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# AN EVALUATING CHARACTERIZATION OF HOMOMORPHISMS

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Abstract. This work provides an evaluating complete description of positive homomorphisms on an arbitrary algebra of real-valued functions.

## 1. Introduction

The books [5, 7] are adopted in this paper as the unique sources of unexplained terminology and notation. In spite of that, efforts are made to make this short note reasonably self-containing and accessible to readers with a standard first-year graduate background.

The first paragraph of this section is devoted to some preliminaries on algebras of real-valued functions. The symbol $\mathbb{R}^X$ is used throughout to indicate the set of all real-valued functions on a nonvoid set $X$. Under the usual pointwise operations, $\mathbb{R}^X$ is a commutative and associative algebra over the real field $\mathbb{R}$. If $X$ in addition is a topological space, then $C(X)$ denotes the subalgebra of $\mathbb{R}^X$ of all continuous functions on $X$. The constant function on $X$ whose value is the scalar $r$ is denoted again by $r$. Thus, 1 is a multiplicative unity in $\mathbb{R}^X$ as well as in $C(X)$. Below, $A$ stands for an arbitrary subalgebra of $\mathbb{R}^X$. If $A$ contains the constant functions, then $A$ is said to be *bounded inversion closed* if $f \in A$ has an inverse in $A$ whenever $f \geq \mu$ for some $\mu \in (0, \infty)$. Also, if $1 \in A$ then $A$ is known as an *inversion closed* subalgebra of $\mathbb{R}^X$ whenever $f \in A$ and $f(x) \neq 0$ for all $x \in X$ imply that $f$ has an inverse in $A$. Next, we shift our emphasis from algebras of real-valued functions to linear functionals on them.

A nonzero linear functional $\mathfrak{t}$ on $A$ is called a *homomorphism* on $A$ if $\mathfrak{t}$ is multiplicative, that is to say, $\mathfrak{t}(fg) = \mathfrak{t}(f)\mathfrak{t}(g)$ for all $f$, $g \in A$. On the other hand, the linear functional $\mathfrak{t}$ on $A$ is said to be *positive* if $\mathfrak{t}(f) \geq 0$ for all $f \in A$ with $f \geq 0$. Now, let $\mathfrak{m}$ be a natural number and define the linear functional $\mathfrak{t}$ on $A$ to be $\mathfrak{m}$-*evaluating* if for every subset $B$ of $A$ with cardinality $\mathfrak{m}$ there is a point $x_B \in X$ such that $\mathfrak{t}(f) = f(x_B)$ for all $f \in B$. Similarly, the linear functional $\mathfrak{t}$ is said to be *countably evaluating* if for every countable subset $B$ of $A$ there exists a point $x_B \in X$ such that $\mathfrak{t}(f) = f(x_B)$ for all $f \in B$. On the other hand,

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the linear functional $\mathfrak{t}$ on $A$ is said to be $\overline{\mathfrak{m}}$-*evaluating* if for every subset $B$ of $A$ with cardinality $\mathfrak{m}$ and for every $\varepsilon \in (0,\infty)$ there exists a point $x_{\varepsilon,B} \in X$ such that $\left|f(x_{\varepsilon,B}) - \mathfrak{t}(f)\right| < \varepsilon$ for all $f \in B$. Obviously, any $\mathfrak{m}$-evaluating linear functional on $A$ is $\overline{\mathfrak{m}}$-evaluating and any $\overline{\mathfrak{m}}$-evaluating linear functional on $A$ is positive. For more informations and a vast literature about evaluating properties of linear functionals on algebras of real-valued functions, the reader is encouraged to consult the remarkable survey [7, Chapter IV] by Kriegl and Michor. We are in position now to introduce the problem we discuss in this work.

