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A NEW APPROACH FOR DESCRIBING INSTANTANEOUS LINE CONGRUENCE

Rashad A. Abdel-Baky and Ashwaq J. Al-Bokhary

Abstract. Based on the E. Study's map, a new approach describing instantaneous line congruence during the motion of the Darboux frame on a regular non-spherical and non-developable surface, whose parametric curves are lines of curvature, is proposed. Afterward, the pitch of general line congruence is developed and used for deriving necessary and sufficient condition for instantaneous line congruence to be normal. In terms of this, the derived line congruences and their differential geometric invariants were examined.

1. Introduction

A set of two-parameter of lines in space is called a rectilinear line congruence (line congruence for short), where the normals to a surface constitute such a line congruence. Generally, the lines of a congruence are not normal to a surface. Hence, the line congruence of normals forms a special class; which is called normal line congruence. They were the first studied, particularly in investigations of the effects of reflection and refraction upon rays of light (geometric optic). The first purely mathematical treatment of general line congruence was given by Kummer in his memoir; Allgemeine Theorie der Gradlinigen Strahen system, Eisenhart [8]. The lines of a line congruence meet a given plane in such a way that through a point of the plane one line, or at most a finite member, pass. Similar results hold if a surface be taken instead of a plane; this surface is called the surface of reference or director surface of the line congruence. The lines of the line congruence which pass through a curve on the surface form a one-parameter family of lines in the space or ruled surface (parameter ruled surface).

An important analytical tool in the study of line trajectories is the introduction of dual numbers which were first introduced by Clifford [7], and rediscovered by Study [9, 13, 21]. A comprehensive analysis of dual numbers and their applications to the kinematics analysis of spatial linkages were conducted by Yang [23]. Bottema & Roth [6] introduced a treatment of theoretical kinematics using dual numbers. Dual numbers are extremely useful for spatial mechanisms, since there is a vast literatures on this branch of classical differential line geometry and spatial mechanisms (see

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for instance Refs. [14]–[21]). In two-parameter spatial motion, the trajectories of oriented lines embedded in a moving rigid body are line congruences. In kinematics, we are interested in studying the intrinsic properties of line trajectory based on the concepts of differential geometry. The applications of line geometry and dual number representation of line trajectories have been developed by Blaschke [5], Bottema & Roth [6], and Muller [17]. A more recent descriptions of this representation can be found in the works [3, 4], [19]–[20]. The dual number is used to recast the point displacement relationship into relationships of lines.

At present, the differential geometry of line congruence have been widely applied in design and manufacturing, (e.g. Computer Aided Geometric Design/Computer Aided Manufacturing) of products and many other areas such as motion analysis and simulation of rigid bodies and model-based object recognition systems [15]–[20].

The present work includes the following: In Section 2, we present, in brief, the basic definitions of the representation of oriented lines in terms of dual vectors and E. Study's map and the Darboux frame on a regular surface whose parametric curves are lines of curvature as well as line congruence. In Section 3, we show how to describe instantaneous line congruence by using the dual vector calculus. The pitch of general line congruence is developed and used for deriving necessary and sufficient condition for line congruence to be normal. In terms of this, in Kummer's sense, the derived instantaneous line congruences are investigated and the correspondences between their differential geometric invariants are examined.

2. Oriented lines with dual representation

The first application of dual numbers into line congruences is due to Blaschke [5], Bottema & Roth [6], and Muller [17]; since the fundamental idea is to replace points by lines as fundamental concepts of geometry. Points are then defined by the straight lines through them.

We start our discussion by reviewing some of the basic concepts of dual numbers. Dual numbers are the set of all pairs of real numbers written as

$$A = a + \varepsilon a^*, \quad a,\, a^* \in \mathbb{R}, \tag{2.1}$$

where the dual unit ε satisfies the relationships

$$\varepsilon \neq 0\,, \quad \varepsilon 1 = 1\varepsilon\,, \quad \varepsilon^2 = 0\,. \tag{2.2}$$

Given dual numbers $A = a + \varepsilon a^*$, and $B = b + \varepsilon b^*$ the rules for combination can be defined as:

$$\begin{cases} \text{Equality:} & A = B \Longleftrightarrow a = b, \quad a^* = b^*\,, \\ \text{Addition:} & A + B = (a + b) + \varepsilon(a^* + b^*)\,, \\ \text{Multiplication:} & AB = ab + \varepsilon(a^* b + ab^*)\,. \end{cases} \tag{2.3}$$

The set of dual numbers, denoted as D, forms a commutative group under addition. The associative laws hold for multiplication and dual numbers are distributive. As a result, the division of dual numbers is defined as:

$$\frac{A}{B} = \frac{a}{b} + \varepsilon(\frac{a^* b - ab^*}{b^2})\,, \quad b \neq 0\,. \tag{2.4}$$

A dual number is called a pure dual when

$$A = \varepsilon a^* . \tag{2.5}$$

Division by a pure dual number is not defined. An example of dual number is the dual angle between two skew lines in three-dimensional Euclidean space E^3 defined as:

$$\Theta = \theta + \varepsilon\theta^* , \tag{2.6}$$

where θ is projected angle between the lines and θ^* is the minimal distance between the lines along their common perpendicular line.

2.1. **E. Study's Map.**
An oriented line L in the three-dimensional Euclidean space E^3 can be determined by a point $\mathbf{p} \in L$ and a normalized direction vector $\mathbf{a}$ of L, i.e. $\|\mathbf{a}\| = 1$. To obtain components for L, one forms the moment vector

$$\mathbf{a}^* = \mathbf{p} \times \mathbf{a} , \tag{2.7}$$

with respect to the origin point in E^3. If $\mathbf{p}$ is substituted by any point

$$\mathbf{q} = \mathbf{p} + \lambda\mathbf{a} , \quad \lambda \in \Re,$$

on L, equation (2.7) implies that $\mathbf{a}^*$ is independent of $\mathbf{p}$ on L. The two vectors $\mathbf{a}$, and $\mathbf{a}^*$ are not independent of one another; they satisfy the following relationships:

$$\langle \mathbf{a}, \mathbf{a} \rangle = 1 , \quad \langle \mathbf{a}^*, \mathbf{a} \rangle = 0 . \tag{2.8}$$

The six components a_i, a_i^* $(i = 1, 2, 3)$ of $\mathbf{a}$ and $\mathbf{a}^*$ are called the normalized Plücker coordinates of the line L. Hence the two vectors $\mathbf{a}$ and $\mathbf{a}^*$ determine the oriented line L.

