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OD-CHARACTERIZATION OF ALMOST SIMPLE GROUPS RELATED TO $L_2(49)$

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Abstract. In the present paper, we classify groups with the same order and degree pattern as an almost simple group related to the projective special linear simple group $L_2(49)$. As a consequence of this result we can give a positive answer to a conjecture of W. J. Shi and J. X. Bi, for all almost simple groups related to $L_2(49)$ except $L_2(49)\cdot 2^2$. Also, we prove that if M is an almost simple group related to $L_2(49)$ except $L_2(49)\cdot 2^2$ and G is a finite group such that $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$, then $G \cong M$.

1. Introduction

Throughout this paper, groups under consideration are finite. For any group G, we denote by $\pi_e(G)$ the set of orders of its elements and by $\pi(G)$ the set of prime divisors of $|G|$. We associate to $\pi(G)$ a simple graph called prime graph of G, denoted by $\Gamma(G)$. The vertex set of this graph is $\pi(G)$, and two distinct vertices p, q are joined by an edge if and only if $pq \in \pi_e(G)$. In this case, we write $p \sim q$. Denote by $t(G)$ the number of connected components of $\Gamma(G)$ and by $\pi_i = \pi_i(G)$ $(i = 1, 2, \dots, t(G))$ the connected components of $\Gamma(G)$. When $|G|$ is even, then by our convention $2 \in \pi_1(G)$. We also denote by $\pi(n)$ the set of all primes dividing n, where n is a natural number. Then $|G|$ can be expressed as a product of $m_1, m_2, \dots, m_{t(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i$. These m_i's are called the order components of G. In particular, if m_i is an odd number, then we call it an odd component of G. Let $OC(G) = \{m_1, m_2, \dots, m_{t(G)}\}$ be the set of order components of G, and $T(G) = \{\pi_i(G) \mid i = 1, 2, \dots, t(G)\}$.

Let G be a group and $p \in \pi(G)$. We denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all of its Sylow p-subgroups, respectively. We also denote by $\mathrm{Soc}(G)$ the socle of G which is the subgroup generated by the set of all minimal normal subgroups of G. We denote by $A\colon B$ (or $A \cdot B$) a split (or non-split) extension of A by B. Also, $\mathbb{N}$ and $\mathbb{P}$ denote the set of natural numbers and the set of primes, respectively.

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In particular, this paper itself is accessible only with the basic knowledge of group theory. All further unexplained notations are standard and can be found in [4].

Definition 1.1. Let G be a finite group and $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_i \in \mathbb{P}$ and $\alpha_i \in \mathbb{N}$ for $i = 1, 2, \dots, k$. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G) \mid p \sim q\}|$, called the degree of p. We also define $D(G) := \big(\deg(p_1), \deg(p_2), \dots, \deg(p_k)\big)$, where $p_1 < p_2 < \dots < p_k$. We call it the degree pattern of G.

Definition 1.2. A group M is called *k-fold OD-characterizable* if there exist exactly k non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$. Moreover, a 1-fold OD-characterizable group is simply called an *OD-characterizable group*.

Definition 1.3. A group G is said to be an almost simple related to S if and only if $S \trianglelefteq G \leq \mathrm{Aut}\,(S)$ for some non-abelian simple group S.

Definition 1.4. Let p be a prime. A group G is called a $C_{p,p}$-group if and only if $p \in \pi(G)$ and the centralizers of its elements of order p in G are p-groups.

The significance of the prime graphs of finite groups can be found in many articles, for example [6], [18]–[21]. Therefore, the characterizations of finite groups by their orders and degree patterns may help us to know certain properties of the almost simple groups more clearly. In a series of articles (see [10, 11, 22, 23]), it was shown that many finite almost simple groups are OD-characterizable. We point out some of these results.

Result 1 ([10, 11])**.** All sporadic simple groups and their automorphism groups except $\mathrm{Aut}\,(J_2)$ and $\mathrm{Aut}\,(M^cL)$ are OD-characterizable.