Garrido, Gómez, and Jaramillo proved in [4] that if $A$ contains the constant functions and is bounded inversion closed, then any homomorphism on $A$ is $\overline{\mathfrak{1}}$-evaluating. Also, if $A$ is inversion closed then any homomorphism on $A$ is $\mathfrak{1}$-evaluating. These implications are not reversible, however. To see this, consider the linear functional $\mathfrak{t}$ defined on $C\big([0,1]\big)$ by $\mathfrak{t}(f) = \int_0^1 f(x)\mathrm{d}x$ for all $f \in A$. The classical Intermediate Value Theorem implies directly that $\mathfrak{t}$ is $\mathfrak{1}$-evaluating but it is far from being a homomorphism, of course. Also, the condition of (bounded) inversion closure cannot be dropped. Indeed, assume $A$ to be the subalgebra of $\mathbb{R}^{(0,\infty)}$ such that if $f \in \mathbb{R}^{(0,\infty)}$, then $f \in A$ if and only if there exists a real polynomial $p_f$ and a point $x_f \in X$ for which $f(x) = p_f(x)$ for all $x \in X$ with $x \geq x_f$. Observe now that the linear functional $\mathfrak{t}$ on $A$ defined by $\mathfrak{t}(f) = p_f(0)$ for all $f \in A$ is a homomorphism on $A$. However, there is no natural number $\mathfrak{m}$ for which $\mathfrak{t}$ is $\mathfrak{m}$-evaluating because $\mathfrak{t}$ is not positive. In other words, neither the conditions of being $\overline{\mathfrak{1}}$-evaluating nor of being $\mathfrak{1}$-evaluating characterize positive homomorphisms on subalgebras of $\mathbb{R}^X$. The main objective of this work is to provide an evaluating characterization of positive homomorphisms on the arbitrary subalgebra $A$ of $\mathbb{R}^X$. A short synopsis of the content of this paper seems to be in order.

Let $A$ be an arbitrary subalgebra of $\mathbb{R}^X$ and $\mathfrak{t}$ be a linear functional on $A$. The main result of this note asserts that $\mathfrak{t}$ is a positive homomorphism if and only if $\mathfrak{t}$ is $\overline{\mathfrak{2}}$-evaluating. Two particular cases are then discussed. Recall from [3] that a function $f \in A$ is said to have a *quasi-inverse* in $A$ if the equality $fg = f + g$ holds for some function $g \in A$. Say that $A$ is *bounded quasi-inversion closed* if $f \in A$ and $|1 - f| \geq \mu$ for some $\mu \in (0,\infty)$ imply that $f$ has a quasi-inverse in $A$. Furthermore, $A$ is said to be *quasi-inversion closed* if $f \in A$ has a quasi-inverse whenever $f(x) \neq 1$ for all $x \in A$. A short's moment thought reveals that if $1 \in A$ then $A$ is (bounded) quasi-inversion closed if and only if $A$ is (bounded) inversion closed. As consequences of the central theorem of this paper, we prove that if $A$ is bounded quasi-inversion closed (respectively, quasi-inversion closed), then $\mathfrak{t}$ is a homomorphism if and only if $\mathfrak{t}$ is $\overline{\mathfrak{2}}$-evaluating (respectively, $\mathfrak{2}$-evaluating). These facts turn out to be extensions of the aforementioned results by Garrido, Gómez, and Jaramillo. The final result of the paper is topological in nature. More details are given in the last paragraph of this section.

Assume $X$ to a topological space. Recall from [8] that $X$ is said to be *Lindelöf* if from every open cover of $X$, that is, a family of open sets whose union equals $X$, one can select a countable subfamily whose union also equals $X$. In [1], Biström, Bjon, and Lindström proved that if $A$ is a point-separating unital subalgebra of $C(X)$ with $X$ Lindelöf, then any countably evaluating homomorphism on $A$ is evaluating (see also [2] by Blasco). The central result of this note allows us to generalize this fact in two directions and in an elementary way. Indeed, we prove that if $X$ is Lindelöf then any countably evaluating linear functional on any subalgebra of $C(X)$ is evaluating.

## 2. The results and their proofs

This section begins straightforwardly with the main result of this note since all the needed ingredients are gathered in the above preliminaries.

**Theorem 1.** *Let $X$ be a nonvoid set and $A$ be a subalgebra of $\mathbb{R}^X$. A linear functional $\mathfrak{t}$ on $A$ is a positive homomorphism if and only if $\mathfrak{t}$ is $\overline{2}$-evaluating.*

