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RULED W-SURFACES IN MINKOWSKI 3-SPACE $\Re_1^3$

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Abstract. In this paper, we study a spacelike (timelike) ruled W-surface in Minkowski 3-space which satisfies nontrivial relation between elements of the set $\{K,\ K_{II},\ H,\ H_{II}\}$, where (K,H) and (K_{II},H_{II}) are the Gaussian and mean curvatures of the first and second fundamental forms, respectively. Finally, some examples are constructed and plotted.

1. Introduction

Let $\Re^3 = \{(x_i) \mid x_i \in \Re,\ i = 1,2,3\}$ be a 3-dimensional vector space, $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$ be two vectors in $\Re^3$. The Lorentz scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 \,.$$

The space $\Re_1^3 = \big(\Re^3, \langle\ ,\ \rangle\big)$ is called a Lorentz 3-space or Minkowski 3-space. The Lorentz vector product of the vectors $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2)\,.$$

This yields

$$\mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_3,\ \ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1,\ \ \mathbf{e}_3 \times \mathbf{e}_1 = -\mathbf{e}_2\,,$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, are the base of the space $\Re_1^3$. A vector $\mathbf{x}$ in $\Re_1^3$ is spacelike, lightlike (isotropic) or timelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, respectively. Moreover, for $\mathbf{x} \in \Re_1^3$ the norm is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$, then the vector $\mathbf{x}$ is called a spacelike and a timelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = +1$ and -1 respectively [1]. The unit spheres in Minkowski 3-space $\Re_1^3$ are defined by

$$S_1^2 = \left\{\mathbf{x} \in \Re_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = 1\right\}\ ,$$

$$H_0^2 = \left\{\mathbf{x} \in \Re_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = -1, x_1 > 0\right\}\ .$$

It is well known that the set of all Lorentzian transformations which preserve the Lorentzian metric form a group which is called Lorentzian group. A transformation in this group which lies in the connected component of the identity transformation is called a proper rotation and is denoted by A where $\det A = 1$ [4, 10]. Using linear algebra concepts, it is easy to see that the proper rotation preserves a fixed vector

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invariant which is called the axis of rotation with causal character in $\Re^3_1$. Depending on the causal character of the axis of rotation i.e., being timelike, spacelike, there are two types of Lorentzian group of transformations generated by:

$$\bar{X} = A_1 X \,, \qquad \text{(elliptic group)}, \tag{1.1}$$

$$\bar{X} = A_2 X \,, \qquad \text{(hyperbolic group)}, \tag{1.2}$$

where,

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos u^1 & -\sin u^1 \\ 0 & \sin u^1 & \cos u^1 \end{pmatrix} \quad A_2 = \begin{pmatrix} \cosh u^1 & \sinh u^1 & 0 \\ \sinh u^1 & \cosh u^1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

$$\mathbf{X} = (X_i)^t \;, \quad \overline{\mathbf{X}} = \left(\overline{X}_i\right)^t \,.$$

Similarly, as for Euclidean 3-space E^3 we could define a Lorentzian screw motion as a Lorentzian rotation around an axis with causal character together with a translation in the direction of the axis. Using (1.1) and (1.2) , we give Lorentzian screw motion as a non-trivial one-parameter group of Lorentzian transformations as the following :

$$\overline{X} = A_1 X + T_1 \,, \tag{1.3}$$

$$\overline{X} = A_2 X + T_2 \,, \tag{1.4}$$

where T_1 and T_2 are translations along the axis of rotation and given by $T_1 = \left(bu^1, 0, 0\right)^t$ and $T_2 = \left(0, 0, bu^1\right)^t$, $b \neq 0$, according to the axis of rotation is timelike or spacelike, respectively.

Now we consider orbits of straight lines under the above two one-parameter groups of Lorentzian screw motions. Let L be a straight line which meets the axis of the screw motion orthogonally at a point p. If the screw motion is a cubic screw motion, then we require furthermore that L also cuts the orbit of p by the screw motion orthogonally at p. According to these conditions the orbit of L under the screw motion is a surface with vanishing mean curvature. The transformation (1.3) leads to the so-called helicoid of first kind which in these coordinates, has the same equations as the ordinary right helicoid in Euclidean 3-space. While, the transformation (1.4) leads to the so called helicoid of second and third kind [3], depending on the type spacelike or timelike of the generators. Without loss of generality we take $X = \left(0, u^2, 0\right)$ and $X = \left(u^2, 0, 0\right)$, respectively as the direction of L. Thus, we have three types of helicoids namely:

$$\text{first kind :} \quad \overline{X}(u^1, u^2) = (bu^1, u^2 \cos u^1, u^2 \sin u^1), \; b = \text{const.}, \tag{1.5}$$

$$\text{second kind :} \quad \overline{X}(u^1, u^2) = (u^2 \sinh u^1, u^2 \cosh u^1, bu^1) \,, \tag{1.6}$$

$$\text{third kind :} \quad \overline{X}(u^1, u^2) = (u^2 \cosh u^1, u^2 \sinh u^1, bu^1) \,. \tag{1.7}$$

The helicoid of the third kind is timelike but the other two types of helicoids consists of spacelike and timelike pieces. A surface in the Minkowski 3-space $\Re^3_1$ is called a spacelike or timelike if the induced metric on the surface is a Riemannian or Lorentzian metric [10, 11].