Result 2 ([10])**.** The alternating groups A_p, A_{p+1}, A_{p+2} and the symmetric groups S_p and S_{p+1}, where p is a prime, are OD-characterizable.

Result 3 ([10, 11])**.** The simple groups of Lie type $L_2(q)$, $L_3(q)$, $U_3(q)$, ${}^2B_2(q)$ and ${}^2G_2(q)$ are OD-characterizable for certain $q \in \mathbb{N}$.

Result 4 ([10])**.** All finite simple $C_{2,2}$-groups are OD-characterizable.

Result 5 ([23])**.** All finite simple groups with exactly four prime divisors except A_{10} are OD-characterizable.

2. Lemmas

Lemma 2.1 ([9, Table 1])**.** *Let G be an almost simple group related to $L := L_2(49)$. Then G is isomorphic to one of the following groups: L, $L\colon 2_1 \big(\cong PGL(2,49)\big)$, $L\colon 2_2$, $L \cdot 2_3$, $L \cdot 2^2 \big(\cong \mathrm{Aut}\,(L_2(49))\big)$. Moreover, $\pi_e(L) = \{25, 24, 7\}$, $\pi_e(L\colon 2_1) = \{50, 48, 7\}$, $\pi_e(L\colon 2_2) = \{25, 24, 14\}$, $\pi_e(L \cdot 2_3) = \{25, 24, 16, 7\}$, and $\pi_e(L \cdot 2^2) = \{50, 48, 14\}$. More information about the algorithm can be obtained in* [8].

Lemma 2.2 ([5, Theorem 1])**.** *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leq 2$.*

Lemma 2.3 ([9, Table 1])**.** *If S is a finite non-abelian simple groups such that $\pi(S) \subseteq \{2,3,5,7\}$, then S is isomorphic to one of the following simple groups in Table 1. In particular, $\{2,3\} \subset \pi(S)$ and $\pi(\mathrm{Out}(S)) \subseteq \{2,3\}$ if $S \neq S_6(2)$.*

Table 1. Finite non-abelian simple groups S such that $\pi(S) \subseteq \{2,3,5,7\}$

S	Order of S	Out(S)	S	Order of S	Out(S)
A_5	$2^2 \cdot 3 \cdot 5$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	S_3
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	D_8
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	D_{12}	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	S_3

Now we quote two lemmas on Frobenius groups.

Lemma 2.4 ([1, Theorem 1])**.** *Let G be a Frobenius group of even order with H and K its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and T(G)= $\{\pi(K), \pi(H)\}$.*

Lemma 2.5 ([4, 12])**.** *Let G be a Frobenius group with kernel F and complement C. Then the following assertions are true.*

(a) *F is a nilpotent group.*

(b) $|F| \equiv 1(\mathrm{mod} \ \ |C|)$.

(c) *Every subgroup of C of order $p \cdot q$, with p, q (not necessarily distinct) primes, is cyclic. In particular, every Sylow subgroup of C of odd order is cyclic and Sylow 2-subgroup of C is either cyclic or generalized quaternion group. If C is a non-solvable group, then C has a subgroup of index at most 2 isomorphic to $\mathrm{SL}(2,5) \times M$, where M has cyclic Sylow p-subgroups and $(|M|, 30) = 1$; in particular, 15, 20 $\notin \pi_e(C)$. If C is solvable and $O(C) = 1$, then either C is a 2-group or C has a subgroup of index at most 2 isomorphic to $\mathrm{SL}(2,3)$.*

A group G is a 2-Frobenius group if there exists a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Now we quote a lemma on 2-Frobenius groups.

Lemma 2.6 ([1, Theorem 2])**.** *Let G be a 2-Frobenius group of even order, which has a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then*

(a) $t(G) = 2$ *and* $T(G)$= $\{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$.

