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A NOTE ON A THEOREM OF MEGIBBEN

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Abstract. We prove that pure subgroups of thick Abelian p-groups which are modulo countable are again thick. This generalizes a result due to Megibben (Michigan Math. J. 1966). Some related results are also established.

Throughout this short paper, let G be an additively written Abelian p-group, where p is a prime number fixed for the duration, with socle $G[p] = \{g \in G : pg = 0\}$ and with first Ulm subgroup $G^1 = \cap_{i<\omega} p^i G$ where $p^i G = \{p^i g : g \in G\}$ is the subgroup of G consisting of all elements of G with height no less than i. We will also assume some rudimentary knowledge of the basic terminology in Abelian group theory. Nevertheless, for the convenience of these readers which are not familiar with the chief concepts, we shall give some details.

Following [1], a group G is said to be *thick* if there exists $t \in \mathbb{N}$ with the property $(p^t G)[p] \subseteq T$ whenever G/T is a direct sum of cyclic groups. This class of groups properly contains the class of quasi-complete groups and, in particular, torsion-complete groups. Moreover, it is easily checked that thick direct sums of cyclic groups are themselves bounded. Likewise, it is worthwhile noticing for a further application that divisible groups are thick and finite direct sums of thick groups are again thick (see, e.g., [3]).

Megibben established in ([6, Theorem 3.5]) the following result.

Theorem (Megibben, 1966). *If P is a pure and dense subgroup of the torsion-complete group G such that $|G/P| \leq \aleph_0$, then P is thick.*

Moreover, Wallace obtained in ([9, Theorem 1]) the following result.

Theorem (Wallace, 1971). *Let H be a totally projective subgroup of the reduced group G such that G/H is countable. Then G is totally projective.*

With the aid of Wallace theorem, we shall generalize here the preceding Megibben's assertion to arbitrary thick groups. Actually, we shall demonstrate much more by finding a suitable criterion for thickness of groups under countable pure extensions.

We can now prove a striking consequence of the last theorem.

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Corollary. *Suppose K is an isotype subgroup of the group G such that G/K is countable. If K is simply presented, then G is simply presented.*

Proof. One may decompose $G = G_d \oplus G_r$ and, by a reason of symmetry, $K = K_d \oplus K_r$ into divisible and reduced parts, respectively. Since $K/K_d = K/(K \cap G_d) \cong (K + G_d)/G_d \subseteq G/G_d$ is totally projective and $G/G_d/(K + G_d)/G_d \cong G/(K + G_d)$ is countable as an epimorphic image of the countable factor-group G/K, we plainly observe that Wallace's theorem works to infer that G/G_d is totally projective, i.e., that G is simply presented as claimed. $\square$

The following is elementary but for completeness of the exposition we include a proof.

Lemma. *If $E \leq A$ are Abelian p-groups and A/E is reduced, then A is reduced if and only if E is reduced.*

Proof. The necessity is straightforward.

As for the sufficiency, we observe that $(A_d + E)/E \subseteq (A/E)_d = \{0\}$ whence $A_d \subseteq E$. But it follows at once that $A_d = E_d = 0$ and we are finished. $\square$

We may also extend the latter theorem in a different way without K being isotype in G; for another general strengthening of Wallace theorem, see [4] as well. The proof of the next assertion requires the same sort of combinatorial game with quotients played in the proof of the previous claim. Actually, however, things are somewhat simpler.

Proposition. *Suppose H is a simply presented subgroup of the group G such that G/H is countable reduced. Then G is simply presented.*

Proof. Observe that $G/H_d/H/H_d \cong G/H$ is countable reduced, whence by the Lemma above we derive that G/H_d is reduced. Thus by Wallace theorem we have that G/H_d is totally projective since so is H/H_d. But H_d is a direct summand of G and hence it is readily checked that G is, in fact, simply presented, as desired. $\square$

Remark. If H is a direct sum of cocyclic groups and G/H is countable, then G is simply presented. Indeed, observe that $(G/H_d)_r/\big[(G/H_d)_r \cap H/H_d\big] \cong \big((G/H_d)_r + H/H_d\big)/H/H_d \subseteq G/H_d/H/H_d \cong G/H$ is countable. But $(G/H_d)_r \cap H/H_d$ is a direct sum of cyclic groups because H/H_d is a direct sum of cyclic groups and so, with Wallace theorem at hand, we obtain that $(G/H_d)_r$ is totally projective. So, G/H_d is simply presented. Since H_d is a direct summand of G we are done.