Conversely, any six-tuple a_i, a_i^* $(i = 1, 2, 3)$ with

$$a_1^2 + a_2^2 + a_3^2 = 1 , \quad a_1 a_1^* + a_2 a_2^* + a_3 a_3^* = 0 . \tag{2.9}$$

represents a line in the three-dimensional Euclidean space E^3. Thus, the set of all oriented lines in the three-dimensional Euclidean space E^3 is in one-to-one correspondence with pairs of vectors in E^3 subject to the relationships in equation (2.8).

For vectors $(\mathbf{a}^*, \mathbf{a}) \in E^3 \times E^3$ we define the set

$$D^3 = D \times D \times D = \{\mathbf{A} = \mathbf{a} + \varepsilon\mathbf{a}^*;\ \varepsilon \neq 0,\ \varepsilon 1 = 1\varepsilon,\ \varepsilon^2 = 0\} . \tag{2.10}$$

Then for any vectors $\mathbf{A}$ and $\mathbf{B}$ in D^3, the scalar product is defined by:

$$\langle \mathbf{A}, \mathbf{B} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \varepsilon\big(\langle \mathbf{a}^*, \mathbf{b} \rangle + \langle \mathbf{a}, \mathbf{b}^* \rangle\big) ,$$

and the norm of $\mathbf{A}$ is defined by:

$$\|\mathbf{A}\| = \|\mathbf{a}\| + \varepsilon \frac{\langle \mathbf{a}^*, \mathbf{a} \rangle}{\|\mathbf{a}\|} , \qquad \|\mathbf{a}\| \neq 0 . \tag{2.11}$$

Hence, we may write the dual vector $\mathbf{A}$ as a dual multiplier of a dual vector in the form

$$\mathbf{A} = \|\mathbf{A}\|\, \mathbf{U} , \tag{2.12}$$

where $\mathbf{U}$ is referred to as the axis. The ratio

$$P = \frac{\langle \mathbf{a}^*, \mathbf{a} \rangle}{\|\mathbf{a}\|^2}, \tag{2.13}$$

is called the pitch along the axis $\mathbf{U}$. If $P = 0$ and $\|\mathbf{a}\| = 1$, $\mathbf{A}$ is an oriented line, and when P is finite, $\mathbf{A}$ is a proper screw, and when P is infinite, $\mathbf{A}$ is called a couple. A dual vector with norm equal to unit is called a dual unit vector. From equation (2.13) it is easy to show that a dual unit vector satisfy the relationships in equation (2.11). Hence, each oriented line $L = (\mathbf{a}, \mathbf{a}^*) \in E^3$ is represented by dual unit vector

$$\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}^*. \tag{2.14}$$

The dual unit sphere in D^3 is defined as the following:

$$K = \left\{ \mathbf{A} \in D^3 \mid \|\mathbf{A}\|^2 = \langle \mathbf{a}, \mathbf{a} \rangle + \varepsilon \langle \mathbf{a}^*, \mathbf{a} \rangle = 1 \right\}. \tag{2.15}$$

It follows that relations (2.8) and (2.15) are corresponding. Via this we have the following map (Gugenheimer [9] E. Study's Map): The set of all oriented lines in Euclidean 3-space E^3 is in one-to-one correspondence with set of points of dual unit sphere in dual 3-space D^3.

This dualized form of line representation along with the E. Study's map leads to a new interpretation of the scalar and vectorial products of two lines. For two directed lines $\mathbf{X}$ and $\mathbf{Y}$ the dual angle $\Theta = \theta + \varepsilon\theta^*$combines the angle θ and the minimal distance θ^*. This gives rise to geometric interpretations of the following products of the dual unit vectors:

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \cos\Theta = \cos\theta - \varepsilon\theta^* \sin\theta. \tag{2.16}$$

The following special cases can be given:

(1) If $\langle \mathbf{X}, \mathbf{Y} \rangle = 0$, then $\theta = \frac{\pi}{2}$ and $\theta^* = 0$; this means that the two lines $\mathbf{X}$ and $\mathbf{Y}$ meet at right angle.

(2) If $\langle \mathbf{X}, \mathbf{Y} \rangle$ = pure dual, then $\theta = \frac{\pi}{2}$ and $\theta^* \neq 0$; the lines $\mathbf{X}$ and $\mathbf{Y}$ are orthogonal skew lines,

(3) If $\langle \mathbf{X}, \mathbf{Y} \rangle$ = pure real, then $\theta \neq \frac{\pi}{2}$ and $\theta^* = 0$; the lines $\mathbf{X}$ and $\mathbf{Y}$ are intersect.

(4) If $\langle \mathbf{X}, \mathbf{Y} \rangle = 1$, then $\theta = 0$ and $\theta^* = 0$; the lines $\mathbf{X}$ and $\mathbf{Y}$ are coincident (their directions are the same or opposite).

The vectorial product of $\mathbf{X}$ and $\mathbf{Y}$ is defined by

$$\mathbf{X} \times \mathbf{Y} = \mathbf{N} \sin\Theta, \tag{2.17}$$

where $\mathbf{N}$ represents a direct common perpendicular of the lines $\mathbf{X}$ and $\mathbf{Y}$, and the signs of θ and θ^* are related to the orientation of $\mathbf{N}$. If oriented lines $\mathbf{X}$ and $\mathbf{Y}$ meet at right angle, then

$$\mathbf{Z} = \cos\Phi \mathbf{X} + \sin\Phi \mathbf{Y}, \tag{2.18}$$

define a line which is the image of $\mathbf{X}$ under a helicial motion about the axis $\mathbf{X} \times \mathbf{Y}$ with dual angle Φ.

