $R$, we denote $\sum_{1\le i\ne j\le n} A_{i,j}E_{i,j}$ by $w$.

**Theorem 4.1.** *Let $R$ be an arbitrary commutative ring with identity, $n \ge 1$,*

$$L_\Phi = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}E_{i,j}$$

*a subalgebra of $\mathrm{gl}(n,R)$ containing $d(n,R)$ with $\Phi = \{A_{i,j} \in I(R) \mid 1 \le i < j \le n\}$ a flag of ideals of $R$. Then every derivation of $L_\Phi$ may be uniquely written as the* *sum of an inner derivation induced by an element in $w$, a transpose derivation and a ring derivation.*

**Proof.** If $n = 1$, then it's easy to determine $\operatorname{Der} L_\Phi$. From now on, we assume that $n > 1$. Let $\phi$ be a derivation of $L_\Phi$. In the following we give the proof by steps.

**Step 1:** There exists $W_0 \in w$ such that $d(n, R)$ is stable under $\phi + \text{ad } W_0$.

For $k = 1, 2, \ldots, n$, we set $v_k = \sum_{i=k}^{n} \sum_{j=1}^{i-k+1} A_{i,j} E_{i,j}$. Denote $L_\Phi \cap t(n, R)$ by $t$. For any $H \in d(n, R)$, suppose that

$$\phi(H) \equiv (\sum_{1 \le i < j \le n} a_{j,i}(H) E_{j,i}) (\mod t),$$

where $a_{j,i}(H) \in A_{j,i}$ are relative to $H$. By $[D_1, H] = 0$, we have that

$$\big[H, \phi(D_1)\big] = \big[D_1, \phi(H)\big],$$

which follows that

$$\sum_{1 \le i < j \le n} \big(\chi_j(H) - \chi_i(H)\big) a_{j,i}(D_1) E_{j,i} = \sum_{1 \le i < j \le n} \big(\chi_j(D_1) - \chi_i(D_1)\big) a_{j,i}(H) E_{j,i}.$$

This yields that

$$\big(\chi_j(H) - \chi_i(H)\big) a_{j,i}(D_1) = \big(\chi_j(D_1) - \chi_i(D_1)\big) a_{j,i}(H), \quad \forall\, i, j (1 \le i < j \le n-1).$$

In particular, we have that

$$a_{i+1,i}(H) = \big(\chi_{i+1}(H) - \chi_i(H)\big) a_{i+1,i}(D_1), \quad i = 1, 2, \ldots, n.$$

Let $X_1 = \sum_{i=1}^{n-1} a_{i+1,i}(D_1) E_{i+1,i} \in L_\Phi$, then $(\phi + \text{ad } X_1)\big(d(n, R)\big) \subseteq t + v_3$. If $n = 2$, this step is completed. If $n > 2$, for any $H \in d(n, R)$, we now suppose that

$$(\phi + \text{ad } X_1)(H) \equiv \Big(\sum_{1 \le i < j \le n-1} b_{j+1,i}(H) E_{j+1,i}\Big) (\mod t),$$

where $b_{j+1,i}(H) \in A_{j+1,i}$ are relative to $H$. By $[D_2, H] = 0$, we have that

$$\big[H, (\phi + \text{ad } X_1)(D_2)\big] = \big[D_2, (\phi + \text{ad } X_1)(H)\big],$$

which follows that

$$\sum_{1 \le i < j \le n-1} (\chi_{j+1}(H) - \chi_i(H)) b_{j+1,i}(D_2) E_{j+1,i} = \sum_{1 \le i < j \le n-1} (\chi_{j+1}(D_2) - \chi_i(D_2)) b_{j+1,i}(H) E_{j+1,i}.$$

This yields that

$$(\chi_{j+1}(H) - \chi_i(H)) b_{j+1,i}(D_2) = (\chi_{j+1}(D_2) - \chi_i(D_2)) b_{j+1,i}(H),$$

for all $i, j (1 \le i < j \le n-1)$. In particular, we have that

$$b_{i+2,i}(H) = \big(\chi_{i+2}(H) - \chi_i(H)\big) b_{i+2,i}(D_2), \quad i = 1, 2, \ldots, n-2.$$