(b) G/K *and* K/H *are cyclic,* $|G/K| \mid |\,\mathrm{Aut}\,(K/H)|$, *and* $\big(|G/K|, |K/H|\big) = 1$.

(c) H *is a nilpotent group and* G *is a solvable group.*

The structure of a finite group with disconnected prime graph is described in the following lemma. Though this lemma is a useful tool for the groups with disconnected prime graphs, we should not use it if a finite group has only one connected component.

Lemma 2.7 ([7, 17, Theorem A]). *Let G be a finite group with $t(G) \geq 2$, then G is one of the following groups:*

(a) *G is a Frobenius or 2-Frobenius group;*

(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a finite non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|G/H| \mid |\,\mathrm{Aut}\,(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.*

Lemma 2.8 ([2, Theorem]). *Let G be a finite non-abelian simple $C_{p,p}$-group, where $p \in \mathbb{P}$.*

(a) *If $p = 5$, then G is isomorphic to one of the following simple groups: A_5, A_6, A_7, M_{11}, M_{22}, $L_3(4)$, $S_4(3)$, $S_4(7)$, $U_4(3)$, $Sz(8)$, $Sz(32)$, $L_2(49)$, $L_2(5^m)$, $L_2(2 \cdot 5^m \pm 1)$, where $m \in \mathbb{N}$ and $2 \cdot 5^m \pm 1 \in \mathbb{P}$.*

(b) *If $p = 7$, then G is isomorphic to one of the following simple groups: A_7, A_8, A_9, J_1, M_{22}, J_2, HS, $L_3(4)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $G_2(13)$, $U_3(3)$, $U_3(5)$, $U_3(19)$, $U_4(3)$, $U_6(2)$, $Sz(8)$, $L_2(8)$, $L_2(7^m)$, $L_2(2 \cdot 7^m - 1)$, where $m \in \mathbb{N}$ and $2 \cdot 7^m - 1 \in \mathbb{P}$.*

Lemma 2.9 ([22, Theorem]). *If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where $M = U_4(3) : 2_2$ or $U_4(3) \cdot 2_3$, then $G \cong U_4(3) : 2_2$ or $U_4(3) \cdot 2_3$.*

3. OD-characterization of almost simple groups related to $L_2(49)$

Theorem 3.1. *If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where M is an almost simple group related to $L := L_2(49)$, then the following assertions are true:*

(a) *If $M = L, L : 2_1, L : 2_2$ or $L \cdot 2_3$, then $G \cong M$.*

(b) *If $M = L \cdot 2^2$, then $G \cong L \cdot 2^2$, $\mathbb{Z}_2 \times (L : 2_1)$, $\mathbb{Z}_2 \times (L : 2_2)$, $\mathbb{Z}_2 \times (L \cdot 2_3)$, $\mathbb{Z}_2 \cdot (L : 2_1)$, $\mathbb{Z}_2 \cdot (L : 2_2)$, $\mathbb{Z}_2 \cdot (L \cdot 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.*

In particular, L, L: 2_1, L: 2_2 and $L \cdot 2_3$ are OD*-characterizable; $L \cdot 2^2$ is 9-fold* OD*-characterizable.*

Proof. By Lemma 2.1, first we list the prime graphs of the almost simple groups related to L as follows:

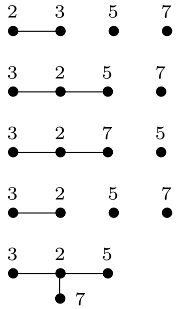
$\Gamma(L)$:

$\Gamma(L:2_1)$:

$\Gamma(L:2_2)$:

$\Gamma(L\cdot 2_3)$:

$\Gamma(L\cdot 2^2)$:



Moreover, we break the proof into a number of separate cases.

Case 1. If $M = L$, then $G \cong L$ by Result 5.

Case 2. If $M = L\colon 2_1$, then $G \cong L\colon 2_1$.