2.2. **Line congruence in Euclidean 3-space $\mathbf{E}^3$.**
As stated earlier, a line congruence is a two-parameter set of lines in E^3 described as lines through a surface $\mathbf{r} = \mathbf{r}(u_1, u_2)$ and in the direction of $\mathbf{e} = \mathbf{e}(u_1, u_2)$ is parametrized by [11, 15, 23]:

$$K\colon \mathbf{Y}(u_1, u_2, \mu) = \mathbf{r}(u_1, u_2) + \mu \mathbf{e}(u_1, u_2)\,, \quad \mu \in \mathbb{R}\,, \tag{2.19}$$

where $\mathbf{r} = \mathbf{r}(u_1, u_2)$ is its director surface and $\mathbf{e} = \mathbf{e}(u_1, u_2)$ is the unit vector along the direction of the generating lines of the congruence, i.e. $\|\mathbf{e}\| = 1$. μ is the parameter of its points indicating the singed distance of the corresponding point on $\mathbf{r} = \mathbf{r}(u_1, u_2)$. On each generator of K, there are two special real (complex identical) points, called focal points. The locus of these focal points are called the focal surfaces of the line congruence; in general there are two parts of the focal surfaces.

Let the vector function $\mathbf{r} = \mathbf{r}(u_1, u_2)$ represent a regular non-spherical and-non developable surface M, and suppose that the u_1-and u_2 curves of this parametrization are lines of curvature, i.e., the elements g_{12} and h_{12} of the first and second fundamental forms vanish identically ($g_{12}{=}h_{12} = 0$). Consider now the unit vectors $\mathbf{e}_1 = \mathbf{e}_1(u_1, u_2)$, $\mathbf{e}_2 = \mathbf{e}_2(u_1, u_2)$, are the tangents of the parametric curves $u_2 = \text{const.}$, $u_1 = \text{const.}$, and the unit vector $\mathbf{e}_3 = \mathbf{e}_3(u_1, u_2)$ of the normal to the surface M at any regular point, then we have [1, 2]:

$$\mathbf{e}_1 = \frac{1}{\sqrt{g_{11}}} \frac{\partial \mathbf{r}}{\partial u_1}\,, \quad \mathbf{e}_2 = \frac{1}{\sqrt{g_{22}}} \frac{\partial \mathbf{r}}{\partial u_2}\,, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2\,, \tag{2.20}$$

which are invariants vector functions on the surface. Using that u_1-and u_2 curves are curvature lines on the surface, we can calculate $ds = \sqrt{g_{11}}\, du_1$ and $d\bar{s} = \sqrt{g_{22}}\, du_2$, the arc length parameters of the curves $u_2 = \text{const.}$, $u_1 = \text{const.}$, respectively. The moving frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ on the surface M at every regular point is then called the Darboux frame. Hence, by means of the derivatives with respect to the arc length parameter of the curves $u_2 = \text{const.}$ with tangent $\mathbf{e}_1$ on M, the derivative formula with respect to the Darboux frame, may be stated as:

$$\frac{\partial}{\partial s}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} = \begin{pmatrix}0 & q & k\\ -q & 0 & 0\\ -k & 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}\,, \tag{2.21}$$

where $k = \frac{h_{11}}{g_{11}} = \left\langle \frac{\partial \mathbf{e}_1}{\partial s}, \mathbf{e}_3 \right\rangle$, and $q = \frac{-(g_{11})_{u_2}}{2g_{11}\sqrt{g_{22}}} = \left\langle \frac{\partial \mathbf{e}_1}{\partial s}, \mathbf{e}_2 \right\rangle$ are the normal and geodesic curvatures of the curves $u_2 = \text{const.}$, respectively. Similarly, the derivative formula of the Darboux frame of the curves $u_1 = \text{const.}$, with tangent $\mathbf{e}_2$ on M is

$$\frac{\partial}{\partial \bar{s}}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} = \begin{pmatrix}0 & \bar{q} & 0\\ -\bar{q} & 0 & \bar{k}\\ 0 & -\bar{k} & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}\,, \tag{2.22}$$

also $\bar{k} = \frac{h_{22}}{g_{22}} = \left\langle \frac{\partial \mathbf{e}_2}{\partial \bar{s}}, \mathbf{e}_3 \right\rangle$, and $\bar{q} = \frac{(g_{22})_{u_1}}{2g_{22}\sqrt{g_{11}}} = -\left\langle \frac{\partial \mathbf{e}_2}{\partial \bar{s}}, \mathbf{e}_1 \right\rangle$ have the same meaning as in (2.21), for the curves $u_1 = \text{const.}$ on the surface M. Here, g_{ij} and h_{ij} are the coefficients of the first and second fundamental forms of the surface M. We shall denote $\partial/\partial s$ and $\partial/\partial \bar{s}$ by the suffixes 1 and 2.

Since k, $\bar{k}$, and q, $\bar{q}$ are the invariant quantities of curvature on M, these invariants and their derivatives must fulfill the Gauss-Codazi equations [23]:

$$-q^2 + q_2 - k\bar{k} = \bar{q}_1 + \bar{q}^2\,, \quad q(\bar{k} - k) + k_2 = 0\,, \quad \bar{q}(\bar{k} - k) + \bar{k}_1 = 0\,. \tag{2.23}$$

3. Instantaneous line congruences

The relation between line geometry and kinematics in the Euclidean 3-space E^3 can be used for understanding the line congruences from a practical point of view. Then, according to E. Study's map equation (2.19) can be rewritten as:

$$\mathbf{E}(u_1, u_2) = \mathbf{e}(u_1, u_2) + \varepsilon\mathbf{r}(u_1, u_2) \times \mathbf{e}(u_1, u_2)\,. \tag{3.1}$$