Let $X_2 = \sum_{i=1}^{n-2} b_{i+2,i}(D_2) E_{i+2,i}$, then $(\phi + \text{ad } X_1 + \text{ad } X_2)\big(d(n, R)\big) \subseteq t + v_4$. If $n = 3$, this step is completed. If $n > 3$, we repeat above process. After $n - 2$ steps, we may assume that $\big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\big)\big(d(n,R)\big)\subseteq t+v_n$. For any $H\in d$, suppose that $\big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\big)(H)\equiv c_{n,1}(H)E_{n,1}(\mod t)$, where $c_{n,1}(H)\in A_{n,1}$ is relative to $H$. By $[D_{n-1},H]=0$, we have that

$$\Big[H,\Big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\Big)(D_{n-1})\Big]=\Big[D_{n-1},\Big(\phi+\sum_{i=1}^{n-2}\text{ad } X_i\Big)(H)\Big],$$

which follows that

$$\big(\chi_n(H)-\chi_1(H)\big)c_{n,1}(D_{n-1})=\big(\chi_n(D_{n-1})-\chi_1(D_{n-1})\big)c_{n,1}(H)\,.$$

So we have that

$$c_{n,1}(H)=\big(\chi_n(H)-\chi_1(H)\big)c_{n,1}(D_{n-1})\,.$$

Let $X_{n-1}=c_{n,1}(D_{n-1})E_{n,1}$, then $\big(\phi+\sum_{i+1}^{n-1}\text{ad } X_i\big)\big(d(n,R)\big)\subseteq t$. If we choose $X_0=\sum_{i=1}^{n-1}X_i$, then $(\phi+\text{ad } X_0)\big(d(n,R)\big)\subseteq t$.

Similarly, we may further choose $Y_0\in\sum_{j=1}^{n}\sum_{i=1}^{j-1}A_{i,j}E_{i,j}$ (the process is omitted) such that $(\phi+\text{ad } X_0+\text{ad } Y_0)\big(d(n,R)\big)\subseteq d(n,R)$.

Thus we may choose $W_0=X_0+Y_0\in w$ such that $(\phi+\text{ad } W_0)\big(d(n,R)\big)\subseteq d(n,R)$. Denote $\phi+\text{ad } W_0$ by $\phi_1$, then $\phi_1\big(d(n,R)\big)\subseteq d(n,R)$.

**Step 2:** If $k\neq l$, then $A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$ is stable under $\phi_1$.

For any fixed $b_{k,l}\in A_{k,l}$, we suppose that $\phi_1(b_{k,l}E_{k,l})=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}$, where $a_{i,j}\in A_{i,j}$. By applying $\phi_1$ to $[E_{k,k},b_{k,l}E_{k,l}]=b_{k,l}E_{k,l}$, we have that

$$\big[\phi_1(E_{k,k}),b_{k,l}E_{k,l}]+[E_{k,k},\phi_1(b_{k,l}E_{k,l})\big]=\phi_1(b_{k,l}E_{k,l})\,.$$

This follows that

$$(*)\qquad \big[\phi_1(E_{k,k}),b_{k,l}E_{k,l}\big]+\Big[E_{k,k},\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}\Big]=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}\ldots$$

Note that $\phi_1(E_{k,k})\in d(n,R)$ (by Step 1), thus $\big[\phi_1(E_{k,k}),b_{k,l}E_{k,l}\big]\in A_{k,l}E_{k,l}$. It is easy to see that $\big[E_{k,k},\sum_{i=1}^{n}\sum_{j=1}^{n}a_{i,j}E_{i,j}\big]=\sum_{j=1}^{n}a_{k,j}E_{k,j}-\sum_{i=1}^{n}a_{i,k}E_{i,k}$. By comparing the two sides of $(*)$, we see that $a_{i,j}=0$ when $i\neq k$ and $j\neq k$. For the same reason, we know that $a_{i,j}=0$ when $i\neq l$ and $j\neq l$. Hence $\phi_1(b_{k,l}E_{k,l})\in A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$, which leads to $\phi_1(A_{k,l}E_{k,l})\subseteq A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$. Similarly, $\phi_1(A_{l,k}E_{l,k})\subseteq A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$. So $A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$ is stable under $\phi_1$.