These mentioned surfaces are called Lorentzian helicoids or, more precisely, the right Lorentzian helicoids. In equations (1.5)–(1.7), if we replace the linear function bu^1 by a smooth function $f\left(u^1\right)$ such that f' never vanishes, then those surfaces are called right Lorentzian conoid [5]. Whenever one applies a Lorentzian screw motion to a straight line which is not orthogonal to the axis of rotation or does not meet it, one obtains the so called general helicoidal surface [4].

In Euclidean 3-space E^3 the relations $\varphi(K,H)=0$, $\varphi(K_{II},H)=0$, and $aK_{II}+bH=$ const., where a, b are constants, on ruled surfaces were investigated in [2, 9]. For surfaces with $K_{II}=H,\ K_{II}=\sqrt{K},\ K_{II}=$ const.; we refer to [7, 8, 9] for the history and general results in this problem. In Minkowski 3-space $\mathfrak{R}_1^3$, for non-developable ruled surfaces the linear relations $aK_{II}+bH+cH_{II}+dK=$ const., $a^2+b^2+c^2\neq 0$ and $aK_{II}+bH+cK=$ const., $a^2+b^2\neq 0$ along each generator, were studied in [5] and [12], respectively.

In this paper, the Minkowski versions of the major results proved in [2, 9] are given and additional results are obtained. Moreover, for some special cases new examples are constructed and plotted.

2. Ruled surfaces in Minkowski 3-space $\mathfrak{R}_1^3$

A ruled surface M in $\mathfrak{R}_1^3$ is a surface generated by a straight line moving along a curve. The various positions of the generating lines are called the generators of the surface. Then, as in the Euclidean case, one can introduce the standard parameters in the ruled form:

$$M\colon \mathbf{R}(u^1,u^2)=\mathbf{r}(u^1)+u^2\mathbf{e}(u^1)\,, \tag{2.1}$$

such that

$$\langle\mathbf{e},\mathbf{e}\rangle=\varepsilon\,,\quad \langle\mathbf{e}',\mathbf{e}'\rangle=\eta\,,\quad \langle\mathbf{e}',\mathbf{t}\rangle=0\,,\qquad \mathbf{t}=\mathbf{r}'\,,\quad {}'=\frac{d}{du^1}$$

and $\varepsilon,\eta\in\{1,-1\}$. In this case the base curve $\mathbf{r}(u^1)$ is the striction curve, and the parameter u^1 is the arclength of the spherical image or indicatrix $\mathbf{e}=\mathbf{e}(u^1)\in S_1^2$ or H_0^2. Excluding $\mathbf{e}$ is constant or null or $\mathbf{e}'$ null.

Here, and following, we assume that the indices $\{i,j,\alpha,\beta\}$ run over the ranges $\{1,2\}$ unless otherwise stated. The Einstein summation convention will be used, that is, repeated indices, with one upper index and one lower index, denoted summation over its range.

Then, the elements of the first fundamental form g_{ij} are given by:

$$g_{11}=\|\mathbf{t}\|^2+\eta\left(u^2\right)^2\,,\quad g_{12}=\langle\mathbf{e},\mathbf{t}\rangle\,,\quad g_{22}=\varepsilon\,. \tag{2.2}$$

In terms of the moving frame $\{\mathbf{e},\mathbf{e}',\mathbf{e}\times\mathbf{e}'\}$ with signs ε, η, $-\varepsilon\eta$, the curve $\mathbf{r}=\mathbf{r}(u^1)$ can be reconstructed from

$$\mathbf{t}=\varepsilon g_{12}\mathbf{e}-\varepsilon\eta\lambda\mathbf{e}\times\mathbf{e}'\,, \tag{2.3}$$

where $\lambda=\langle\mathbf{t},\mathbf{e}\times\mathbf{e}'\rangle$ is the distribution parameter of the ruled surface. One obtains

$$\mathrm{Det}(g_{ij})=g=\eta\big(\varepsilon(u^2)^2-\lambda^2\big)\,. \tag{2.4}$$

The generator $\mathbf{e} = \mathbf{e}(u^1)$ can be reconstructed from

$$\mathbf{e}'' = \varepsilon\eta(-\mathbf{e} + \kappa_g \mathbf{e} \times \mathbf{e}') \,, \tag{2.5}$$

where $\kappa_g = \det(\mathbf{e}'', \mathbf{e}', \mathbf{e})$ is called the geodesic curvature of $\mathbf{e}(u^1)$. Let

$$\sqrt{|g|} = \|\mathbf{R}_1 \times \mathbf{R}_2\| = \sqrt{\lambda^2 - \varepsilon\,(u^2)^2} \,, \quad \mathbf{R}_i = \frac{\partial \mathbf{R}}{\partial u^i} \,. \tag{2.6}$$

The unit normal vector of the surface is given by:

$$\mathbf{n} = \frac{1}{\sqrt{|g|}}\,(\eta\lambda\mathbf{e}' - u^2\mathbf{e} \times \mathbf{e}') \,. \tag{2.7}$$

This leads to the elements of the second fundamental form h_{ij} where

$$\begin{aligned} h_{11} &= \frac{1}{\sqrt{|g|}}\,\{\varepsilon\lambda(-\lambda\kappa_g + g_{12}) - \lambda' u^2 + \kappa_g (u^2)^2\} \,, \\ h_{12} &= \frac{\lambda}{\sqrt{|g|}} \,, \quad h_{22} = 0 \,, \quad \sqrt{|g|} \neq 0 \,. \end{aligned} \tag{2.8}$$

2.1. **Curvature functions.** A surface M in an Euclidean 3-space E^3 with positive Gaussian curvature K possesses a positive definite second fundamental form II if appropriately orientated. Therefore, the second fundamental form defines a new Riemannian metric on M. In turn, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a Riemannian metric. If a surface has non-zero Gaussian curvature everywhere, K_{II} can be defined formally and it is the curvature of the Riemannian or pseudo-Riemannian manifold (M, II) depending on $\mathrm{Det}(h_{ij}) > 0$ or < 0, respectively.