Since the spherical image $\mathbf{e}(u_1, u_2)$ is a unit vector, then the dual vector $\mathbf{E}(u_1, u_2)$ also has unit magnitude as is seen from the computations:

$$\begin{aligned} \langle \mathbf{E}, \mathbf{E} \rangle &= \langle \mathbf{e} + \varepsilon\mathbf{r} \times \mathbf{e}, \mathbf{e} + \varepsilon\mathbf{r} \times \mathbf{e} \rangle \\ &= \langle \mathbf{e}, \mathbf{e} \rangle + 2\varepsilon\langle \mathbf{r} \times \mathbf{e}, \mathbf{e} \rangle + \varepsilon^2\langle \mathbf{r} \times \mathbf{e}, \mathbf{r} \times \mathbf{e} \rangle \\ &= \langle \mathbf{e}, \mathbf{e} \rangle = 1\,. \end{aligned} \tag{3.2}$$

Thus the line congruence fills a domain on dual unit sphere in D^3. Hence, the line congruence can be viewed as a two-dimensional surface in D^3-space. It follows that there are resemblances between theory of surfaces and theory of line congruences. This provides the ability to have the dual versions of the derivative formulae of (2.21), and (2.22), respectively, in the following forms:

$$\frac{\partial}{\partial s}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix} = \begin{pmatrix} 0 & q & k \\ -q & 0 & \varepsilon \\ -k & -\varepsilon & 0 \end{pmatrix}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}, \tag{3.3}$$

and

$$\frac{\partial}{\partial \bar{s}}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix} = \begin{pmatrix} 0 & \bar{q} & -\varepsilon \\ -\bar{q} & 0 & \bar{k} \\ \varepsilon & -\bar{k} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}. \tag{3.4}$$

From equations (3.3), and (3.4), we see that:

$$\begin{aligned} \boldsymbol{\Omega}(u_1, u_2) &= \omega + \varepsilon\omega^* = \varepsilon\mathbf{E}_1 - k\mathbf{E}_2 + q\mathbf{E}_3\,, \\ \bar{\boldsymbol{\Omega}}(u_1, u_2) &= \omega + \varepsilon\bar{\omega}^* = \bar{k}\mathbf{E}_1 + \varepsilon\mathbf{E}_2 + \bar{q}\mathbf{E}_3\,, \end{aligned} \tag{3.5}$$

are the instantaneous dual Darboux vectors along the curves $u_2 =$ const., $u_1 =$ const., respectively. These vectors allows to collect the derivations formulae (3.3), and (3.4) by:

$$\frac{\partial \mathbf{E}_i}{\partial s} = \boldsymbol{\Omega} \times \mathbf{E}_i\,, \quad \frac{\partial \mathbf{E}_i}{\partial \bar{s}} = \bar{\boldsymbol{\Omega}} \times \mathbf{E}_i \qquad i = 1, 2, 3\,. \tag{3.6}$$

Then

$$\begin{aligned}\mathbf{H}(u_1,u_2)&=\mathbf{h}+\varepsilon\mathbf{h}^*=\frac{\varepsilon\mathbf{E}_1-k\mathbf{E}_2+q\mathbf{E}_3}{\sqrt{k^2+q^2}}\,,\\ \bar{\mathbf{H}}(u_1,u_2)&=\bar{\mathbf{h}}+\varepsilon\bar{\mathbf{h}}^*=\frac{\bar{k}\mathbf{E}_1+\varepsilon\mathbf{E}_2+\bar{q}\mathbf{E}_3}{\sqrt{\bar{k}^2+\bar{q}^2}}\,,\end{aligned}\tag{3.7}$$

are the dual unit vectors along the instantaneous axes of the Darboux frame on the surface M for the curves $u_2=$ const., $u_1=$ const., respectively. Now, in view of equations (2.13), from (3.5) we have that:

$$P=\frac{\langle\omega^*,\omega\rangle}{\|\omega\|^2}=0\,,\qquad \bar{P}=\frac{\langle\bar{\omega}^*,\bar{\omega}\rangle}{\|\bar{\omega}\|^2}=0\,.\tag{3.8}$$

Equations (2.16), and (3.7) show that:

$$\begin{cases}\langle\mathbf{H},\mathbf{E}_1\rangle=\text{ pure dual}, & \langle\mathbf{H},\mathbf{E}_2\rangle=\text{ pure real }=\langle\mathbf{H},\mathbf{E}_3\rangle\,,\\ \langle\bar{\mathbf{H}},\mathbf{E}_2\rangle=\text{ pure dual}, & \langle\bar{\mathbf{H}},\mathbf{E}_1\rangle=\text{ pure real }=\langle\bar{\mathbf{H}},\mathbf{E}_3\rangle\,.\end{cases}\tag{3.9}$$

If we calculate the real and dual parts of equations (3.7), respectively, we have:

$$\begin{cases}\mathbf{h}=\dfrac{-k\mathbf{e}_2+q\mathbf{e}_3}{\sqrt{k^2+q^2}}\,,\\ \bar{\mathbf{h}}=\dfrac{\bar{k}\mathbf{e}_1+\bar{q}\mathbf{e}_3}{\sqrt{\bar{k}^2+\bar{q}^2}}\,,\end{cases}\tag{3.10}$$

and

$$\begin{cases}\mathbf{h}^*=\dfrac{\mathbf{e}_1-k\mathbf{e}_2^*+q\mathbf{e}_3^*}{\sqrt{k^2+q^2}}\,,\\ \bar{\mathbf{h}}^*=\dfrac{\bar{k}\mathbf{e}_1^*+\mathbf{e}_2+\bar{q}\mathbf{e}_3^*}{\sqrt{\bar{k}^2+\bar{q}^2}}\,.\end{cases}\tag{3.11}$$

Therefore, in view of (2.7), the position vectors of the intersection points of $\mathbf{H}$ with $\mathbf{E}_3$, and $\mathbf{E}_2$ are found as results of real and dual parts as follows:

$$\begin{cases}F\colon\mathbf{x}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\dfrac{1}{k}\mathbf{e}_3(u_1,u_2)\,, & k\neq 0\,,\\ C\colon\mathbf{z}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\dfrac{1}{q}\mathbf{e}_2(u_1,u_2)\,, & q\neq 0\,.\end{cases}\tag{3.12}$$