**Step 3:** There exists a ring derivation $\phi_\Sigma$ such that each $A_{k,l}E_{k,l}$ $(k\neq l)$ is send by $\phi_1-\phi_\Sigma$ to $A_{l,k}E_{l,k}$ and $d(n,R)$ is send by it to 0.

We denote the the restriction of $\phi_1$ to $d(n,R)$ by $\sigma$, and let $\sigma_{i,i}\colon A_{i,i}\to A_{i,i}$ be zero. By Step 2, we know that $A_{k,l}E_{k,l}+A_{l,k}E_{l,k}$ is stable under $\phi_1$ if $k\neq l$. Now for any $k,l$ $(1\leq k,l\leq n)$ we define the map $\sigma_{k,l}$ from $A_{k,l}$ to itself according to the following rule:

(a) $\sigma_{k,l}=0$ when $k=l$;

(b) If $k\neq l$, define $\sigma_{k,l}\colon A_{k,l}\to A_{k,l}$ such that for any $a_{k,l}\in A_{k,l}$, $\sigma_{k,l}(a_{k,l})$ satisfies the condition: $\phi_1(a_{k,l}E_{k,l})\equiv\sigma_{k,l}(a_{k,l})E_{k,l}\ (\mod A_{l,k}E_{l,k})$.

Then $\sigma$, $\sigma_{k,l}$ $(k \neq l)$ are all endomorphism of the $R$-modules. Set $\Sigma = \big\{\sigma_{i,j} \in \mathrm{Hom}_R(A_{i,j}), \sigma \mid 1 \leq i,j \leq n\big\}$. We intend to prove that $\Sigma$ is suitable for ring derivations.

For all $D \in d(n,R)$, $a_{i,j} \in A_{i,j}$, by applying $\phi_1$ to $[D, a_{i,j}E_{i,j}] = \big(\chi_i(D) - \chi_j(D)\big)a_{i,j}E_{i,j}$, we have that $a_{i,j}\big(\chi_i(\sigma(D)) - \chi_j(\sigma(D))\big) = 0$, leads to $\chi_i\big(\sigma(D)\big) - \chi_j(\sigma(D) \in B_{i,j}$. Similarly, we may prove that $\chi_i\big(\sigma(D)\big) - \chi_j\big(\sigma(D)\big) \in B_{j,i}$.

For all $i,j$ $(1 \leq i,j \leq n)$, $\forall a_{i,j} \in A_{i,j}$, $a_{j,i} \in A_{j,i}$, by applying $\phi_1$ to $[a_{i,j}E_{i,j}, a_{j,i}E_{j,i}] = a_{i,j}a_{j,i}(E_{i,i}-E_{j,j})$, we have that $\sigma\big(a_{i,j}a_{j,i}(E_{i,i}-E_{j,j})\big) = \big(\sigma_{i,j}(a_{i,j})a_{j,i} + a_{i,j}\sigma_{j,i}(a_{j,i})\big)(E_{i,i} - E_{j,j})$.

