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LATTICE-VALUED BOREL MEASURES III

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Abstract. Let X be a completely regular T_1 space, E a boundedly complete vector lattice, $C(X)$ ($C_b(X)$) the space of all (all, bounded), real-valued continuous functions on X. In order convergence, we consider E-valued, order-bounded, σ-additive, τ-additive, and tight measures on X and prove some order-theoretic and topological properties of these measures. Also for an order-bounded, E-valued (for some special E) linear map on $C(X)$, a measure representation result is proved. In case E_n^* separates the points of E, an Alexanderov's type theorem is proved for a sequence of σ-additive measures.

1. Introduction and notation

All vector spaces are taken over reals. E, in this paper, is always assumed to be a Dedekind complete Riesz space (and so, necessarily Archimedean) ([1], [15], [14]). For a completely regular T_1 space X, υX is the real-compactification, $\tilde{X}$ is the Stone-Čech compactification of X, $B(X)$ is the space of all real-valued bounded functions on X, $C(X)$ (resp. $C_b(X)$) is the space of all real-valued, (resp. real-valued and bounded) continuous functions on X; sets of the form $\left\{f^{-1}(0); f \in C_b(X)\right\}$ are called zero-sets of X and their complements positive subsets of X, and the elements of the σ-algebra generated by zero-sets are called Baire sets ([20], [19]); $\mathcal{B}(X)$ and $\mathcal{B}_1(X)$ will denote the classes of Borel and Baire subsets of X and $\mathcal{F}(X)$ will be the algebra generated by the zero-sets of X. $\beta_1(X)(\beta(X))$ are, respectively the spaces of bounded Baire (Borel) measurable functions on X. It is easily verified that the order σ-closure of $C_b(X)$ in $\beta_1(X)$, in the topology of pointwise convergence, is $\beta_1(X)$ and the order σ-closure, in $\beta(X)$, of the vector space generated by bounded lower semi-continuous functions on X, is $\beta(X)$ ([3], [4]).

In ([21], [23]), the author discussed the positive measures taking values in Dedekind complete Riesz spaces and proved some basic results about the integration relative to these measures; he also proves some Riesz representation type theorems; it was proved there that when X is a compact Hausdorff space and $\mu\colon C(X) \to E$ is a positive linear mapping then μ arises from a unique quasi-regular Borel measure $\mu\colon \mathcal{B}(X) \to E$ which is countably additive in order convergence (quasi-regular means that the measure of any open set is inner regular by the compact subsets

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of X). In ([7], [8]) new proofs were given for these Riesz representation theorems for positive measures and then the study was extended to completely regular T_1 spaces and σ-additive, τ-additive and tight positive measures were studied on these spaces. In ([17], [18]), some decomposition theorems for measures, which take values in Dedekind complete Riesz spaces and are not necessarily positive, were proved. In [16], the authors proved some results about the countable additivity of the order-theoretic modulus of a countable additive measures taking values in a Banach lattice.

In the present paper, we consider measures, not necessarily positive, on completely regular T_1 spaces, taking values in Dedekind complete Riesz spaces. In Section 2, some order-theoretic and topological properties of σ-additive, τ-additive and tight measures are proved. In Section 3, a well-known result about the measure representation of real-valued, order-bounded linear map on $C(X)$ is extended to the case when the order-bounded linear map on $C(X)$ takes values in $C(S)$, S being a Stone space. In Section 4, assuming that the continuous order dual E_n^* separates the points of E, an Alexanderov's type theorem is proved about a sequence of σ-additive measures.

For locally convex spaces and vector lattices, we will be using notations and results for ([15], [1], [13]). For a locally convex space E with E' its dual, with an $x \in E$ and $f \in E'$, $\langle f, x\rangle$ will stand for $f(x)$. For measures, results and notations from ([21], [10], [2]) will be used, and for lattice-valued measures, results of ([17], [18]) will be used.

2. Order-bounded measures on completely regular T_1 space in order convergence

We start with a compact Hausdorff space X and an order-bounded, countably additive (countable additivity in the order convergence of E) Borel measure $\mu\colon \mathcal{B}(X) \to E$. Further assume that for any decreasing net $\{C_\alpha\}$ of closed subsets of X, $\mu(\cap C_\alpha) = o - \lim \mu(C_\alpha)$ (if μ has this property then we say μ is τ-smooth). We first prove the following theorem.

