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SOME INEQUALITIES CONNECTED WITH THE EXPONENTIAL FUNCTION

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Abstract. The paper is devoted to some functional inequalities related to the exponential mapping.

1. Introduction

It is well known that the logarithmic mean L lies between the geometric mean G and the arithmetic mean A, i.e.:

$$\sqrt{s\cdot t} \le \frac{t-s}{\log t - \log s} \le \frac{s+t}{2}\,, \tag{1}$$

for each s, $t>0$ such that $s \neq t$ (see B. C. Carlson [3]). On substituting $s := e^x$ and $t := e^y$ we infer that the exponential function satisfies the inequalities:

$$e^{\frac{x+y}{2}} \le \frac{e^y - e^x}{y-x} \le \frac{e^x + e^y}{2}\,, \tag{2}$$

for each x, $y \in \mathbb{R}$ such that $x \neq y$.

It may be interesting to find a characterization of the exponential function by means of functional inequalities. In 1988 B. Poonen [4], answering a problem proposed by D. J. Shelupsky [5], has shown that the general solution $f\colon I \to \mathbb{R}$ of the system:

$$\min\{f(x), f(y)\} \le \frac{f(y)-f(x)}{y-x} \le \max\{f(x), f(y)\}\,, \quad x, y \in I\,,\ x \neq y \tag{3}$$

is of the form $f(x) = ce^x$, where $c \ge 0$ is an arbitrary constant.

The result of B. Poonen was developed by C. Alsina and J. L. Garcia-Roig [1]. In particular they investigated functional inequalities:

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(x)+f(y)}{2}\,, \quad x, y \in \mathbb{R}\,,\ x \neq y\,, \tag{4}$$

and

$$0 \le \frac{f(y)-f(x)}{y-x} \le \frac{f(x)+f(y)}{2}\,, \quad x, y \in \mathbb{R}\,,\ x \neq y\,. \tag{5}$$

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They have proved that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies (4) if and only if there exists a nonincreasing function $d\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for $x \in \mathbb{R}$ [1, Theorem 1]. Further, $f\colon \mathbb{R} \to \mathbb{R}$ is a solution of (5) if and only if there exists a continuous nonincreasing function $d\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for $x \in \mathbb{R}$ and $d(x+t) \geq e^{-t}d(x)$ for $x \in \mathbb{R}$ and $t > 0$ [1, Theorem 2].

In view of (2) and the above-mentioned results of C. Alsina and J. L. Garcia-Roig, it is of interest to solve the following functional inequality:

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(y)-f(x)}{y-x}\,, \quad x<y\,, \tag{6}$$

or, equivalently, taking into account $-f$ instead of f in (6), the inequality:

$$\frac{f(y)-f(x)}{y-x} \leq f\Big(\frac{x+y}{2}\Big)\,, \quad x<y\,. \tag{7}$$

This two functional inequalities have been investigated by C. Alsina and R. Ger [2]. In particular, they have proved that a nonnegative function $f\colon I \to \mathbb{R}$ defined on an open interval I satisfies (6) if and only if there exists a nondecreasing nonnegative function $i\colon I \to \mathbb{R}$ such that $f(x) = i(x)e^x$ for $x \in I$ [2, Lemma 3]. Furthermore, a nonnegative, nondecreasing concave function $f\colon I \to \mathbb{R}$ satisfies (7) if and only if there exists a nonnegative nonincreasing function $d\colon I \to \mathbb{R}$ such that $f(x) = d(x)e^x$ for $x \in I$ and the mapping $I \ni x \mapsto d(x)e^x$ is concave [2, Lemma 5] (note that each nonnegative solution of (6) is nondecreasing). Thus, for inequality (6) the situation seems to be more difficult than for (4) and (5). In [2] C. Alsina and R. Ger have stated an open problem if it is possible to solve (6) in a wider class of functions. In what follows we will solve (6) under less restrictive assumptions imposed upon f and thus we provide an affirmative answer to this question. Then, we derive from (6) a functional-integral inequality (see (14) below) and we discuss it.

2. Results

From now on we will be assuming that $I \subset \mathbb{R}$ is an open nonempty interval.