Naturally, we can extend such a notion to surfaces in a Minkowski 3-space $\Re^3_1$. Thus we define the second Gaussian curvature K_{II} by [6]

$$K_{II} = \frac{1}{h^2}\left(\begin{vmatrix} -\frac{h_{11,22}}{2} + h_{12,12} - \frac{h_{22,11}}{2} & \frac{h_{11,1}}{2} & h_{12,1} - \frac{h_{11,2}}{2} \\ h_{12,2} - \frac{h_{22,1}}{2} & h_{11} & h_{12} \\ \frac{h_{22,2}}{2} & h_{12} & h_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{h_{11,2}}{2} & \frac{h_{22,1}}{2} \\ \frac{h_{11,2}}{2} & h_{11} & h_{12} \\ \frac{h_{22,1}}{2} & h_{12} & h_{22} \end{vmatrix} \right) , \tag{2.9}$$

where $h = \mathrm{Det}(h_{ij})$, $h_{ij,\alpha} = \frac{\partial h_{ij}}{\partial u^\alpha}$ and $h_{ij,\alpha\beta} = \frac{\partial^2 h_{ij}}{\partial u^\alpha \partial u^\beta}$.

It is well known that a minimal surface has vanishing second Gaussian curvature but a surface with vanishing second Gaussian curvature need not be minimal.

Since Brioschi's formulas in Euclidean and Minkowski 3-spaces are the same, we are able to define H_{II} of M by replacing the components of the first fundamental form g_{ij} by the components of the second fundamental form h_{ij} respectively in Brioschi's formula. Consequently, the second mean curvature H_{II} is given by [5]:

$$H_{II} = H - \frac{1}{2}\Delta\big(\ln\sqrt{|K|}\big) \,, \tag{2.10}$$

where Δ is the Laplacian with respect to the second fundamental form of M, expressed as:

$$\Delta = -\frac{1}{\sqrt{|h|}}\,\frac{\partial}{\partial u^i}\left[\sqrt{|h|}h^{ij}\,\frac{\partial}{\partial u^j}\right] , \quad (h^{ij}) = (h_{ij})^{-1} \,. \tag{2.11}$$

The Gaussian curvature K and the mean curvature H are given by [1]:

$$K = \varepsilon\,\frac{h}{g} \quad \text{and} \quad H = \frac{1}{2}\,\varepsilon\; tr(g^{ij}\,h_{jk})\,, \tag{2.12}$$

respectively.

From Eqs. (1.5)–(1.7) we have the following

Corollary 1. *For the right Lorentzian helicoids of first, second and third kind (Figs. 1, 2 and 3), one can get:*

$$\kappa_g = g_{12} = 0 \quad \textit{and} \quad \lambda = \varepsilon b = \text{const.}\,, \tag{2.13}$$

and thus

$$K = \frac{b^2}{(b^2 - \varepsilon (u^2)^2)^2}\,, \quad H = H_{II} = K_{II} = 0\,. \tag{2.14}$$

Corollary 2. *For the right Lorentzian conoids of first, second and third kind (Figs. 4, 5 and 6), one can obtain:*

$$\kappa_g = g_{12} = 0 \quad \textit{and} \quad \lambda = \varepsilon f'(u^1)\,, \tag{2.15}$$

and thus

$$\begin{gathered} K = \frac{f'^2(u^1)}{(f'^2(u^1) - \varepsilon (u^2)^2)^2}\,, \quad H = \frac{-\varepsilon f''(u^1 t)}{2(f''(u^1) - \varepsilon (u^2)^2 t)^{\frac{3}{2}}} u^2\,, \\ -6H = 2H_{II} = 3K_{II}\,. \end{gathered} \tag{2.16}$$

3. Ruled Weingarten-surfaces

In this section, we study ruled Weingarten-surfaces (W-surfaces) M of which there exists a nontrivial functional relation between a pair $\{A, B\}$, $A \neq B$, of the curvatures K, K_{II}, H and H_{II}. A nondevelopable ruled surface M, is called an $\{A, B\}$-W-surface if there exist a nontrivial functional relation $\varphi(A, B) = 0$, or, equivalently, the corresponding Jacobian determinant is identically zero, i.e.,

$$\frac{\partial(A, B)}{\partial(u^1, u^2)} \equiv 0\,. \tag{3.1}$$

Using Equations (2.2)–(2.8) and in view of Eqs. (2.9)–(2.12) one can obtain the curvature functions K, H, K_{II} and H_{II} in the following form:

$$K=\frac{\lambda^2}{|g|^2}\,,\quad H=\frac{1}{2\sqrt{|g|^3}}\big\{\varepsilon\kappa_g(u^2)^2-\varepsilon\lambda' u^2-\lambda(\lambda\kappa_g+g_{12})\big\}\,,\tag{3.2}$$

$$\begin{aligned}K_{II}=&\frac{1}{2\lambda^2\sqrt{|g|^3}}\big\{\kappa_g(u^2)^4+\varepsilon\lambda(g_{12}-2\lambda\kappa_g)\,(u^2)^2\\&+2\varepsilon\lambda^2\lambda' u^2+\lambda^3(g_{12}+\lambda\kappa_g)\big\}\end{aligned}\tag{3.3}$$

$$\begin{aligned}H_{II}=&\frac{1}{2\lambda^2\sqrt{|g|^3}}\big\{-2\kappa_g(u^2)^4+\varepsilon\lambda(5\lambda\kappa_g+2g_{12})\,(u^2)^2+3\varepsilon\lambda^2\lambda' u^2\\&+\lambda^3(g_{12}-3\lambda\kappa_g)\big\}\,.\end{aligned}\tag{3.4}$$