If $M = L\colon 2_1$, then $\Gamma(G) = \Gamma(M)$ by our assumptions.

First let G be a solvable group. Then G has a solvable Hall $\{3,5,7\}$-subgroup H. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in H are of prime power order. Hence $t(H) \leq 2$ by Lemma 2.2, a contradiction. Thus G is not solvable, which implies that G is not a 2-Frobenius group by Lemma 2.6(c). If G is a non-solvable Frobenius group with H and K being its Frobenius complement and Frobenius kernel, respectively, then, by Lemma 2.5(c), it follows that H has a normal subgroup H_0 with $|H\colon H_0| \leq 2$ such that $H_0 = \mathrm{SL}(2,5) \times Z$, where the Sylow subgroups of Z are cyclic and $\big(|Z|, 30\big) = 1$. Thus $7 \in \pi(K)$ since $5 \nsim 7$ in $\Gamma(G)$. Since $|G| = |M| = 2^5 \cdot 3 \cdot 5^2 \cdot 7^2$ and $|\,\mathrm{SL}(2,5)| = 2^3 \cdot 3 \cdot 5$, it follows that $5 \in \pi(K)$ too. Because K is nilpotent by Lemma 2.5(a), it follows that $5 \sim 7$ in $\Gamma(K)$, an obvious contradiction. Hence G is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G_1/N is a finite simple $C_{7,7}$-group and G/G_1 is a solvable π_1-group. By Lemmas 2.3 and 2.8(b), we obtain that G_1/N must be isomorphic to L.

Since $G/N \lesssim \mathrm{Aut}\,(G_1/N)$, it follows that $L \lesssim G/N \lesssim \mathrm{Aut}\,(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lesssim \mathrm{Aut}\,(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_7 \in \mathrm{Syl}_7(G)$. Then NG_7 is a subgroup of G, which implies that $2 \sim 7$ in $\Gamma(NG_7)$, an obvious contradiction. Therefore $G/N \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ since $|G| = 2|L|$. It follows that $G \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ by Lemma 2.1. Obviously, $G \cong L\colon 2_1$ since $2 \nsim 5$ in $\Gamma(L\colon 2_2)$ and $\Gamma(L \cdot 2_3)$.

Case 3. If $M = L\colon 2_2$, then $G \cong L\colon 2_2$.

If $M = L\colon 2_2$, then $\Gamma(G) = \Gamma(M)$.

First let G be a solvable group. Then G has a solvable Hall $\{3,5,7\}$-subgroup H. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in H are of prime power order. Hence $t(H) \leq 2$ by Lemma 2.2, a contradiction. Thus G is not solvable, which implies that G is not a 2-Frobenius group by Lemma

2.6(c). If G is a non-solvable Frobenius group with H and K being its Frobenius complement and Frobenius kernel, respectively, then, by Lemma 2.5(c), it follows that H has a normal subgroup H_0 with $|H\colon H_0| \leq 2$ such that $H_0 = \mathrm{SL}(2,5) \times Z$, where the Sylow subgroups of Z are cyclic and $\big(|Z|, 30\big) = 1$. Thus $7 \in \pi(K)$ since $5 \nsim 7$ in $\Gamma(G)$. Since $|G| = |M| = 2^5 \cdot 3 \cdot 5^2 \cdot 7^2$ and $|\,\mathrm{SL}(2,5)| = 2^3 \cdot 3 \cdot 5$, it follows that $5 \in \pi(K)$ too. Because K is nilpotent by Lemma 2.5(a), it follows that $5 \sim 7$ in $\Gamma(K)$, an obvious contradiction. Hence G is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G_1/N is a finite simple $C_{5,5}$-group and G/G_1 is a solvable π_1-group. By Lemmas 2.3 and 2.8(a), we obtain that G_1/N must be isomorphic to L.