Lemma 1. *If a sequence $(\xi_n)_{n\in\mathbb{N}}$ of real mappings is defined by the formula*

$$\xi_n(h) := \frac{1}{2\sqrt{h^2+1}}\left[\Big(h+\sqrt{h^2+1}\Big)^n - \Big(h-\sqrt{h^2+1}\Big)^n\right], \quad h \geq 0,\ n \in \mathbb{N}, \tag{8}$$

then

$$\xi_0(h) = 0\,, \quad \xi_1(h) = 1\,, \quad \xi_n(h) = \xi_{n-2}(h) + 2h\xi_{n-1}(h)\,, \quad h \geq 0\,,\ n \in \mathbb{N}$$

and

$$\lim_{n\to+\infty}\left[\xi_{n+1}\Big(\frac{h}{n}\Big) + \xi_n\Big(\frac{h}{n}\Big)\right] = e^h\,, \quad h \geq 0\,.$$

This lemma can be verified by an elementary, but rather tedious computation, and therefore we will omit the motivation.

Theorem 1. *Assume that* $f\colon I \to \mathbb{R}$ *satisfies* (6) *and*

$$\limsup_{h\to 0+} f(x+h) \geq f(x)\,, \quad x \in I\,. \tag{9}$$

Then

$$f(y) \geq e^{y-x} f(x)\,, \quad x, y \in I\,,\ x \leq y\,, \tag{10}$$

i.e. the mapping $I \ni x \mapsto f(x)e^{-x} \in \mathbb{R}$ *is nondecreasing.*

Proof. Put $y = x + 2h$ in (6) to get

$$f(x+2h) \geq f(x) + 2hf(x+h)\,, \quad x \in I\,,\ h > 0\,,\ x+2h \in I\,. \tag{11}$$

From this, on setting $x + (n-1)h$ instead of x, we derive the inequality

$$f\big(x+(n+1)h\big) \geq f\big(x+(n-1)h\big) + 2hf(x+nh)\,, \tag{12}$$

for each $x \in I$, $h > 0$ and $n \in \mathbb{N}$ such that $x + (n+1)h \in I$. Now, check inductively that

$$f\big(x+(n+1)h\big) \geq \xi_n(h)f(x) + \xi_{n+1}(h)f(x+h)\,, \tag{13}$$

for each $n \in \mathbb{N}$, $x \in I$ and $h > 0$ such that $x + (n+1)h \in I$. Indeed, by the aid of (12), we have

$$\begin{aligned} f\big(x+(n+1)h\big) &\geq f\big(x+(n-1)h\big) + 2hf(x+nh) \\ &\geq \xi_{n-2}(h)f(x) + \xi_{n-1}(h)f(x+h) \\ &\quad + 2h\big[\xi_{n-1}(h)f(x) + \xi_n(h)f(x+h)\big] \\ &= \xi_n(h)f(x) + \xi_{n+1}(h)f(x+h)\,. \end{aligned}$$

Now, joining (13) with Lemma 1 and making use of (9) we finally arrive at

$$\begin{aligned} f(x+h) &= f\Big(x + \frac{n+1}{n+1}h\Big) \\ &\geq \limsup_{n\to+\infty} \Big[\xi_n\Big(\frac{h}{n+1}\Big)f(x) + \xi_{n+1}\Big(\frac{h}{n+1}\Big)f\Big(x + \frac{h}{n+1}\Big)\Big] \\ &\geq f(x)e^h\,, \quad x \in I\,,\ h > 0\,,\ x+2h \in I\,. \end{aligned}$$

To finish the proof it is enough to fix $x, y \in I$ such that $x < y$ and put $h = y - x$. □

Remark 1. The converse of Theorem 1 generally is not true. To see this take $f(x) = -e^x$ for $x \in I = \mathbb{R}$. We see that f satisfies (10) and, as a continuous mapping (9), but not (6).

However, it is possible to obtain a partial converse. For nonnegative (and thus nondecreasing) functions Theorem 1 has been already proved by C. Alsina and R. Ger [2, Lemma 3]. They have also checked that in this situation the converse theorem holds true, i.e. if $f \geq 0$ and (10) is valid, then (6) is satisfied.