**Proof.** First, assume that $\mathfrak{t}$ is a positive homomorphism on $A$. Choose $e$ in $A$ such that $\mathfrak{t}(e) \neq 0$ and observe that $\mathfrak{t}(e^2) = \mathfrak{t}(e)^2 > 0$. Such a function $e$ exists in $A$ because $\mathfrak{t}$ does not vanish. Now, let $f, g \in A$ and $\varepsilon \in (0, \infty)$. Clearly, if

$$h = \varepsilon^{-1}\left(\left(f - \mathfrak{t}(f)\right)^2 + \left(g - \mathfrak{t}(g)\right)^2\right) e\,,$$

then $h \in A$ and $\mathfrak{t}(h) = 0$. It follows that

$$\mathfrak{t}(h^2 - e^2) = -\mathfrak{t}(e^2) < 0\,.$$

Since $\mathfrak{t}$ is positive, the inequality $h(x)^2 - e(x)^2 < 0$ holds for some point $x \in X$. But then $e(x) \neq 0$ and

$$\left(f(x) - \mathfrak{t}(f)\right)^2 + \left(g(x) - \mathfrak{t}(g)\right)^2 = \left(e(x)\right)^{-1} h(x)\varepsilon < \varepsilon\,.$$

This yields that $\overline{2}$-evaluating.

Conversely, suppose $\mathfrak{t}$ to be $\overline{2}$-evaluating. So $\mathfrak{t}$ is positive. We claim that $\mathfrak{t}$ is a homomorphism on $A$. To this end, let $f \in A$ and $\varepsilon \in (0, \infty)$. There exists a point $x \in X$ such that

$$\left|f^2(x) - \mathfrak{t}(f^2)\right| < \varepsilon \quad \text{and} \quad \left|f(x) - \mathfrak{t}(f)\right| < \varepsilon\,.$$

Accordingly,

$$\begin{aligned} \left|\mathfrak{t}(f^2) - \left(\mathfrak{t}(f)\right)^2\right| &= \left|\mathfrak{t}(f^2) - f^2(x) + \left(f(x)\right)^2 - \mathfrak{t}(f)^2\right| \\ &\leq \left|\mathfrak{t}(f^2) - f^2(x)\right| + \left|\left(f(x)\right)^2 - \mathfrak{t}(f)^2\right| \\ &\leq \varepsilon + \varepsilon\left|f(x) + \mathfrak{t}(f)\right| \leq \varepsilon\left(1 + \varepsilon + 2|\mathfrak{t}(f)|\right). \end{aligned}$$

It follows that $\mathfrak{t}(f^2) = \mathfrak{t}(f)^2$ as $\varepsilon$ is arbitrary in $(0, \infty)$. Finally, if $f, g \in A$ then the latter result together with the identity $4fg = (f+g)^2 - (f-g)^2$ leads directly to $\mathfrak{t}(fg) = \mathfrak{t}(f)\mathfrak{t}(g)$. This completes the proof. □

Next, we show that the situation in Theorem 1 improves if $A$ is assumed to be bounded quasi-inversion closed. First of all, notice that if $\mathfrak{t}$ is a homomorphism on $A$ and $f \in A$ has a quasi-inverse in $A$, then $\mathfrak{t}(f) \neq 1$. This obvious observation turns out to be useful for later purposes and is used there without further mention.

**Corollary 2.** *Let $X$ be a nonvoid set and $A$ be a bounded quasi-inversion closed subalgebra of $\mathbb{R}^X$. A linear functional $\mathfrak{t}$ on $A$ is a homomorphism if and only if $\mathfrak{t}$ is $\overline{2}$-evaluating.*

**Proof.** By Theorem 1, it suffices to show that if $\mathfrak{t}$ is a homomorphism on $A$ then $\mathfrak{t}$ is positive. To do this, let $f \in A$ with $f \geq 0$ and $\mathfrak{t}(f) \neq 0$. Put $g = \left(\mathfrak{t}(f)\right)^{-1} f$ and notice that $\mathfrak{t}(g) = 1$. It follows that $g$ has no quasi-inverse in $A$. Hence, fix $\varepsilon \in (0, \infty)$ and observe that there exists $x \in X$ such that $\left|1 - g(x)\right| < \left|\mathfrak{t}(f)\right|^{-1} \varepsilon$. We derive that $\left|\mathfrak{t}(f) - f(x)\right| < \varepsilon$, so $-\varepsilon < \mathfrak{t}(f)$ for all $\varepsilon \in (0, \infty)$. This yields that $\mathfrak{t}(f) \geq 0$ and we are done. $\square$

Isbell in [6] showed that any homomorphism on a bounded inversion closed (and then unital) subalgebra of $\mathbb{R}^X$ is positive. Thus, Corollary 2 is in part an extension of Isbell's well-know result.