Similarly, the position vectors of the intersection points of $\bar{\mathbf{H}}$ with the oriented lines $\mathbf{E}_3$, and $\mathbf{E}_1$ are:

$$\begin{cases}\bar{F}\colon\bar{\mathbf{x}}(u_1,u_2)=\mathbf{r}(u_1,u_2)+\dfrac{1}{\bar{k}}\mathbf{e}_3(u_1,u_2)\,, & \bar{k}\neq 0\,,\\ \bar{C}\colon\bar{\mathbf{z}}(u_1,u_2)=\mathbf{r}(u_1,u_2)-\dfrac{1}{\bar{q}}\mathbf{e}_1(u_1,u_2)\,, & \bar{q}\neq 0\,.\end{cases}\tag{3.13}$$

In consequence, we need the following derivatives:

$$
(3.14)\qquad
\begin{cases}
\dfrac{\partial \mathbf{h}}{\partial s} = \mathbf{h}_1 = \dfrac{(q_1 k - q k_1)}{k^2+q^2}\Big(\dfrac{q\mathbf{e}_2 + k\mathbf{e}_3}{\sqrt{k^2+q^2}}\Big), \\[2ex]
\dfrac{\partial \mathbf{h}}{\partial \bar{s}} = \mathbf{h}_2 = \dfrac{k\bar{q}}{\sqrt{k^2+q^2}}\,\mathbf{e}_1 + k\Big(\dfrac{\bar{q}^2+\bar{q}_1}{k^2+q^2}\Big)\Big(\dfrac{q\mathbf{e}_2 + k\mathbf{e}_3}{\sqrt{k^2+q^2}}\Big), \\[2ex]
\dfrac{\partial \bar{\mathbf{h}}}{\partial s} = \bar{\mathbf{h}}_1 = \dfrac{\bar{q}\bar{k}(q^2-q_2)}{(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}}\,\mathbf{e}_1 + \dfrac{\bar{k}q}{\sqrt{k^2+q^2}}\,\mathbf{e}_2 - \dfrac{\bar{k}^2(q^2-q_2)}{(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}}\,\mathbf{e}_3\,, \\[2ex]
\dfrac{\partial \bar{\mathbf{h}}}{\partial \bar{s}} = \bar{\mathbf{h}}_2 = \dfrac{\bar{q}(\bar{q}\bar{k}_2 - \bar{q}_2\bar{k})}{(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}}\,\mathbf{e}_1 - \dfrac{\bar{k}(\bar{q}\bar{k}_2 - \bar{q}_2\bar{k})}{(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}}\,\mathbf{e}_3\,.
\end{cases}
$$

and

$$
(3.15)\qquad
\begin{cases}
\dfrac{\partial \bar{\mathbf{z}}}{\partial s} = \bar{\mathbf{z}}_1 = \dfrac{(\bar{q}^2+\bar{q}_1)}{\bar{q}^2}\mathbf{e}_1 - \dfrac{1}{\bar{q}}(q\mathbf{e}_2 + k\mathbf{e}_3)\,, \quad \dfrac{\partial \bar{\mathbf{z}}}{\partial \bar{s}} = \bar{\mathbf{z}}_2 = \dfrac{\bar{q}_2}{\bar{q}^2}\mathbf{e}_1\,, \\[2ex]
\dfrac{\partial \mathbf{z}}{\partial s} = \mathbf{z}_1 = -\dfrac{q_1}{q^2}\mathbf{e}_2\,, \quad \dfrac{\partial \mathbf{z}}{\partial \bar{s}} = \mathbf{z}_2 = -\dfrac{\bar{q}}{q}\mathbf{e}_1 + \dfrac{(q^2-q_2)}{q^2}\mathbf{e}_2 + \dfrac{\bar{k}}{q}\mathbf{e}_3\,, \\[2ex]
\dfrac{\partial \bar{\mathbf{x}}}{\partial s} = \bar{\mathbf{x}}_1 = (1-\dfrac{k}{\bar{k}})\mathbf{e}_1 - \dfrac{\bar{k}_1}{\bar{k}^2}\mathbf{e}_3\,, \quad \dfrac{\partial \bar{\mathbf{x}}}{\partial \bar{s}} = \bar{\mathbf{x}}_2 = -\dfrac{\bar{k}_2}{\bar{k}^2}\mathbf{e}_3\,, \\[2ex]
\dfrac{\partial \mathbf{x}}{\partial s} = \mathbf{x}_1 = -\dfrac{k_1}{k^2}\mathbf{e}_3\,, \quad \dfrac{\partial \mathbf{x}}{\partial \bar{s}} = \mathbf{x}_2 = \Big(1-\dfrac{\bar{k}}{k}\Big)\mathbf{e}_2 - \dfrac{k_2}{k^2}\mathbf{e}_3\,.
\end{cases}
$$

In view of equations (2.21), and (2.22). Hence, the unit normal vectors to the surfaces C, and $\bar{C}$ are:

$$
\frac{\bar{\mathbf{z}}_1 \times \bar{\mathbf{z}}_2}{\|\bar{\mathbf{z}}_1 \times \bar{\mathbf{z}}_2\|} = \mathbf{h} \qquad \text{and} \qquad \frac{\mathbf{z}_2 \times \mathbf{z}_1}{\|\mathbf{z}_2 \times \mathbf{z}_1\|} = \bar{\mathbf{h}}\,.
$$