Theorem 1. *Suppose X is a compact Hausdorff space and $\mu\colon \mathcal{B}(X) \to E$ be an order-bounded, countably additive* (*countable additivity in the order convergence of E*) *Borel measure on X, having the propety that for any decreasing net $\{C_\alpha\}$ of closed subsets of X, $\mu(\cap C_\alpha) = o - \lim \mu(C_\alpha)$. Let $\{f_\alpha\}$ be a net of $[0,1]$-valued,* usc (*upper semi-continuous*) *functions on X, decreasing pointwise to a function f on X. Then $o - \lim \mu(f_\alpha) = \mu(f)$.*

Proof. Since μ is order-bounded, we can take $E = C(S)$, S being a compact Stone space and $\left|\mu\big(\mathcal{B}(X)\big)\right| \le 1 \in C(S)$; this implies, that for any Borel function $h\colon X \to [-1,1]$, $|\mu(h)| \le 1$. Fix a $k \in N$ and let $Z_\alpha^i = f_\alpha^{-1}\left[\frac{i}{k}, 1\right]$ and $Z^i = f^{-1}\left[\frac{i}{k}, 1\right]$, for $i = 1, 2, \dots, (k-1)$. By hypothesis, $o - \lim_\alpha \mu(Z_\alpha^i) = \mu(Z^i)$, $\forall i$. We have $\frac{1}{k}\sum_{i=1}^{k-1} Z_\alpha^i \le f_\alpha \le \frac{1}{k} + \frac{1}{k}\sum_{i=1}^{k-1} Z_\alpha^i$ and $\frac{1}{k}\sum_{i=1}^{k-1} Z^i \le f \le \frac{1}{k} + \frac{1}{k}\sum_{i=1}^{k-1} Z^i$. This implies $\left|f_\alpha - \frac{1}{k}\sum_{i=1}^{k-1} Z_\alpha^i\right| \le \frac{1}{k}$ and $\left|f - \frac{1}{k}\sum_{i=1}^{k-1} Z^i\right| \le \frac{1}{k}$. This gives $\big|\mu(f_\alpha) - \frac{1}{k}\sum_{i=1}^{k-1} \mu(Z_\alpha^i)\big| \le \frac{1}{k}$ and $\left|\mu(f) - \frac{1}{k}\sum_{i=1}^{k-1} \mu(Z^i)\right| \le \frac{1}{k}$. So $-\frac{1}{k} + \frac{1}{k}\sum_{i=1}^{k-1} \mu(Z_\alpha^i) \le$

$\mu(f_\alpha) \le \frac{1}{k} + \frac{1}{k}\sum_{i=1}^{k-1}\mu(Z^i_\alpha)$. Putting $p = \frac{1}{k}\sum_{i=1}^{k-1} Z^i$ and taking order limits, we get $|o - \limsup_\alpha \mu(f_\alpha) - p| \le \frac{1}{k}$ and $|o - \liminf_\alpha \mu(f_\alpha) - p| \le \frac{1}{k}$. Combining these two, we get $o - \limsup_\alpha \mu(f_\alpha) - o - \liminf_\alpha \mu(f_\alpha) \le \frac{2}{k}$. Letting $k \to \infty$, $o - \lim \mu(f_\alpha)$ exists. Using the fact that $|\mu(f) - p| \le \frac{1}{k}$, we get $|o - \lim \mu(f_\alpha) - \mu(f)| \le \frac{2}{k}$. Letting $k \to \infty$, we get the result. □

We denote by $M_{(o)}(X, E)$ the set of all order-bounded linear mappings $\mu\colon C(X) \to E$. Now we come to the next theorem.

Theorem 2. *Suppose X is a compact Hausdorff space and $\mu\colon C(X) \to E$ be an order-bounded, linear mapping.*

(i) *Then there is a unique countably additive Baire measure, which again we denote by μ, on X, such that the corresponding linear mapping $\mu\colon \beta_1(X) \to E$ extends the given mapping. Further μ can also be uniquely extended to a countably additive τ-smooth Borel measure.*

(ii) *The modulus of the Baire measure μ, determined from $\mu\colon C(X) \to E$ and $\mu\colon \beta_1(X) \to E$ are equal and also modulus of the Borel measure μ, determined from $\mu\colon C(X) \to E$ and $\mu\colon \beta(X) \to E$ are equal. Thus μ can be written as $\mu = \mu^+ - \mu^-$. For every τ-smooth Borel measure μ on X, there is the largest open set $V \subset X$ such that $|\mu|(V) = 0$; $C = X \setminus V$ is called the support of μ and has the property that any open $U \subset X$ such that $U \cap C \neq \emptyset$, we have $|\mu|(U) > 0$.*