By a straightforward calculation, we get:

$$K_2=\frac{4\lambda^2}{|g|^3}u^2\,,\tag{3.5}$$

$$\begin{aligned}K_1=&\frac{2}{|g|^3}\big\{(-\lambda\lambda'(u^2)^2+\lambda^3\lambda')+\varepsilon(3\lambda\lambda''-\lambda^2\lambda'')\,u^2\\&+\lambda^2(\lambda\lambda'\kappa_g-\lambda^2\kappa_g'+2\lambda' g_{12}-\lambda g_{12}')\big\}\,,\end{aligned}\tag{3.6}$$

$$H_2=\frac{1}{2\sqrt{|g|^5}}\big\{\kappa_g(u^2)^3-2\lambda'(u^2)^2-\varepsilon\lambda(\lambda\kappa_g+3g_{12})u^2-\varepsilon\lambda^2\lambda'\big\}\,,\tag{3.7}$$

$$\begin{aligned}H_1=&\frac{1}{2\sqrt{|g|^5}}\big\{-\kappa_g'(u^2)^4+\lambda''(u^2)^3\\&+\varepsilon(-\lambda\lambda'\kappa_g+2\lambda^2\kappa_g'+\lambda' g_{12}+\lambda g_{12}')(u^2)^2\\&+\varepsilon(3\lambda\lambda'^2-\lambda^2\lambda'')u^2+\lambda^2(\lambda\lambda'\kappa_g-\lambda^2\kappa_g'+2\lambda' g_{12}-\lambda g_{12}')\big\}\,,\end{aligned}\tag{3.8}$$

$$\begin{aligned}(K_{II})_2=&\frac{1}{2\lambda^2\sqrt{|g|^5}}\big\{-\varepsilon\kappa_g(u^2)^5+\lambda(g_{12}+2\lambda\kappa_g)\,(u^2)^3\\&+4\lambda^2\lambda'(u^2)^2+\varepsilon\lambda^3(5g_{12}-\lambda\kappa_g)u^2+2\varepsilon\lambda^4\lambda'\big\}\,,\end{aligned}\tag{3.9}$$

$$\begin{aligned}(K_{II})_1=&\frac{1}{2\lambda^3\sqrt{|g|^5}}\big\{-\varepsilon(\lambda\kappa_g'-2\lambda'\kappa_g)(u^2)^6\\&+\big(\lambda^2(3\lambda\kappa_g'-5)\lambda'\kappa_g+\lambda(\lambda' g_{12}-\lambda g_{12}')\big)\,(u^2)^4\\&-2\lambda^3\lambda''(u^2)^3+\big(\lambda^3\lambda'(4\lambda\kappa_g-5g_{12})-3\lambda^5\kappa_g'\big)\,(u^2)^2\\&+2\varepsilon\lambda^4(\lambda\lambda''-3\lambda'^2)u^2+\lambda^7\kappa_g'-\lambda^6\lambda'\kappa_g+\lambda^6 g_{12}'-2\lambda^5\lambda' g_{12}\big\}\,,\end{aligned}\tag{3.10}$$

$$(H_{II})_2 = \frac{1}{2\lambda^2\sqrt{|g|^5}}\big\{2\varepsilon\kappa_g(u^2)^5 - 3\lambda^2\kappa_g(u^2)^3 + 6\lambda^2\lambda'(u^2)^2 + \varepsilon\lambda^3(7g_{12} + \lambda\kappa_g)u^2 + 3\varepsilon\lambda^4\lambda'\big\}\,, \tag{3.11}$$

where $\phi_i = \frac{\partial\phi}{\partial u^i}$.

Applying formula (3.1) when $A = K$ and $B = H$, we have:

$$K_1H_2 - K_2H_1 = 0\,, \tag{3.12}$$

then from Equations (3.5)–(3.8), we get

$$K_1H_2 - K_2H_1 = \frac{1}{\sqrt{|g|^{11}}}\, a_i(u^2)^i\,, \quad i = 0,\dots,5,\ a_i = a_i(u^1)\,, \tag{3.13}$$

where

$$\begin{cases} a_0 = -\lambda^5\lambda'\,, & a_1 = \varepsilon(-5\lambda^5\lambda'\kappa_g - 2\lambda^6\kappa_g' + \lambda^4\lambda' g_{12} - 2\lambda^5 g_{12}')\,,\\ a_2 = 3\lambda^3\lambda'^2 - 2\lambda^4\lambda''\,, & a_3 = -2\lambda^3\lambda'\kappa_g + 4\lambda^4\kappa_g' - 8\lambda^2\lambda' g_{12} + 2\lambda^3 g_{12}'\,,\\ a_4 = \varepsilon(2\lambda^2\lambda'' - \lambda\lambda'^2)\,, & a_5 = \varepsilon(-2\lambda^2\kappa_g' + \lambda\lambda'\kappa_g)\,. \end{cases} \tag{3.14}$$

Now we assume $\lambda \neq 0$ and $K \neq 0$. Therefore in a neighborhood of any point with $\lambda \neq 0$, the vanishing of the coefficients of $(u^2)^0$, $(u^2)^5$ implies $\lambda' = 0$, and $\kappa_g' = 0$. Then the vanishing of the coefficient of $(u^2)^3$ implies $g_{12}' = 0$.