Since $G/N \lesssim \mathrm{Aut}\,(G_1/N)$, it follows that $L \lesssim G/N \lesssim \mathrm{Aut}\,(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lesssim \mathrm{Aut}\,(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_5 \in \mathrm{Syl}_5(G)$. Then NG_5 is a subgroup of G, which implies that $2 \sim 5$ in $\Gamma(NG_5)$, an obvious contradiction. Therefore $G/N \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ since $|G| = 2|L|$. It follows that $G \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ by Lemma 2.1. Obviously, $G \cong L\colon 2_2$ since $2 \nsim 7$ in $\Gamma(L\colon 2_1)$ and $\Gamma(L \cdot 2_3)$.

Case 4. If $M = L \cdot 2_3$, then $G \cong L \cdot 2_3$.

If $M = L \cdot 2_3$, then $\Gamma(G) = \Gamma(M)$. Thus $t(G) = t(M) = 3$. By Lemmas 2.4 and 2.6(a), G is neither a Frobenius group nor a 2-Frobenius group.

By Lemma 2.7, G has a normal series $1 \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G$ such that N is a nilpotent π_1-group, G_1/N is a finite simple $C_{5,5}$- and $C_{7,7}$-group, and G/G_1 is a solvable π_1-group. By Lemmas 2.3 and 2.8, we obtain that G_1/N must be isomorphic to L.

Since $G/N \lesssim Aut(G_1/N)$, it follows that $L \lesssim G/N \lesssim \mathrm{Aut}\,(L)$. If $G/N \cong L$, then $|N| = 2$. Since $G/C_G(N) \lesssim \mathrm{Aut}\,(N) = 1$, it follows that $N \leq Z(G)$. Suppose $G_5 \in \mathrm{Syl}_5(G)$. Then NG_5 is a subgroup of G, which implies that $2 \sim 5$ in $\Gamma(NG_5)$, an obvious contradiction. Therefore $G/N \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ since $|G| = 2|L|$. It follows that $G \cong L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$ by Lemma 2.1. Obviously, $G \cong L \cdot 2_3$ since $2 \sim 5$ in $\Gamma(L\colon 2_1)$ and $2 \sim 7$ in $\Gamma(L\colon 2_2)$, respectively.

Case 5. If $M = L \cdot 2^2$, then $G \cong L \cdot 2^2$, $\mathbb{Z}_2 \times (L\colon 2_1)$, $\mathbb{Z}_2 \times (L\colon 2_2)$, $\mathbb{Z}_2 \times (L \cdot 2_3)$, $\mathbb{Z}_2 \cdot (L\colon 2_1)$, $\mathbb{Z}_2 \cdot (L\colon 2_2)$, $\mathbb{Z}_2 \cdot (L \cdot 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-subgroup. In particular, G is non-solvable.

If $M = L \cdot 2^2$, then $\Gamma(G) = \Gamma(M)$.

First assume that $\{5,7\} \subseteq \pi(K)$. Let T be a Hall $\{5,7\}$-subgroup of K. It is easy to see that T is an abelian subgroup of order $5^i \cdot 7^j$, where $i,j = 1$ or 2. Thus $5 \cdot 7 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction. Next, we assume that $5 \in \pi(K)$ and $7 \notin \pi(K)$. Then K is a $\{2,3,5\}$-group. Let $R \in \mathrm{Syl}_5(K)$. By Frattini argument $G = KN_G(R)$. Therefore, the normalizer $N_G(R)$ contains an element of order 7, say x. Now $< x > R$ is a subgroup of G of order $5^i \cdot 7$, where $i = 1$ or 2. Hence $\langle x\rangle R$ is an abelian group. Thus $5 \cdot 7 \in \pi_e(\langle x\rangle R) \subseteq \pi_e(G)$, a contradiction. Finally, we assume $7 \in \pi(K)$ and $5 \notin \pi(K)$. In this case, K is a $\{2,3,7\}$-subgroup and we consider the Sylow 7-subgroup P of K. As before, we see that $G = KN_G(P)$ and by a similar argument we get $5 \cdot 7 \in \pi_e(G)$, which is a contradiction. Thus K is a $\{2,3\}$-subgroup.