Let us note also that from [2, Lemma 5] of C. Alsina and R. Ger it follows in particular that there exist nonpositive solutions of (6).

On joining Theorem 1 with a result of C. Alsina and J. L. Garcia-Roig [1, Theorem 1] we immediately get the following corollary:

Corollary 1. *A map $f\colon I \to \mathbb{R}$ satisfies* (9) *and is a joint solution of* (4) *and* (6) *if and only if f is of the form*

$$f(t) = f(0) \cdot \exp(t)\,, \quad t \in I\,.$$

Theorem 2. *If $f\colon I \to \mathbb{R}$ is a Riemann-integrable solution of* (6), *then it satisfies a functional-integral inequality*

$$\frac{1}{y-x}\int_x^y f(t)\,dt \le \frac{f(y)-f(x)}{y-x}\,, \quad x, y \in I\,,\ x < y\,. \tag{14}$$

Proof. Fix $x \in I$ and $h > 0$ such that $x + h \in I$. On applying (6) for $y := x + h$ we obtain

$$f(x+h) - f(x) \ge hf\Big(x + \frac{h}{2}\Big)\,.$$

Replace in this inequality x by $x + nh$ for $n = 1, 2, \dots, N$ and sum up side by side the inequalities obtained to arrive at

$$f(x+Nh) - f(x) \ge h\sum_{n=1}^{N+1} f\Big(x + \frac{2n-1}{2}h\Big)\,, \quad N \in \mathbb{N}\,.$$

Now, put $\frac{h}{N}$ instead of h and let N tends to $+\infty$. Using the definition of the Riemann integral we get

$$f(x+h) - f(x) \ge \int_x^{x+h} f(t)\,dt\,, \quad x \in I\,,\ h > 0\,,\ x + h \in I\,.$$

After setting $h = y - x$ we eventually obtain (14). This completes the proof. □

Remark 2. One may observe that for a convex map f inequality (14) is sharper than (6). Moreover, each nonnegative solution of (14) is nondecreasing but needs not to be convex, since there exist nonnegative and continuous solutions of (6) which are not convex. Note also that if f is differentiable and satisfies (14), then

$$f(x) \le f'(x)\,, \quad x \in I\,. \tag{15}$$

Inequality (15) has been investigated by C. Alsina and R. Ger [2, Lemma 1], and earlier by C. Alsina and J. L. Garcia-Roig [1, Corollary 2 & Corollary 3]. In particular, they have proved that if f satisfies (15), then there exists a nondecreasing map $i\colon I \to \mathbb{R}$ such that $f(x) = i(x)e^x$ for each $x \in I$ (note that the same representation for solutions of (6) follows from Theorem 1). Therefore, one may expect the general solution of (14) has a similar representation. For f being an arbitrary continuous solution of (14) we may proceed as follows: denote by F the primitive function of f. Thus, from (14) we get

$$F(y) - F(x) \le f(y) - f(x)\,, \quad x, y \in I\,,\ x < y\,,$$

i.e. $f - F$ is a nondecreasing mapping. Fix $x \in I$ and put $c := f(x) - F(x)$. On replacing F by $F + c$ and using the fact that $F' = f$ we get $F(y) \le F'(y)$ for $y \in I$, i.e. F satisfies (15). Thus, there exists a nondecreasing map $i\colon I \to \mathbb{R}$ such that $F(y) = i(y)e^y$ for $y \in I$. Since i is differentiable, then we get $f(y) = \big[i(y) + i'(y)\big]e^y$ for $y \in I$. Clearly, the map $i + i'$ needs not to be nondecreasing, unless i is nondecreasing and convex. We do not know whether each continuous solution of (14) is of the form $f(y) = i(y)e^y$, where i is a nondecreasing mapping.

Our next result describes nonnegative continuous solutions of (14). From the previous remark it follows that if f is nonnegative then there exists $\lambda \in \mathrm{cl}I$ such that f restricted to $I \cap (-\infty, \lambda)$ is equal to zero, whereas f restricted to $I \cap (\lambda, +\infty)$ is positive. Therefore, in case of nonnegative solutions of (14) we may assume that f is positive on the interval I.