Hence we have the following theorem:

Theorem 1. *Suppose M be any non-developable ruled surface in $\Re_1^3$ such that the generator is nowhere null. A necessary and sufficient condition that M be a W-surface is that the distribution parameter and the geodesic curvature of spherical indicatrix of the generators are constant as well the generators be inclined at constant angle to the striction curve, which consequently is a geodesic curve.*

As, it is a well known result for the general helicoidal surface [4] that the quantities λ, κ_g, g_{12} are constants, and that vice versa. Then in view of the expressions K and H in Eq. (3.2), we obtain the following result:

Theorem 2. *Suppose M be any non-developable ruled surface in $\Re_1^3$ such that the generator is nowhere null. Then the following conditions are equivalent:*

(i) *M is a W-ruled surface,*

(ii) *the quantities λ, κ_g, g_{12} are constants,*

(iii) $2H = \mp\frac{\kappa_g}{|\lambda|^{\frac12}}K^{\frac14} - \frac{g_{12}}{|\lambda|^{\frac12}}K^{\frac34}$, *for constants λ, κ_g, g_{12},*

(iv) *M is a congruent to a part of a general helicoidal surface.*

Corollary 3. *In the particular case $\kappa_g = 0$ or $g_{12} = 0$, one can get the following relation:*

$$K^3 = 16\frac{\lambda^2}{g_{12}^4}\,H^4\,, \quad \text{or} \quad K = 16\frac{\lambda^2}{\kappa_g^4}\,H^4\,, \tag{3.15}$$

respectively.

Similarly, according to (3.1), we have:

$$(K_{II})_1 H_2 - (K_{II})_2 H_1 = 0\,, \tag{3.16}$$

substituting Eqs. (3.7)–(3.10) in (3.16), and after straightforward computations, the following polynomial of degree 9 in the variable u^2 is obtained as follows:

$$\frac{1}{4\lambda^3|g|^5} a_i (u^2)^i = 0\,, \quad i = 1, \ldots, 9, \quad a_i = a_i(u^1)\,, \tag{3.17}$$

where

$$a^9 = 2\kappa_g(\lambda'\kappa_g - \lambda\kappa_g')\,, \tag{3.18}$$

$$a^8 = 2\lambda\lambda'\kappa_g' - 4\lambda'^2\kappa_g + \lambda\lambda''\kappa_g\,, \tag{3.19}$$

$$a^7 = \lambda\big[\lambda\kappa_g^{'}(4g_{12} + 8\lambda\kappa_g) + \lambda'\kappa_g\ (-4g_{12} - 8\lambda\kappa_g)\big]\,, \tag{3.20}$$

$$a^6 = \lambda\big[\lambda'(2\lambda g_{12} - \lambda^2\kappa_g') + \lambda'^2(2g_{12} + 8\lambda\kappa_g)\big]\,, \tag{3.21}$$

$$\begin{aligned} a^5 = -\lambda^2\big[&2\varepsilon\lambda g_{12}'(\lambda\kappa_g - g_{12}) + 3\lambda^2\kappa_g'(\lambda\kappa_g + 2\varepsilon g_{12} + 3\varepsilon\lambda\kappa_g) \\ &+ \lambda'(4\varepsilon g_{12} - 4\lambda^2\kappa_g^2 - 15\varepsilon\lambda\kappa_g g_{12} - 8\varepsilon\lambda^2\kappa_g^2 + 5\lambda\kappa_g g_{12})\big]\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} a^4 = -\lambda^3\big[&3\lambda\lambda'(\varepsilon g_{12} - 2\lambda\kappa_g^{'} + 3\varepsilon\lambda\kappa_g') + \lambda'^2(8\lambda\kappa_g - 10g_{12} \\ &+ 5\varepsilon g_{12} - 3\varepsilon\lambda\kappa_g) + \lambda''(3\varepsilon g_{12} - 2\varepsilon\lambda g_{12} + 6\varepsilon\lambda\kappa_g - 7\varepsilon\lambda^2\kappa_g)\big]\,, \end{aligned} \tag{3.23}$$

$$\begin{aligned} a^3 = -\lambda^4\big[&\lambda'\big(g_{12}^2(7 - 15\varepsilon) + \lambda\kappa_g g_{12}(1 + 7\varepsilon) + 4\lambda^2\kappa_g^2(1 + \varepsilon) \\ &- 12\varepsilon(\lambda'^2 + \lambda'')\big) + 4\lambda g_{12}'(g_{12} - \lambda\kappa_g) \\ &+ \lambda^2\kappa_g'(9g_{12} - 9\varepsilon g_{12} - 3\varepsilon\lambda\kappa_g - 5\lambda\kappa_g)\big]\,, \end{aligned} \tag{3.24}$$

$$\begin{aligned} a^2 = -\lambda^5\big[&\big(- \lambda^2\kappa_g'(3\varepsilon - 2) + 4g_{12}\lambda'\big) + \lambda'^2\big(\lambda\kappa_g(4\varepsilon - 6) - \varepsilon g_{12}(5 + 16\varepsilon)\big) \\ &+ \lambda''\big(g_{12}(15 + \lambda) - 3\lambda\kappa_g(1 - \lambda)\big)\big]\,, \end{aligned} \tag{3.25}$$

$$\begin{aligned} a = -2\varepsilon\lambda^6\big[&\big(\lambda\kappa_g - g_{12})\lambda g_{12}' + \lambda^2\kappa_g'\big) + \lambda'(2g_{12}^2 - \lambda\kappa_g(g_{12} + \lambda\kappa_g) \\ &- 3\varepsilon(\lambda'^2 - \lambda'')\big]\,, \end{aligned} \tag{3.26}$$

$$a^0 = \lambda^7\lambda'(\lambda\lambda'\kappa_g + \lambda^2\kappa_g' + \lambda g_{12}' - 2\lambda' g_{12})\,. \tag{3.27}$$