It is known that the consecutive normals along a line of curvature on $M\colon \mathbf{r} = \mathbf{r}(u_1, u_2)$ intersect, the points of intersection being the corresponding center of curvature. The locus of the centers of curvature for all points of the surface M is called the surface of centers or centro-surface of M. In general it consists of two sheets, conjugated to the two families of lines of curvature and called focal surfaces of M. The normals to M are tangents to family of geodesics on each sheet of centro-surface. Thus according to Beltrami's theorem [23], the surface M is an involute of its centro-surface with respect to this family of geodesics, and the centro-surface is one sheet of the evolute with respect to M. The family of geodesic curves on one of the focal surface is edge of regression of the developable ruled surfaces generated by the normals along one family of lines of curvature on M. The orthogonal trajectories of the geodesics on the focal surface correspond to the lines of curvature on M along which one of the radii of normal curvature is constant. The locus of the centers of geodesic curvature of the line of curvature on M is the second sheet of evolute; this second sheet could be called complementary surface for M. Therefore, there are two complementary surfaces conjugates to the two families of lines of curvature on M, the position vectors of which are $\bar{\mathbf{z}} = \bar{\mathbf{z}}(u_1, u_2)$, and $\mathbf{z} = \mathbf{z}(u_1, u_2)$, corresponding to $\frac{1}{k}$, and $\frac{1}{k}$ constants, respectively. From the above discussions, we record the following theorem:

Theorem 3.1. *Under the above notations, the motion of the Darboux frame along a line of curvature on a regular non-spherical and non-developable surface in Euclidean 3-space E^3 is revolution. The instantaneous revolution axis of the Darboux frame and the instantaneous tangent of this line of curvature are orthogonal skew lines. Additionally, the instantaneous revolution axis of the Darboux frame satisfies the following:*

(i) *it normals to the complementary surface which is conjugate to the orthogonal path of this line of curvature,*

(ii) *it tangent to the focal surface which is conjugate to this line of curvature.*

4. Properties of instantaneous line congruences

Let us start again with the regular surface M defined at the beginning of our work. For this purpose, we are going to investigate in this paper the following: By the motion of the Darboux frame on the surface M, and according to equations (3.12), and (3.13), we can introduce the following line congruences:

$$\begin{cases} T\colon \mathbf{Y}(u_1, u_2, \mu) = \mathbf{x}(u_1, u_2) + \mu\mathbf{h}(u_1, u_2)\,, \\ \bar{T}\colon \bar{\mathbf{Y}}(u_1, u_2, \mu) = \bar{\mathbf{x}}(u_1, u_2) + \mu\bar{\mathbf{h}}(u_1, u_2)\,, \end{cases} \tag{4.1}$$

and

$$\begin{cases} Q\colon \mathbf{Y}(u_1, u_2, \mu) = \mathbf{z}(u_1, u_2) + \mu\mathbf{h}(u_1, u_2)\,, \\ \bar{Q}\colon \bar{\mathbf{Y}}(u_1, u_2, \mu) = \bar{\mathbf{z}}(u_1, u_2) + \mu\bar{\mathbf{h}}(u_1, u_2)\,. \end{cases} \tag{4.2}$$

In order to investigate the properties of the line congruence T at first, if the coefficients of the first and second fundamental forms of T, in the Kummer sense are e, f, g; a, b, b', c, then we have, from equations (3.14), (3.15), and with attention to Gauss-Coodazi equations, that:

$$\begin{cases} e := g_{11}\langle \mathbf{h}_1, \mathbf{h}_1\rangle = g_{11}\Big(\dfrac{q_1 k - q k_1}{k^2 + q^2}\Big)^2, \\ f := \sqrt{g_{11}g_{22}}\langle \mathbf{h}_1, \mathbf{h}_2\rangle = -\sqrt{g_{11}g_{22}}\dfrac{k(q_1 k - q k_1)(\bar{q}^2 + \bar{q}_1)}{(k^2 + q^2)^2}\,, \\ g := g_{22}\langle \mathbf{h}_2, \mathbf{h}_2\rangle = g_{22}\dfrac{k^2[\bar{q}^2(k^2 + q^2) + (\bar{q}^2 + \bar{q}_1)^2]}{(k^2 + q^2)^2}\,, \end{cases} \tag{4.3}$$

and

$$\left\{\begin{aligned} a &:= g_{11}\langle \mathbf{h}_1, \mathbf{x}_1\rangle = g_{11}\frac{k_1(q_1k - qk_1)}{k(k^2+q^2)^{\frac{3}{2}}}\,,\\ b &:= \sqrt{g_{11}g_{22}}\langle \mathbf{h}_1, \mathbf{x}_2\rangle = 0\,,\\ b' &:= \sqrt{g_{11}g_{22}}\langle \mathbf{h}_2, \mathbf{x}_1\rangle = -\sqrt{g_{11}g_{22}}\frac{k_1(\bar{q}^2+\bar{q}_1)}{(k^2+q^2)^{\frac{3}{2}}}\,,\\ c &:= g_{22}\langle \mathbf{h}_2, \mathbf{x}_2\rangle = 0\,. \end{aligned}\right. \tag{4.4}$$

The same arguments can be valid for the line congruence $\bar{T}$. Therefore, if $\bar{e}$, $\bar{f}$, $\bar{g}$; $\bar{a}$, $\bar{b}$, $\bar{b}'$, $\bar{c}$ are the coefficients of the first and second fundamental forms of $\bar{T}$, we have:

$$\left\{\begin{aligned} \bar{e} &= g_{11}\frac{\bar{k}^2[q^2(\bar{k}^2+\bar{q}^2)+(q^2-q_2)^2]}{(\bar{k}^2+\bar{q}^2)^2}\,,\\ \bar{f} &= \sqrt{g_{11}g_{22}}\frac{\bar{k}(\bar{q}_2\bar{k}-\bar{q}\bar{k}_2)(q^2-q_2)^2}{(\bar{k}^2+\bar{q}^2)^2}\,,\\ \bar{g} &= g_{22}\Big(\frac{\bar{q}_2\bar{k}-\bar{q}\bar{k}_2}{\bar{k}^2+\bar{q}^2}\Big)^2\,, \end{aligned}\right. \tag{4.5}$$

and

$$\left\{\begin{aligned} \bar{a} &= 0\,,\\ \bar{b} &= \sqrt{g_{11}g_{22}}\frac{\bar{k}_2(q^2-q_2)}{(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}}\,,\\ \bar{b}' &= 0\,,\\ \bar{c} &= -g_{22}\frac{\bar{k}_2(\bar{q}_2\bar{k}-\bar{q}\bar{k}_2)}{\bar{k}(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}}\,. \end{aligned}\right. \tag{4.6}$$