When $i \neq j$, for all $a_{i,k} \in A_{i,k}$, $a_{k,j} \in A_{k,j}$, by applying $\phi_1$ to $[a_{i,k}E_{i,k}, a_{k,j}E_{k,j}] = a_{i,k}a_{k,j}E_{i,j}$, we have that

$$\big[\sigma_{i,k}(a_{i,k})E_{i,k}, a_{k,j}E_{k,j}\big] + \big[a_{i,k}E_{i,k}, \sigma_{k,j}(a_{k,j})E_{k,j}\big] = \sigma_{i,j}(a_{i,k}a_{k,j})E_{i,j}\,.$$

This shows that

$$\sigma_{i,j}(a_{i,k}a_{k,j}) = \sigma_{i,k}(a_{i,k})a_{k,j} + a_{i,k}\sigma_{k,j}(a_{k,j})\,.$$

Now we see that $\Sigma$ is suitable for ring derivations. Using $\Sigma$ we construct the ring derivation $\phi_\Sigma$ as in Section 3, and denote $\phi_1 - \phi_\Sigma$ by $\phi_2$. Then we see that $\phi_2(A_{k,l}E_{k,l}) \subseteq A_{l,k}E_{l,k}$ for all $k$, $l$ satisfy $k \neq l$ and $\phi_2$ sends $d(n,R)$ to 0.

**Step 4:** $\phi_2$ exactly is a transpose derivation.

By Step 3, we know that $A_{k,l}E_{k,l}$ is send by $\phi_2$ to $A_{l,k}E_{l,k}$ when $k \neq l$ and $d(n,R)$ is send by it to 0. Now for any $k$, $l$ $(1 \leq k,\, l \leq n)$ we define the map $\pi_{k,l}$ from $A_{k,l}$ to $A_{l,k}$ according to the following rule:

(a) $\pi_{k,l} = 0$ when $k = l$;

(b) If $k \neq l$, define $\pi_{k,l}\colon A_{k,l} \to A_{l,k}$ such that for any $a_{k,l} \in A_{k,l}$, $\sigma_{k,l}(a_{k,l})$ satisfies the condition: $\phi_2(a_{k,l}E_{k,l}) = \pi_{k,l}(a_{k,l})E_{l,k}$.

Then $\sigma_{k,l}$ is an homomorphism from the $R$-module $A_{k,l}$ to $A_{l,k}$. Set $\Pi = \big\{\pi_{i,j} \in \mathrm{Hom}_R(A_{i,j}, A_{j,i}) \mid 1 \leq i,j \leq n\big\}$. We intend to prove that $\Pi$ is suitable for transpose derivations. If $i \neq j$, for $\forall a_{i,k} \in A_{i,k}$, $\forall a_{k,j} \in A_{k,j}$, by applying $\phi_2$ to $[a_{i,k}E_{i,k}, a_{k,j}E_{k,j}] = a_{i,k}a_{k,j}E_{i,j}$, we have that

$$\big[\pi_{i,k}(a_{i,k})E_{k,i}, a_{k,j}E_{k,j}\big] + \big[a_{i,k}E_{i,k}, \pi_{k,j}(a_{k,j})E_{j,k}\big] = \pi_{i,j}(a_{i,k}a_{k,j})E_{j,i}\,.$$

If $k \neq i$, $k \neq j$, we see that the left side of above is 0, then $\pi_{i,j}(a_{i,k}a_{k,j}) = 0$, leads to $\pi_{i,j}(A_{i,k}A_{k,j}) = 0$.

If $i \neq k$, $i \neq j$, $\forall a_{i,k} \in A_{i,k}$, $\forall a_{i,j} \in A_{i,j}$, by applying $\phi_2$ to $[a_{i,k}E_{i,k}, a_{i,j}E_{i,j}] = 0$, we see that

$$\big[\pi_{i,k}(a_{i,k})E_{k,i}, a_{i,j}E_{i,j}\big] + \big[a_{i,k}E_{i,k}, \pi_{i,j}(a_{i,j})E_{j,i}\big] = 0\,.$$

This shows that

$$\pi_{i,k}(a_{i,k})a_{i,j}E_{k,j} - a_{i,k}\pi_{i,j}(a_{i,j})E_{j,k} = 0\,.$$

Thus $a_{i,k}\pi_{i,j}(a_{i,j}) = 0$, leads to $A_{i,k}\pi_{i,j}(A_{i,j}) = 0$ for $i \neq k$. Similarly, $A_{k,j}\pi_{i,j}(A_{i,j}) = 0$ for $k \neq j$.