Thus, we give the following theorem:

Theorem 3. *Suppose M be any non-developable ruled surface in $\Re_1^3$ such that the generator is nowhere null. Then M is a II-W surface if and only if one of the following cases is satisfied:*

(i) λ, κ_g, g_{12} *are constants,* $\lambda \neq 0$,

(ii) $\lambda'\kappa_g = \lambda\kappa_g'$, $2\lambda' g_{12} = \lambda g_{12}'$, $2\lambda'^2 = \lambda\lambda''$, $\lambda \neq 0$,

or, if $\lambda',\ \kappa_g,\ g_{12} \neq 0 : \frac{g_{12}'}{g_{12}} = 2\frac{\kappa_g'}{\kappa_g} = 2\frac{\lambda'}{\lambda} = \frac{\lambda''}{\lambda'}$,

(iii) $\kappa_g = g_{12} = 0$, $\lambda \neq 0$ *arbitrary (right Lorentzian conoids).*

Proof. The Eqs. (3.18)–(3.27) vanish identically $\forall\ u^2 \in R$, so we discuss the following cases:
Case (i): $\lambda' = 0$. The Eq. (3.18) implies $\kappa_g' = 0$. Furthermore, one calculates in this case

$$(K_{II})_1 H_2 - (K_{II})_2 H_1 = \frac{1}{4\lambda^3 |g|^5}\, b_i (u^2)^i\,, \quad i = 1, 3, 5, 7,\ b_i = b_i(u^1)\,, \tag{3.28}$$

which vanishes if and only if $g_{12}' = 0$. Therefore λ, κ_g, g_{12} are constants.
Case (ii): $\lambda' \neq 0$. Let us consider a neighborhood of a point with $\lambda' \neq 0$. The Eq. (3.18) implies $\kappa_g = 0$ or $\lambda'\kappa_g - \lambda\kappa_g' = 0$. Let us assume for the moment that $\lambda'\kappa_g = \lambda\kappa_g'$ and $\kappa_g \neq 0$. Then the Eq. (3.19) implies $\lambda\lambda'' = 2\lambda'^2$, and the Eq. (3.27) implies $\lambda g_{12}' = 2\lambda' g_{12}$. Thus, we see that the Eqs. (3.20)–(3.26) are vanished. Using the relations (3.18), (3.19) and (3.27), we get

$$(K_{II})_1 = -u^2 \frac{\lambda'}{\lambda}\,(K_{II})_2\,, \quad H_1 = -u^2 \frac{\lambda'}{\lambda}\, H_2\,, \tag{3.29}$$

and thus $(K_{II})_1 H_2 - (K_{II})_2 H_1 = 0$.
Case (iii): $\lambda' \neq 0$ and $\kappa_g = 0$. In this case we have

$$(K_{II})_1 H_2 - (K_{II})_2 H_1 = \frac{1}{4\lambda^3 |g|^5}\, c_i (u^2)^i\,, \quad i = 0, \ldots, 6,\ c_i = c_i(u^1)\,. \tag{3.30}$$

Therefore, the coefficients c_i vanishing identically if $g_{12} = 0$ with arbitrary λ. This is the Case (iii) of the theorem. If $g_{12} \neq 0$, then we get the equations $2\lambda' g_{12} = \lambda g_{12}'$, $2\lambda'^2 = \lambda\lambda''$ which again leads to the Case (ii). The converse is clear, and the theorem is proved. □

As a result of the above theorem, we have the following theorem which is analogous to Theorem 1:

Theorem 4. *Suppose M be any non-developable ruled surface in $\mathfrak{R}_1^3$ such that the generator is nowhere null. A necessary and sufficient condition that M be a II-W-surface is that the distribution parameter and the geodesic curvature of spherical indicatrix of the generators are constant as well the generators be inclined at constant angle to the striction curve, which consequently is a geodesic curve.*

For contribution of the works in [5] and [12], the following theorem can be given:

Theorem 5. *For non-developable ruled surface in $\mathfrak{R}_1^3$ whose generator is nowhere null and $aK_{II} + bH_{II} + cH$ is constant along each generator, then M is congruent to one of the right Lorentzian conoids, where $2a + 3b - c \neq 0$, $a, b, c \in R$.*

Proof. By using Eqs. (3.7), (3.9) and (3.11), the proof of the theorem is clear, since the above case satisfy the relations (2.15) of the (*right*) Lorentzian conoids. □

As a result of the above theorem the following linear relations are hold:

Corollary 4.
(i) $aK_{II} + bH_{II} = \text{const.}$ *where* $2a + 3b \neq 0$,
(ii) $aK_{II} + cH = \text{const.}$ *where* $2a - c \neq 0$,
(iii) $bH_{II} + cH = \text{const.}$ *where* $3b - c \neq 0$.

3.1. **Examples.** Here three examples of ruled *II-W* surfaces (which are not among the Lorentzian helicoids or conoids) are constructed for the special case:

$$\kappa_g = 0\,, \quad \frac{g'_{12}}{g_{12}} = 2\frac{\lambda'}{\lambda} = \frac{\lambda''}{\lambda'}\,. \tag{3.31}$$

The solutions for λ and g_{12} are

$$\lambda(u^1) = \frac{\ell}{u^1}\,, \quad g_{12}(u^1) = \frac{m}{(u^1)^2}\,, \tag{3.32}$$

where ℓ and m are non-vanishing constants.