A characterization of homomorphisms on quasi-inversion closed subalgebras of $\mathbb{R}^X$ is furnished next.

**Corollary 3.** *Let $X$ be a nonvoid set and $A$ be a quasi-inversion closed subalgebra of $\mathbb{R}^X$. A linear functional $\mathfrak{t}$ on $A$ is a homomorphism if and only if $\mathfrak{t}$ is 2-evaluating.*

**Proof.** Assume $\mathfrak{t}$ to be a homomorphism on $A$. Since $\mathfrak{t}$ is non-zero, there exists $e \in A$ such that $\mathfrak{t}(e) \neq 0$. Now, choose $f, g \in A$ and set

$$h = 2\left(\mathfrak{t}(f)f + \mathfrak{t}(g)g + \mathfrak{t}(e)e\right) - \left(f^2 + g^2 + e^2\right).$$

Hence,

$$\mathfrak{t}(h) = \mathfrak{t}(f^2 + g^2 + e^2) = \mathfrak{t}(f)^2 + \mathfrak{t}(g)^2 + \mathfrak{t}(e)^2 \geq \mathfrak{t}(e)^2 > 0\,.$$

Accordingly, we can write $\mathfrak{t}\left(\left(\mathfrak{t}(f^2 + g^2 + e^2)\right)^{-1} h\right) = 1$. But then the function $\left(\mathfrak{t}(f^2 + g^2 + e^2)\right)^{-1} h$ has no quasi-inverse in $A$. As $A$ is quasi-inversion closed, there exists $x \in X$ such that $\left(\mathfrak{t}(f^2 + g^2 + e^2)\right)^{-1} h(x) = 1$. It follows that $h(x) = \mathfrak{t}(f^2 + g^2 + e^2)$ and thus,

$$\left(f(x) - \mathfrak{t}(f)\right)^2 + \left(g(x) - \mathfrak{t}(g)\right)^2 + \left(e(x) - \mathfrak{t}(e)\right)^2 = 0\,.$$

Therefore, $\mathfrak{t}(f) = f(x)$ and $\mathfrak{t}(g) = g(x)$, meaning that $\mathfrak{t}$ is 2-evaluating.

Notice finally that the converse follows straightforwardly from the previous corollary. $\square$

The result of Corollary 3 fails if $A$ is not assumed to be quasi-inversion closed. Indeed, let $A$ denote the subalgebra of $\mathbb{R}^{(0,\infty)}$ such that $f \in \mathbb{R}^{(0,\infty)}$ is a member of $A$ if and only if $f$ is bounded and $f(x)$ has a finite limit as $x$ approaches $\infty$. We check easily that $A$ is bounded inversion closed but not inversion closed. Now, let $\mathfrak{t}$ be the homomorphism on $A$ defined by $\mathfrak{t}(f) = \lim_{\infty} f$ for all $f \in A$. Clearly, $\mathfrak{t}$ is not 1-evaluating and — a fortiori — $\mathfrak{t}$ is not 2-evaluating.

We arrive at this point to the topological application of Theorem 1, which is the final result of this note.

**Corollary 4.** *Let $X$ be a Lindelöf topological space and $A$ be a subalgebra of $C(X)$. Then any countably evaluating linear functional on $A$ is evaluating.*

**Proof.** Let $\mathfrak{t}$ be a finitely countably evaluating linear functional on $A$. From Theorem 1, it follows quickly that $\mathfrak{t}$ is a homomorphism on $A$. Our next goal is to show that $\mathfrak{t}$ is evaluating. Arguing by contradiction, assume that for every $x \in X$ there exists $f_x \in A$ such that $\mathfrak{t}(f_x) = 0$ and $f_x(x) \neq 0$. Hence, for each $x \in X$

there is an open set $\omega_x$ such that $f_x(y) \neq 0$ for all $y \in Y$. The family $(\omega_x)_{x\in X}$ is an open cover of $X$. There exists therefore a countable subset $Y$ of $X$ such that the family $(\omega_y)_{y\in Y}$ is again an open cover of $X$. Since $\mathfrak{t}$ is countably evaluating, there exists $x \in X$ such that $f_y(x) = \mathfrak{t}(f_y) = 0$ for all $y \in Y$. Choose $y \in Y$ so that $x \in \omega_y$ and observe that $f_y(x) \neq 0$. This leads to a contradiction and the corollary follows. $\square$

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