(iii) *$M_{(o)}(X, E)$ is a Dedekind-complete vector lattice.*

Proof. (i) Since μ is order-bounded and E is a boundedly order-complete, we can write $\mu = \mu^+ - \mu^-$ ([13, Theorem 1.3.2, p. 24]). Now μ^+ and μ^- can be uniquely extended to E^+-valued, countably additive Baire measures and also to E^+-valued, countably additive τ-smooth Borel measures ([7], [21], [24]). Thus we get a countably additive Baire measure $\mu\colon \beta_1(X) \to E$ and a countably additive τ-smooth Borel measure $\mu\colon \beta(X) \to E$. Since the order σ-closure, in $\beta_1(X)$, of $C(X)$ is $\beta_1(X)$, for Baire measure, the uniqueness follows. Now we consider the case of Borel measure. Suppose two τ-smooth Borel measures μ_1, μ_2 are equal on $C(X)$. By Theorem 1, they are equal on bounded lower semi-continuous functions and so they are equal on the vector space generated by lower semi-continuous functions. Since the order σ-closure, in $\beta(X)$, of the vector space generated by lower semi-continuous functions is $\beta(X)$, by countable additivity they are equal on $\beta(X)$.

(ii) Let μ_1, μ_2 be the μ^+'s coming from $\mu\colon C(X) \to E$ and $\mu\colon \beta_1(X) \to E$ respectively. Evidently $\mu_2 \ge \mu_1$. Fix a $g \in C(X), g \ge 0$ and take an $h \in \beta_1(X)$, $0 \le h \le g$. Since $\mu(h) \le \mu_1(g)$, taking $\sup_{0 \le h \le g}$, we get $\mu_2(g) \le \mu_1(g)$. By ([18], Theorem 2.3, p.25), μ_2 is countably additive. Since $\mu_1 = \mu_2$ on $C(X)$, we get $\mu_1 = \mu_2$ on $\beta_1(X)$. The result follows now. The other result about the support of μ is easily verified.

(iii) It is a simple verification. □

Now we consider the case when X is a completely regular T_1 space and $\mu\colon \mathcal{F}(X) \to E$ a finitely additive, order-bounded measure. Because of order-boundedness, order modulus $|\mu|$ exists. μ will be called regular if for any $A \in \mathcal{F}(X)$, there exists an increasing net $\{Z_\alpha\}$ of zero-sets in X, $Z_\alpha \subset A$, $\forall\alpha$, and a deceasing net $\{\eta_\alpha\}$ in E such that $\eta_\alpha \downarrow 0$ and $|\mu|(A \setminus Z_\alpha) < \eta_\alpha$, $\forall\alpha$.

Theorem 3. *Suppose X be a completely regular T_1 space and $\mu\colon C_b(X) \to E$ be an order-bounded, linear mapping. Then there is unique, finitely additive, order-bounded measure, regular measure $\nu\colon \mathcal{F}(X) \to E$ such that $\mu(f) = \int f d\nu$, $\forall f \in C_b(X)$. $M_{(o)}(X, E)$ is a Dedekind-complete vector lattice.*

Proof. When μ is positive, then result is proved in ([12], p. 353). Since $\mu = \mu^+ - \mu^-$, using the result ([12], p. 353), we get a ν with the required properties. We denote ν by μ also

Uniqueness: Let $\mu\colon \mathcal{F}(X) \to E$ be an order-bounded, finitely additive, order-bounded measure, regular measure such that $\mu = 0$ on $C_b(X)$. Denoting by $S(X)$ the norm closure of $\mathcal{F}(X)$-simple real valued functions on X, we have $S(X) \supset C_b(X)$. Thus μ extends to $\mu\colon S(X) \to E$, is linear and order-bounded. Split $\mu = \mu^+ - \mu^-$. By the definition of regularity, $|\mu|$ is regular and so $\mu^+,\ \mu^-$ are regular and $\mu^+ = \mu^-$ on $C_b(X)$. Since both are regular, there is unique extension to $\mathcal{F}(X)$. This means $\mu^+ = \mu^-$ on $\mathcal{F}(X)$ and consequently $\mu^+ = \mu^-$ on $S(X)$. This proves uniqueness. It is easy to verify that $M_{(o)}(X, E)$ is a Dedekind-complete vector lattice. □

We come to countably additive (in order convergence), of order-bounded Baire measures on a completely regular T_1 space X. A countably additive, order-bounded $\mu\colon \mathcal{B}_1(X) \to E$ is called an order-bounded Baire measure on X. The collection of all such measures will be denoted by $M_{(o,\sigma)}(X, E)$.