Using Eqs. (3.31), (3.32) into Eqs. (2.3) and (2.5), we can give:

Example 1. In the case of $\varepsilon = 1$ and $\eta = -1$, the curve $\mathbf{e}(u^1)$ is

$$\mathbf{e}(u^1) = (\sinh u^1, \cosh u^1, 0)\,, \tag{3.33}$$

and the curve $\mathbf{r}(u^1)$ can be obtained from

$$\mathbf{r}(u^1) = \int \mathbf{t}(u^1)\,du^1 = \Big(m\int_{u_0^1}^{u^1} \frac{\sinh u}{u^2}\,du, m\int_{u_0^1}^{u^1} \frac{\cosh u}{u^2}\,du, \ell\log|u^1|\Big)\,. \tag{3.34}$$

Example 2. In the case of $\varepsilon = -1$ and $\eta = 1$, the curve $\mathbf{e}(u^1)$ is

$$\mathbf{e}(u^1) = (\cosh u^1, \sinh u^1, 0)\,, \tag{3.35}$$

and the curve $\mathbf{r}(u^1)$ is given as

$$\mathbf{r}(u^1) = -\Big(m\int_{u_0^1}^{u^1} \frac{\cosh u}{u^2}\,du, m\int_{u_0^1}^{u^1} \frac{\sinh}{u^2}\,du, \ell\log|u^1|\Big)\,. \tag{3.36}$$

Example 3. In the case of $\varepsilon = 1$ and $\eta = 1$, the curve $\mathbf{e}(u^1)$ is

$$\mathbf{e}(u^1) = (0, \cos u^1, \sin u^1)\,, \tag{3.37}$$

and the curve $\mathbf{r}(u^1)$ is given as

$$\mathbf{r}(u^1) = \Big(-\ell\log|u^1|, m\int_{u_0^1}^{u^1} \frac{\cos u}{u^2}\,du, m\int_{u_0^1}^{u^1} \frac{\sin u}{u^2}\,du\Big)\,. \tag{3.38}$$

Remarks. Note that the graphs of the surfaces, given by Examples 1, 2 and 3 are shown in Figs. 7, 8, Figs. (9, 10 and Figs. 11, 12, respectively. And

(i) The ruled II-W surfaces which are shown in Figs. 7, 9 and Figs. 8, 10 are of type spacelike (timelike) surface if $u^1u^2 > \sqrt{\cosh^2 u^1 + \sinh^2 u^1}$ $\big(u^1u^2 < \sqrt{\cosh^2 u^1 + \sinh^2 u^1}\big)$, respectively.

(ii) The ruled II-W surfaces which are shown in Fig. 11 and Fig. 12 are of type spacelike (timelike) surface if $u^1u^2 > 1$ ($u^1u^2 < 1$), respectively.

(iii) Although $\mathbf{r}(u^1)$ are defined only for $u^1 > 0$ (or $u^1 < 0$), the resulting surfaces are completely in the Minkowski 3-space $\Re_1^3$.

(iv) Using Taylor series expansion for the first two components of the base curve $\mathbf{r}(u^1)$. Thus, we plotted these types of surfaces depending on some approximation to Taylor's remainder terms.

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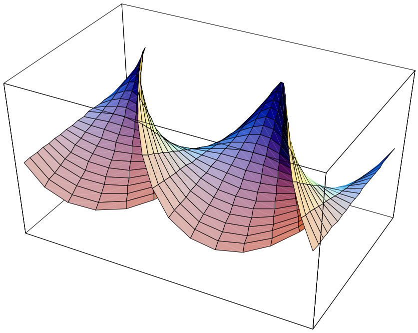

FIG. 1: A helicoid of the 1st kind $b = 2$, $u^1 \in [0, 2\pi]$, $u^2 \in [-4, 4]$

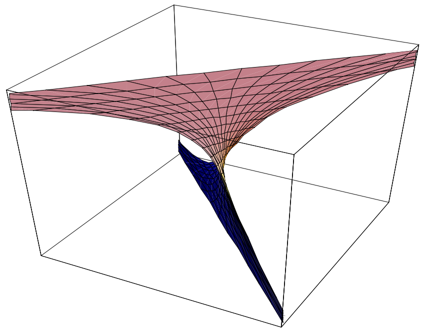

FIG. 2: A helicoid of the 2nd kind $b = 2$, $u^1 \in [-\pi, \pi]$, $u^2 \in [-1, 1]$

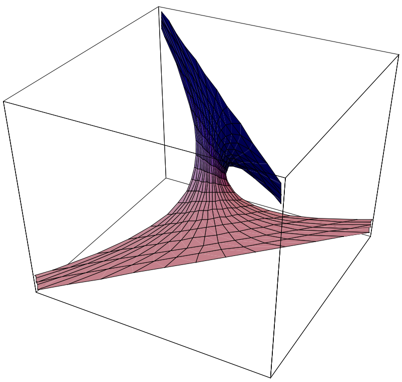

FIG. 3: A helicoid of the 3rd kind $b = -2$, $u^1 \in [-\pi, \pi]$, $u^2 \in [-1, 1]$