For all $i \neq j$, $\forall a_{i,j} \in A_{i,j}$, by applying $\phi_2$ to $[E_{i,i}, a_{i,j}E_{i,j}] = a_{i,j}E_{i,j}$, we have that

$$\big[E_{i,i}, \pi_{i,j}(a_{i,j})E_{j,i}\big] = \pi_{i,j}(a_{i,j})E_{j,i}\,.$$

Since $\big[E_{i,i}, \pi_{i,j}(a_{i,j})E_{j,i}\big] = -\pi_{i,j}(a_{i,j})E_{j,i}$, we see that $\pi_{i,j}(a_{i,j}) = -\pi_{i,j}(a_{i,j})E_{j,i}$. So $2\pi_{i,j}(A_{i,j}) = 0$ for $i \neq j$. Then $2\pi_{i,j}(A_{i,j}) = 0$ for $\forall i,\, j$.

Now we see that $\Pi$ is suitable for transpose derivations. Using $\Pi$ we construct the transpose derivation $\phi_\Pi$ as in Section 3, and denote $\phi_2 - \phi_\Pi$ by $\phi_3$. Then we see that $\phi_3(A_{k,l}E_{k,l}) = 0$ for all $k, l$ satisfy $k \neq l$ and $\phi_3(d(n,R)) = 0$. So $\phi_3 = 0$.

Thus $\phi = \phi_\Pi + \phi_\Sigma - \text{ad }\ W_0$, as desired.

For the uniqueness of the decomposition of $\phi$, we first prove that if $\phi_\Pi + \phi_\Sigma + \text{ad }\ W_0 = 0$, then $\phi_\Pi = \phi_\Sigma = \text{ad }\ W_0 = 0$. Suppose that $\phi_\Pi + \phi_\Sigma + \text{ad }\ W_0 = 0$, where $W_0 \in w$ and $\phi_\Pi, \phi_\Sigma$ are the the transpose and the ring derivation of $L_\Phi$, respectively. By $(\phi_\Pi + \phi_\Sigma + \text{ad }\ W_0)(d(n,R)) = 0$, we easily see that $W_0 = 0$. Then we have that $\phi_\Pi + \phi_\Sigma = 0$. By applying $\phi_\Pi + \phi_\Sigma$ to $a_{i,j}E_{i,j}$ for $1 \leq i \neq j \leq n, a_{i,j} \in A_{i,j}$, we have that $\sigma_{i,j}(a_{i,j})E_{i,j} + \pi_{i,j}(a_{i,j})E_{j,i} = 0$, leads to $\sigma_{i,j}(a_{i,j}) = \pi_{i,j}(a_{i,j}) = 0$. This forces that $\phi_\Pi = \phi_\Sigma = 0$. Now suppose that

$$\phi = \phi_{\Pi_1} + \phi_{\Sigma_1} - \text{ad }\ W_1 = \phi_{\Pi_2} + \phi_{\Sigma_2} - \text{ad }\ W_2\,,$$

is two decompositions of $\phi$. Then we have that

$$(\phi_{\Pi_1} - \phi_{\Pi_2}) + (\phi_{\Sigma_1} - \phi_{\Sigma_2}) + (\text{ad }\ W_2 - \text{ad }\ W_1) = 0\,.$$

Note that $\phi_{\Pi_1} - \phi_{\Pi_2}$ (resp., $\phi_{\Sigma_1} - \phi_{\Sigma_2}$) is also a transpose (resp., ring) derivation of $L_\Phi$ and $\text{ad }\ W_2 - \text{ad }\ W_1 = ad\ (W_2 - W_1)$. This implies that $\phi_{\Sigma_1} = \phi_{\Sigma_2}$, $\phi_{\Pi_1} = \phi_{\Pi_2}$ and $\text{ad }\ W_1 = \text{ad }\ W_2$. □

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