Let G be a solvable group. Then G has a solvable Hall $\{3,5,7\}$-subgroup H. Since there exists no edge between 3, 5 and 7 in $\Gamma(G)$, it implies that all elements in H are of prime power order. Hence $t(H) \leq 2$ by Lemma 2.2, a contradiction. Thus G is not solvable.

Step 2. The quotient G/K is an almost simple group. In fact, $S \lesssim G/K \lesssim \mathrm{Aut}\,(S)$ where S is a finite non-abelian simple group isomorphic to A_5, $L_2(7)$ or L.

Let $\overline{G} := G/K$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times \cdots \times P_m$, where P_i's are finite non-abelian simple groups. It is obvious that $\{2,3\} \subseteq \pi(P_i) \subseteq \{2,3,5,7\}$ by Lemma 2.3, where $i = 1,2,\ldots,m$. Now we assert that $C_{\overline{G}}(S)$ is solvable. In fact, if $C_{\overline{G}}(S)$ is non-solvable, then it can not be a $\{2,3\}$-group by Burnside's Theorem. It follows that $5 \in \pi(C_{\overline{G}}(S))$ or $7 \in \pi(C_{\overline{G}}(S))$, which shows that $3 \cdot 5 \in \pi_e(\overline{G})$ or $3 \cdot 7 \in \pi_e(\overline{G})$ since $\{2,3\} \subseteq \pi(P_i) \subseteq \pi(S)$. It follows that $3 \cdot 5 \in \pi_e(G)$ or $3 \cdot 7 \in \pi_e(G)$, which is a contradiction since $3 \nsim 5$ and $3 \nsim 7$ in $\Gamma(G)$. Suppose $1 \neq T/K =: C_{\overline{G}}(S)$, which is solvable. Then $T/K \neq G/K$ since G/K is non-solvable. Thus $K \trianglelefteq T \trianglelefteq G$, where T is solvable. This is a contradiction by the choice of K. Hence $C_{\overline{G}}(S) = 1$. It follows that $S \lesssim G/K \cong G/K/C_{\overline{G}}(S) \lesssim \mathrm{Aut}\,(S)$.

By Lemma 2.3, it is clear that $m = 1$ since $|G|_3 = 3$, where $|G|_3$ is the 3-part of $|G|$. Using Table 1, S is isomorphic to one of the following simple groups: A_5, $L_2(7)$ or L.

Step 3. $G \cong L \cdot 2^2$, $\mathbb{Z}_2 \times (L\colon 2_1)$, $\mathbb{Z}_2 \times (L\colon 2_2)$, $\mathbb{Z}_2 \times (L \cdot 2_3)$, $\mathbb{Z}_2 \cdot (L\colon 2_1)$, $\mathbb{Z}_2 \cdot (L\colon 2_2)$, $\mathbb{Z}_2 \cdot (L \cdot 2_3)$, $\mathbb{Z}_4 \times L$ or $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$.

By Step 2, $S \lesssim G/K \lesssim \mathrm{Aut}\,(S)$ where S is a finite non-abelian simple group isomorphic to A_5, $L_2(7)$ or L.

If $S \cong A_5$, then $A_5 \lesssim \overline{G} \lesssim \mathrm{Aut}\,(A_5)$. It follows that $|K| = 2^4 \cdot 5 \cdot 7^2$ or $2^3 \cdot 5 \cdot 7^2$ by Lemma 2.3. Obviously, this is a contradiction since K is a $\{2,3\}$-group by Step 1.