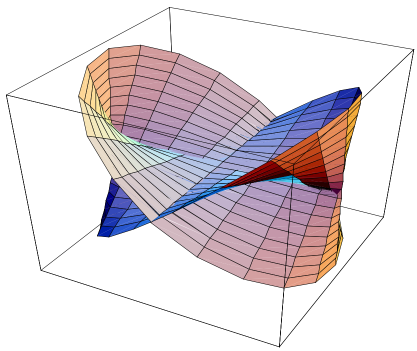

FIG. 4: A canoid of the 1st kind $f(u^1) = \sin u^1$, $u^1 \in [0, 2\pi]$, $u^2 \in [-1, 1]$

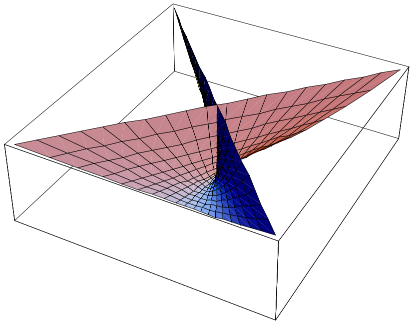

FIG. 5: A canoid of the 2nd kind $f(u^1) = \cos hu^1$, $u^1 \in [-\pi/2, \pi/2]$, $u^2 \in [-1, 1]$

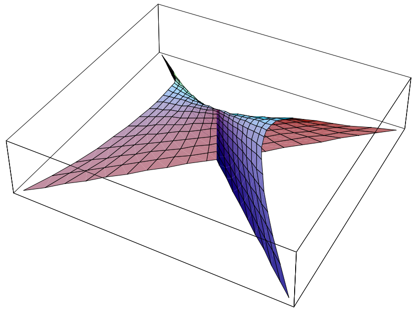

FIG. 6: A canoid of the 3rd kind $f(u^1) = \cos u^1$, $u^1 \in [-\pi/2, \pi/2]$, $u^2 \in [-1, 1]$

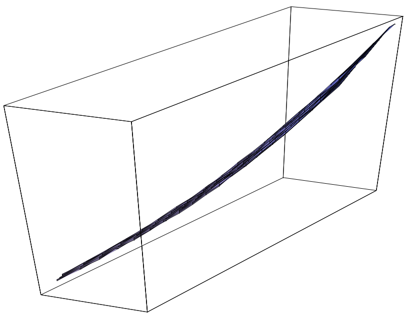

FIG. 7: Spacelike ruled II-W surface $\varepsilon = 1$, $\eta = -1$, $\ell = 10$, $m = 15$, $u^1 \in [1, \pi]$, $u^2 \in [1, 2]$

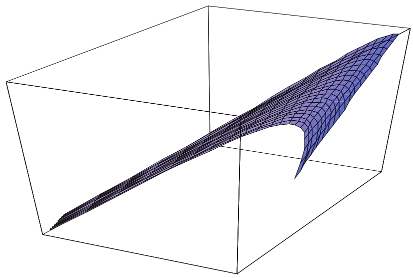

FIG. 8: Timelike ruled II-W surface $\varepsilon = 1$, $\eta = -1$, $\ell = 10$, $m = 15$, $u^1 \in [1, \pi]$, $u^2 \in [-1, -2]$

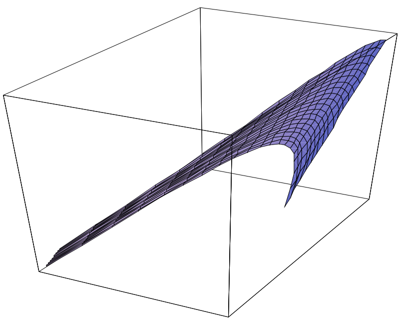

FIG. 9: Spacelike ruled II-W surface $\varepsilon = -1$, $\eta = 1$, $\ell = 10$, $m = 15$, $u^1 \in [1, \pi]$, $u^2 \in [1, 2]$

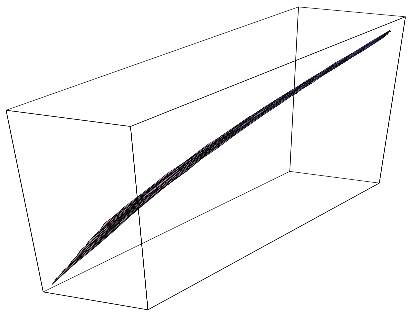

FIG. 10: Timelike ruled II-W surface $\varepsilon = -1$, $\eta = 1$, $\ell = 10$, $m = 15$, $u^1 \in [1, \pi]$, $u^2 \in [-1, -2]$

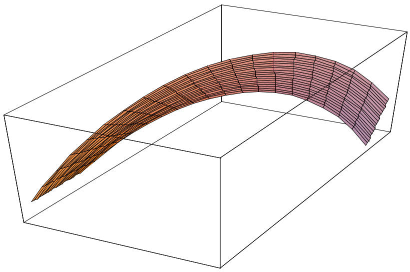

FIG. 11: Spacelike ruled II-W surface $\varepsilon = 1$, $\eta = 1$, $\ell = 2$, $m = 3$, $u^1 \in [1, \pi]$, $u^2 \in [1, 2]$

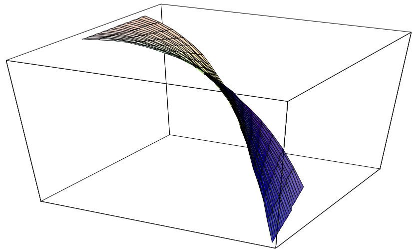

FIG. 12: Timelike ruled II-W surface $\varepsilon = 1$, $\eta = 1$, $\ell = 2$, $m = 3$, $u^1 \in [1, \pi]$, $u^2 \in [-1, -2]$

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