4.1. **Pitch of ruled surface belonging to line congruence.**

A relation such as $f(u_1, u_2) = 0$ between the parameters u_1, u_2 restricts the line congruence to a one-parameter set of lines, that is, to a ruled surface in the congruence. In (2.19), the equations

$$u_1 = u_1(t)\,, \quad u_2 = u_2(t)\,, \quad u_1'^2 + u_2'^2 \neq 0, \tag{4.7}$$

define a ruled surface belonging to these line congruences. Firstly, we can describe the ruled surface belonging to K by the equation

$$\mathbf{Y}\big(u_1(t), u_2(t), \mu\big) = \mathbf{r}\big(u_1(t), u_2(t)\big) + \mu\mathbf{e}\big(u_1(t), u_2(t)\big)\,, \quad t, \mu \in \mathbb{R}\,. \tag{4.8}$$

For this ruled surface

$$\mathbf{Y}\big(u_1(t+2\pi), u_2(t+2\pi), \mu\big) = \mathbf{Y}\big(u_1(t), u_2(t), \mu\big)\,, \tag{4.9}$$

can be taken as a condition for being closed. The pitch of this ruled surface is defined by

$$L_K = \oint d\mu = -\oint \langle d\mathbf{r}, \mathbf{e}\rangle \,. \tag{4.10}$$

An orthogonal trajectory starting from the point p_0 on a $\mathbf{e}$-generator intersects the same generator at another point p_1 which is generally different from p_0; this distance $L_K = p_0 p_1$ is called the pitch of parameter ruled surface in the congruence K. We may write

$$L_K = -\oint \Big\langle \frac{\partial \mathbf{r}}{\partial s}\, ds + \frac{\partial \mathbf{r}}{\partial \bar{s}}\, d\bar{s}, \mathbf{e}\Big\rangle \,, \tag{4.11}$$

or

$$L_K = -\oint \Big[\Big\langle \frac{\partial \mathbf{r}}{\partial s}, \mathbf{e}\Big\rangle\, ds + \Big\langle \frac{\partial \mathbf{r}}{\partial \bar{s}}, \mathbf{e}\Big\rangle\, d\bar{s}\Big] \,. \tag{4.12}$$

We found by application of Green's theorem, that:

$$L_K = \iint \Big[\frac{\partial}{\partial s}\Big(\Big\langle \frac{\partial \mathbf{r}}{\partial \bar{s}}, \mathbf{e}\Big\rangle\Big) - \frac{\partial}{\partial \bar{s}}\Big(\Big\langle \frac{\partial \mathbf{r}}{\partial s}, \mathbf{e}\Big\rangle\Big)\Big]\, ds\, d\bar{s} \,, \tag{4.13}$$

where the double integral is taken along the portion of the director surface bounded by the curve $\mathbf{r} = \mathbf{r}(u_1(t), u_2(t))$. Since $\frac{\partial}{\partial \bar{s}}\big(\frac{\partial \mathbf{r}}{\partial s}\big) = \frac{\partial}{\partial s}\big(\frac{\partial \mathbf{r}}{\partial \bar{s}}\big)$, we have:

$$L_K = \iint \Big[\Big\langle \frac{\partial \mathbf{r}}{\partial s}, \frac{\partial \mathbf{e}}{\partial \bar{s}}\Big\rangle - \Big\langle \frac{\partial \mathbf{r}}{\partial \bar{s}}, \frac{\partial \mathbf{e}}{\partial s}\Big\rangle\Big]\, ds\, d\bar{s} \,. \tag{4.14}$$

Hence, the following theorem can be given:

Theorem 4.1. *In Euclidean 3-space E^3, the pitch vanish for every parameter ruled surface belonging to the line congruence generated by an oriented line connected with the motion of the Darboux frame along a family of curves on the director surface if and only if the line congruence is normal line congruence.*

It is clear now from equations (3.14), and (3.15), for the line congruences T and $\bar{T}$, we have:

$$\begin{cases} L_T = -\displaystyle\iint_F \frac{k_1(\bar{q}^2 + \bar{q}_1)}{(k^2+q^2)^{\frac{3}{2}}}\, ds\, d\bar{s} \,, \\[2ex] L_{\bar{T}} = -\displaystyle\iint_{\bar{F}} \frac{\bar{k}_2(q^2 - q_2)}{(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}}\, ds\, d\bar{s} \,. \end{cases} \tag{4.15}$$

From these equations, we remark that:

$$\begin{cases} L_T = 0 \Leftrightarrow k_1(\bar{q}^2 + \bar{q}_1) = 0 \,, \\ L_{\bar{T}} = 0 \Leftrightarrow \bar{k}_2(q^2 - q_2) = 0 \,. \end{cases} \tag{4.16}$$

Now for the surface M, we have if $k_1 = 0$, $\bar{k}_2 \neq 0$, or $k_1 \neq 0$, $\bar{k}_2 = 0$, then the surface is canal surface. One of the family lines of curvature of a canal surface are plane curves, since they are circular. Therefore, the corresponding central surface of the canal surface becomes a curve and the congruences T, or $\bar{T}$ degenerates into a ruled surface. Therefore, we have to impose:

$$\begin{cases} L_T = 0 \Leftrightarrow (\bar{q}^2 + \bar{q}_1) = 0\,, \\ L_{\bar{T}} = 0 \Leftrightarrow (q^2 - q_2) = 0\,. \end{cases} \tag{4.17}$$