We claim that Wallace theorem is true even for simply presented groups with no any extra limitations (see [4]). However, a more global analysis of the Wallace's proof in [9] must be done (see [3] too).

We require one further technicality before proceeding to the proof of our main theorem. The requisite statement is the following (see [5] or [2], respectively).

Proposition (Keef, 1995 - Danchev, 2005)**.** *A group G is thick if and only if G/G^1 is thick.*

We have now at our disposal all the machinery needed to prove the following.

Theorem. *Let P be a pure subgroup of a group G such that G/P is either countable or a direct sum of cyclic groups. Then G is thick if and only if P is thick and G/P is a direct sum of a divisible group and a bounded group.*

Proof. Foremost, suppose that G/P is a direct sum of a divisible group and a bounded group. Since G/P is obviously thick, the sufficiency follows from ([8, Theorem 5.7]).

As for the necessity, suppose first that G/P is countable. Assume that P/C is a direct sum of cyclic groups, whence $P^1 \subseteq C$ and C is nice in P. Since $G/C/P/C \cong G/P$ is at most countable and P/C is isotype in G/C as separable and pure in G/C, we employ the Corollary to infer that G/C is simply presented. Therefore, owing to [7], we have $G/C/(G/C)^1 = G/C/\cap_{i<\omega}(p^iG + C)/C \cong G/\cap_{i<\omega}(p^iG + C)$ is a direct sum of cyclic groups. Thus, there is $t \in \mathbb{N}$ with $(p^tG)[p] \subseteq \cap_{i<\omega}(p^iG + C)$. Furthermore, with the classical modular law at hand, we calculate that $(p^tP)[p] \subseteq \big(\cap_{i<\omega}(p^iG + C)\big) \cap P = \cap_{i<\omega}\big((p^iG + C) \cap P\big) = \cap_{i<\omega}(p^iG \cap P + C) = \cap_{i<\omega}(p^iP + C) = \cap_{i<\omega}p^iP + C = P^1 + C = C$. That is why, appealing to the definition for thickness alluded to above, P has to be thick. Let now $G/P/L/P \cong G/L$ be a direct sum of cyclic groups for some L with $P \subseteq L \leq G$. Hence, there is $l \in \mathbb{N}$ with $(p^lG)[p] \subseteq L$. Therefore, since P is pure in G, it is easily verified that $\big(p^l(G/P)\big)[p] = \big((p^lG + P)/P\big)[p] = \big((p^lG)[p] + P\big)/P \subseteq L/P$. Hence G/P is also thick (see [3, Proposition 6.1] too). But countable thick groups are of necessity direct sums of divisible groups and bounded groups. In fact, if M is a countable thick Abelian p-group, then it follows from the second Proposition that M/M^1 is a thick direct sum of cyclic groups, thus bounded. Hence, M^1 is divisible and $M \cong M^1 \oplus (M/M^1)$, which substantiates our claim.

Second, suppose that G/P is a direct sum of cyclic groups. Then, we can write $G \cong P \oplus G/P$. Since direct summands of thick groups are again thick groups (see, for instance, [3]), we obtain that both P and G/P are thick. Thus, G/P must be bounded, and we are done. The converse is obvious. $\square$

An alternative proof of the structure of G/P in case that we have proved the same result for pure dense subgroups may be like this: Suppose B is a basic subgroup of G/P, and S is a subgroup of G containing P such that $S/P = B$. Then it is obvious that S is pure in G and G/S is both countable and divisible. Therefore, by our assumption, S must be thick. Now, P is pure in S and S/P is a direct sum of cyclic groups. Consequently, there is a subgroup $F \cong B$ of S such that $S \cong P \oplus F$. It follows that P is thick, since it is easily checked that the class of thick groups is closed under summands (see, for example, [3]). It also follows that $F \cong B$ is bounded since F is a thick direct sum of cyclic groups. Hence G/P must, indeed, be a direct sum of a divisible and a bounded group.

We close with the following (see [3, Problem 6.1] as well)

Problem. Does it follow that the Theorem is true when either P is not pure in G or G/P is an uncountable group which is not a direct sum of cyclic groups? Moreover, whether the same type result follows for essentially finitely indecomposable (efi) groups?