Theorem 3. *Assume that $f\colon I \to \mathbb{R}$ is a continuous function satisfying* (14) *and $f > 0$ on I. Then*

$$f(y) \geq e^{c(y-x)} f(x), \quad x, y \in I,\, x \leq y, \tag{16}$$

where $c \geq 0$ is given by

$$c := \inf\left\{ \frac{1}{(y-x)f(x)} \int_x^y f(t)\, dt,\ x, y \in I,\ x < y \right\}.$$

Proof. Since $f \geq 0$, then f is nondecreasing. Thus, by substituting $y = x + 2h$ in (14) we infer that

$$f(x+2h) \geq \int_x^{x+2h} f(t)\, dt + f(x) \geq 2chf(x) + f(x) = (1+2ch)f(x)\,,$$

for each $x \in I$ and $h > 0$ such that $x + 2h \in I$. From this we derive inductively that

$$f(x+2nh) \geq (1+2ch)^n f(x)\,, \quad x \in I\,,\ h > 0\,,\ n \in \mathbb{N}\,,\ x + 2nh \in I\,.$$

On replacing h by $\frac{1}{2n}h$ and tending with n to $+\infty$ we arrive at

$$f(x+h) \geq e^{ch} f(x)\,, \quad x \in I\,,\ h > 0\,,\ x + h \in I\,,]$$

which is equivalent to (16). This completes the proof. □

Remark 3. Under assumptions of the previous theorem, if f is additionally convex, then $c \geq 1$.

Now, we will show that an analogue to Theorem 2 for Riemann-integrable solutions of (4) is also true.

Theorem 4. *If $f\colon I \to \mathbb{R}$ is a Riemann-integrable solution of* (4) *then f satisfies the following functional-integral inequality:*

$$\frac{f(y) - f(x)}{y - x} \leq \frac{1}{y-x} \int_x^y f(t)\, dt\,, \quad x, y \in I\,,\ x < y\,. \tag{17}$$

Proof. Fix $n \in \mathbb{N}$, $x \in I$ and $h > 0$ such that $x + h \in I$. Then put $y := x + h$ in (4) to get

$$f(x+h) - f(x) \geq \frac{h}{2}\big[f(x+h) + f(x)\big]\,.$$

Now, on replacing x by $x+kh$ for $k=0,\dots,n$ we get

$$\begin{aligned} f(x+h)-f(x) &\geq \frac{h}{2}\big[f(x+h)+f(x)\big]\,, \\ f(x+2h)-f(x+h) &\geq \frac{h}{2}\big[f(x+2h)+f(x+h)\big]\,, \\ &\vdots \\ f(x+nh)-f\big(x+(n-1)h\big) &\geq \frac{h}{2}\big[f(x+nh)+f\big(x+(n-1)h\big)\big]\,. \end{aligned}$$

Sum up the above inequalities side by side to arrive at

$$f(x+nh)-f(x) \geq \frac{h}{2}\Big[2\sum_{k=0}^{n} f(x+kh)-f(x+nh)-f(x)\Big]\,, \quad n\in\mathbb{N}\,.$$

On replacing h by $\frac{1}{n}h$ and letting n tends to $+\infty$ we finally derive the inequality

$$f(x+h)-f(x) \leq \int_x^{x+h} f(t)dt.$$

Now, it is clear that f satisfies (17). $\square$

Remark 4. If f is a solution of (17) and has the following property:

$$\frac{1}{y-x}\int_x^y f(t)\,dt \leq \max\big\{f(x),f(y)\big\}\,, \quad x,y\in I\,, \tag{18}$$

then obviously

$$\frac{f(y)-f(x)}{y-x} \leq \max\big\{f(x),f(y)\big\}\,, \quad x,y\in I\,.$$

This inequality has been investigated by C. Alsina and J. L. Garcia-Roig [1, Remark 1]. They have shown that this inequality is equivalent to (4). Note that condition (18) is in particular satisfied by monotonic mappings and by convex mappings.

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