If $S \cong L_2(7)$, then $L_2(7) \lesssim \overline{G} \lesssim \mathrm{Aut}\,(L_2(7))$. It follows that $|K| = 2^3 \cdot 5^2 \cdot 7$ or $2^2 \cdot 5^2 \cdot 7$ by Lemma 2.3. Obviously, this is a contradiction since K is a $\{2,3\}$-group by Step 1.

Therefore, $S \cong L$. Thus $L \lesssim \overline{G} \lesssim \mathrm{Aut}\,(L)$. Hence $|K| = 1$, 2 or 2^2.

If $|K| = 1$, then $G \cong L \cdot 2^2$ by Lemma 2.1.

If $|K| = 2$, then $K \leq Z(G)$, i.e., G is a central extension of K by $L\colon 2_1$, $L\colon 2_2$ or $L \cdot 2_3$. If G splits over K, we obtain $G \cong \mathbb{Z}_2 \times (L\colon 2_1)$, $\mathbb{Z}_2 \times (L\colon 2_2)$ or $\mathbb{Z}_2 \times (L \cdot 2_3)$. Otherwise we have $G \cong \mathbb{Z}_2 \cdot (L\colon 2_1)$, $\mathbb{Z}_2 \cdot (L\colon 2_1)$ or $\mathbb{Z}_2 \cdot (L \cdot 2_3)$.

If $|K| = 2^2$, then $G/K \cong L$. In this case, we have $G/C_G(K) \lesssim \mathrm{Aut}\,(K) \cong \mathbb{Z}_2$ or S_3. Thus $|G/C_G(K)| = 1, 2, 3$ or 6. If $\left|G/C_G(K)\right| = 1$, then $K \leq Z(G)$, i.e., G is a central extension of K by L. If G is a non-split extension of K by L, then $|K|$ must divide the Schur multiplier of L, which is 2 (see [3]). But this is a contradiction. So we obtain that G splits over K. Hence $G \cong K \times L$. Thus $G \cong \mathbb{Z}_4 \times L$ or

$(\mathbb{Z}_2 \times \mathbb{Z}_2) \times L$ since $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $\big|G/C_G(K)\big| = 2, 3$ or 6, then $K < C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$. Since L is simple, we obtain that $G = C_G(K)$, a contradiction. □

Remark 1. W. J. Shi and J. X. Bi in [16] put forward the following conjecture:

Conjecture. Let G be a finite group and M a finite simple group. Then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$.

This conjecture is valid for the sporadic simple groups (see [14]), alternating groups and some simple groups of Lie type (see [13, 15, 16]). As a consequence of Theorem 3.1, we verify the validity of this conjecture for the groups under discussion.

Theorem 3.2. *If G is a finite group such that $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, where M is an almost simple group related to $L_2(49)$ except $L_2(49) \cdot 2^2$, then $G \cong M$.*

Proof. Since $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, we obtain $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$. It follows that $|G| = |M|$ and $D(G) = D(M)$. By Theorem 3.1, we have $G \cong M$. □

Note that if G is a finite group such that $|G| = |M|$ and $D(G) = D(M)$, where M is a given finite group, then $\pi_e(G)$ is not equal to $\pi_e(M)$ necessarily. Now, we give a counterexample as follows. Let $L := U_4(3)$, then $L\colon 2_2$ is 2-fold OD-characterizable by Lemma 2.9. However, in this case, $\pi_e(L\colon 2_2) = \{18, 12, 10, 8, 7\}$ is not equal to $\pi_e(L \cdot 2_3) = \{24, 10, 9, 7\}$ (see [9]).

Theorem 3.3. *If G is a finite group such that $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$, where M is an almost simple group related to $L_2(49)$ except $L_2(49) \cdot 2^2$, then $G \cong M$.*

Proof. Since $|G| = |M|$ and $\Gamma(G) = \Gamma(M)$, we obtain that $|G| = |M|$ and $D(G) = D(M)$. By Theorem 3.1, we have $G \cong M$. □

Question. Let G be a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where M is an almost simple group. Is G non-solvable, too?

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