If we repeat the above discussions for the line congruences in (4.2), we have the same conditions. In addition, by comparing the equations (4.4), (4.6), and (4.16), we find consecutively that:

$$\begin{cases} L_T = 0 \Leftrightarrow k_1(\bar{q}^2 + \bar{q}_1) = 0 \Leftrightarrow b' \ = 0 = b\,, \\ L_{\bar{T}} = 0 \Leftrightarrow \bar{k}_2(q^2 - q_2) = 0 \Leftrightarrow \bar{b} \ = 0 = \bar{b}'\,. \end{cases} \tag{4.18}$$

Thus, we conclude that holds:

Theorem 4.2. *In Euclidean 3-space E^3, for the line congruences generated by the instantaneous revolution axes* $\mathbf{h}$*, and* $\bar{\mathbf{h}}$ *along the lines of curvature $u_2 =$ const., $u_1 =$ const., respectively, on a regular surface M, which is not a canal surface. The following are equivalent:*

(i) *the pitches of every parameter ruled surfaces belonging to these line congruences vanishes,*

(ii) *the line congruences are normal line congruences,*

(iii) *at each point on the surface M, the following*

$$\bar{q}^2 + \bar{q}_1 = 0\,, \quad q^2 - q_2 = 0\,, \tag{4.19}$$

are satisfied.

4.1.1. *Pitch of instantaneous isotropic line congruence.* If the line congruences in (4.1), (or (4.2)) are isotropic line congruences, then we have:

$$\begin{cases} g_{11}\dfrac{k_1(q_1 k - q k_1)}{k(k^2+q^2)^{\frac{3}{2}}} = 0\,, & \sqrt{g_{11}g_{22}}\dfrac{k_1(\bar{q}^2+\bar{q}_1)}{(k^2+q^2)^{\frac{3}{2}}} = 0\,, \\ \sqrt{g_{11}g_{22}}\dfrac{\bar{k}_2(q^2-q_2)}{(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}} = 0\,, & g_{22}\dfrac{\bar{k}_2(\bar{q}_2\bar{k} - \bar{q}\bar{k}_2)}{\bar{k}(\bar{k}^2+\bar{q}^2)^{\frac{3}{2}}} = 0\,. \end{cases} \tag{4.20}$$

in view of equations (4.4) and (4.6). The surface M, chosen to have the families of curvature lines as parametric curves, should obey to the condition $g_{11}g_{22} \neq 0$. Now from the equations (4.20), for the isotropic conditions, we have $k_1 = 0$, and $\bar{k}_2 = 0$ as common solution. On the contrary $k_1 \neq 0$, and $\bar{k}_2 \neq 0$ according to the Theorem 4.2. Thus, we conclude that holds:

Theorem 4.3. *In Euclidean 3-space E^3, for the line congruences generated by the instantaneous revolution axes* $\mathbf{h}$*, and* $\bar{\mathbf{h}}$ *along the lines of curvature $u_2 =$ const., $u_1 =$ const., respectively, on a regular surface M, which is a canal surface. The following are equivalent:*

(i) *The pitches of every parameter ruled surfaces belonging to these line congruences vanishes.*

(ii) *The line congruences are isotropic line congruences.*

4.1.2. *Instantaneous normal line congruence.* We now proceed to study the following line congruences:

$$\begin{cases} I\colon \mathbf{Y}(u_1,u_2,\mu)=\bar{\mathbf{z}}(u_1,u_2)+\mu\mathbf{h}(u_1,u_2)\,, \\ \bar{I}\colon \bar{\mathbf{Y}}(u_1,u_2,\mu)=\mathbf{z}(u_1,u_2)+\mu\bar{\mathbf{h}}(u_1,u_2)\,, \end{cases} \tag{4.21}$$

which can be introduce in view of the Theorem 3.1. For these line congruences, let us now derive the conditions under which their parameter ruled surfaces are principal ruled surfaces. Equivalently, for this purpose, we show when the parametric curves u_1, u_2 are lines of curvature on the surfaces $\bar{\mathbf{z}} = \bar{\mathbf{z}}(u_1,u_2)$, and $\mathbf{z} = \mathbf{z}(u_1,u_2)$, respectively. Therefore for our aim, respectively, we immediately derive from (3.14), and (3.15), that:

$$\begin{cases} k(q_1k-qk_1)(\bar{q}^2+\bar{q}_1)=0\,, & k\bar{q}_2(\bar{q}^2+\bar{q}_1)=0\,, \\ \bar{k}(\bar{q}\bar{k}_2-\bar{q}_2\bar{k})(q^2-q_2)=0\,, & \bar{k}q_1(q^2-q_2)=0\,. \end{cases} \tag{4.22}$$

Here, for the surface M, the conditions indicate that [1, 2]:

(i) If $k=0$ or $\bar{k}=0$, then the surface is a developable ruled surface;

(ii) if $q=0$ or $\bar{q}=0$, then the surface is a moulding surface;

(iii) if $q_1k-q_1=0$ or $\bar{q}\bar{k}_2-\bar{q}_2\bar{k}=0$, then the parametric curves are plane curves.

Then as in [1, 2], we can give the following theorem:

Theorem 4.4. *In Euclidean 3-space E^3, during the motion of the Darboux frame along the families of the lines of curvature $u_1 = const.$, and $u_2 = const.$, on the surface M which can neither be a developable ruled surface, nor a moulding surface or a canal surface. Let the two complementary surfaces $\bar{\mathbf{z}}=\bar{\mathbf{z}}(u_1,u_2)$, and $\mathbf{z}= \mathbf{z}(u_1,u_2)$, respectively, are director surfaces of the line congruences generated by the instantaneous revolution axes $\mathbf{h}$, and $\bar{\mathbf{h}}$. Then, the necessary and sufficient conditions that their parameter ruled surfaces are taken as principal ruled surfaces are:*

$$\bar{q}^2+\bar{q}_1=0,\ q^2-q_2=0\,. \tag{4.23}$$

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Department of Mathematics
Faculty of Science, University of Assiut
Assiut 71516, Egypt
E-mail: rbaky@hotmail.com

Department of Mathematics
Girls College of Education/Science Department
Um alqura University, Makkah, Saudi Arabia