We conjecture that the answer is, in general, no. However, imitating our proof of the Theorem, we observe that the same claim is true provided that P is a pure and nice subgroup of G such that G/P is simply presented. In fact, using the same notations, P/C should be balanced in G/C and hence $G/C/P/C \cong G/P$ being simply presented will imply that $G/C \cong (P/C) \oplus (G/P)$ is also simply presented. Henceforth, the proof goes on in the same manner because simply presented groups have first Ulm factor which is a direct sum of cyclic groups (e.g., [7]).

In closing, we shall also comment some problems on Abelian p-groups posed in [3].

In fact, Problems 5.1 and 5.2 on p. 130 have both negative answer. Indeed, to see this, suppose G is any separable group with torsion completion T. If C is a direct sum of cyclic groups and P is a pure subgroup of C such that $C/P \cong T$ (i.e., $P \to C \to T$ is a pure-projective resolution of T), then let K be the kernel of the obvious map $G \oplus C \to T$ which is actually surjective. Since, for every natural n, $C[p^n]$ maps onto $T[p^n]$ because of the purity of P, the same holds for $(G \oplus C)[p^n] \to T[p^n]$. That means K is a pure subgroup of $G \oplus C$. In addition, since the map is injective on G, K has trivial intersection with G, so that the projection of K onto C is injective, that is K must also be a direct sum of cyclic groups. This gives a whole bunch of examples that, because of the arbitrariness of G, there exists a non semi-complete group $A = G \oplus C$ (for instance, when G is not semi-complete) with a pure subgroup K which is a direct sum of cyclic groups such that $A/K \cong T$ is torsion-complete.

Moreover, Problem 6.1 on p. 132 has a negative settling as well. In fact, the requirement in Proposition 6.2 on p. 131 on A/G to be a direct sum of cyclic groups is essential and cannot be dropped off. Indeed, owing to ([5, Proposition 20]) there is a pure-exact sequence

$$0 \to G \to A \to \bar{B} \to 0\,,$$

where both B and G are unbounded countable direct sum of cyclic groups, $\bar{B}$ is the torsion completion of B and A is thick. Therefore, both A and $A/G \cong \bar{B}$ are thick, but G is not strongly thick in A since it is not even a thick group, and, in fact, it is an unbounded direct sum of cyclic groups.

Continuing the last example, note that G is a balanced subgroup A because G is separable and pure in A, whence G is isotype in A, and A/G is separable, whence G is nice in A. But A is thick, whereas G is far from thick (i.e., it is hardly thick) being an unbounded direct sum of cyclic groups. Thus, Question 6.1 on p. 134 possesses a negative answering, too.

Finally, we shall prove a claim stated on p. 127 which says that $\aleph_1$-Σ-cyclic p-groups are precisely the separable p-groups, i.e., the p-groups without elements of infinite height. In fact, it is obvious that separable p-groups are $\aleph_1$-Σ-cyclic. Conversely, if A is any group that has an element y of infinite order, then for all $n < \omega$ let x_n satisfy $p^n x_n = y$. Then $G = \langle y, x_1, \ldots, x_n, \ldots \rangle$ is a countable subgroup of A with an element of infinite height, and so it is not a direct sum of cyclic groups, so that A is not $\aleph_1$-Σ-cyclic. This gives the desired conclusion.

By the same token, $G \leq A$ is strongly $\aleph_1$-Σ-cyclic if and only if $G \cap p^\omega A = 0$. Indeed, if $0 \neq x \in G \cap p^\omega A$ and $C = \langle x \rangle$, then C is countable but not a strongly direct sum of cyclic groups in A, so that G is not strongly $\aleph_1$-Σ-cyclic in A.

Reciprocally, if $G \cap p^\omega A = 0$ and C is a countable subgroup of G, then C is the ascending union of a collection of finite subgroups $\{C_i\}_{i<\omega}$. By repeating terms in this union enough times, we may assume that for each $i < \omega$, $p^i A \cap C_i = 0$, which proves that G is strongly $\aleph_1$-Σ-cyclic in A.

A slight extension of these arguments will show that a group with a nonzero element of infinite height will necessarily have a countable isotype subgroup which is not a direct sum of cyclic groups.

Correction. In ([5, p. 3622, line 15]) the phrase "$\ldots G$ itself is not efi" should be written and read as "$\ldots G[p]$ itself is not efi". Moreover, although it is clear from the context, in ([3, p. 136, line 4] and [3, p. 138, line 4 of the proof of Theorem 8.1]), the expression $\left|[\cap_{i<\omega}(p^i A+T)]/T\right|$ should be written and read as $\left|[\cap_{i<\omega}(p^i A+T)]/G\right|$.

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