Theorem 4. *For a be a completely regular T_1 space X, $M_{(o,\sigma)}(X, E)$ is a band in $M_{(o)}(X, E)$.*

Proof. Take a $\mu \in M_{(o,\sigma)}(X, E)$. By ([18], Theorem 2.3, p.25), $|\mu|, \mu^+, \mu^-$ are also in $M_{(o,\sigma)}(X, E)$. so $M_{(o,\sigma)}(X, E)$ is a vector sublattice of $M_{(o)}(X, E)$. Let $\{\mu_\alpha\}$ be positive, bounded, increasing net in $M_{(o,\sigma)}(X, E)$ and $\mu = \sup \mu_\alpha$ in $M_{(o)}(X, E)$. Then μ, defined for every $A \in \mathcal{B}_1(X)$, $\mu(A) = \sup \mu_\alpha(A)$, is finitely additive. Take an increasing sequence $\{A_n\} \subset \mathcal{B}_1(X)$ and let $A = \cup A_n$. Now $\mu(A) = o - \lim_\alpha \mu_\alpha(A) = o - \lim_\alpha \big(o - \lim_n \mu_\alpha(A_n)\big) \le o - \lim_n \mu(A_n) \le \mu(A)$. This proves μ is countably additive. This proves the result. □

We denote by $M_{(o,\tau)}(X, E)$ those $\mu \in M_{(o,\sigma)}(X, E)$ which can be extended to $\mu\colon \mathcal{B}(X) \to E$ and are τ-smooth, in the sense, that for any increasing net $\{V_\alpha\}$ of open subsets of X, $\mu(\cup V_\alpha) = o - \lim \mu(V_\alpha)$ (extension will obviously be unique if it exists).

Theorem 5. *For a completely regular T_1 space X, $M_{(o,\tau)}(X, E)$ is a band in $M_{(o,\sigma)}(X, E)$.*

Proof. Take a $\mu \in M_{(o,\tau)}(X, E)$. This gives a $\tilde{\mu} \in M_{(o)}(\tilde{X}, E)$, $\tilde{\mu}(B) = \mu(B \cap X)$ with the property that $\tilde{\mu}(B) = 0$ if $B \cap X = \emptyset$. It is a routine verification that $(\tilde{\mu})^+$, $(\tilde{\mu})^-$, $|\tilde{\mu}|$ all are $= 0$ on those Borel sets B for which $B \cap X = \emptyset$. For this it easily follows that, for any Borel set $B \subset X$, $\mu^+(B) = (\tilde{\mu})^+(B_0)$, where B_0 is any Borel subset of $\tilde{X}$ with $B_0 \cap X = B$; similar result for μ^- and $|\mu|$. To prove τ-smoothness of $|\mu|$, take a collection $\{V_\gamma; \gamma \in I\}$ of open subsets of X and select open subsets $\{U_\gamma; \gamma \in I\}$ in $\tilde{X}$ such that $U_\gamma \cap X = V_\gamma$. Let J be the collection of all finite subsets of I and order them by inclusion; also denote by α a general element of J. By the τ-smooth property of $|\tilde{\mu}|$ (Theorem 2), we have, $|\tilde{\mu}|(\cup U_\gamma) = o - \lim_\alpha |\tilde{\mu}|(\cup_{\gamma\in\alpha} U_\gamma)$. This means $|\mu|(\cup V_\gamma) = o - \lim_\alpha |\mu|(\cup_{\gamma\in\alpha} V_\gamma)$. This proves $|\mu|$ in τ-smooth. In a similar way μ^+ and μ^- are also τ-smooth.

Now the proof that it is a band in $M_{(o,\sigma)}(X, E)$ is very similar to what is done in Theorem 4. $\square$

We denote by $M_{(o,t)}(X, E)$ those $\mu \in M_{(o,\tau)}(X, E)$ which have the property that, for the measure $|\mu|$, open sets are inner regular by the compact subsets of X. From this definition it follows that if $\mu \in M_{(o,t)}(X, E)$ then μ^+, μ^-, $|\mu|$ are also in $M_{(o,t)}(X, E)$.

Theorem 6. *For a completely regular T_1 space X, $M_{(o,t)}(X, E)$ is a band in $M_{(o,\tau)}(X, E)$.*

Proof. $M_{(o,t)}(X, E)$ is already seen to be a vector sub-lattice of $M_{(o,\tau)}(X, E)$. Let $\{\mu_\alpha\}$ be positive, bounded, increasing net in $M_{(o,t)}(X, E)$ and $\mu = \sup \mu_\alpha$ in $M_{(o,\tau)}(X, E)$. Let V be an open subset of X. Let $\{C_\beta\}$ be the family of all compact subsets of V; this is filtering upwards. $\mu(V) = o - \lim_\alpha \mu_\alpha(V) = o - \lim_\alpha \big(o - \lim_\beta \mu_\alpha(C_\beta)\big) \leq o - \lim_\beta \mu(C_\beta) \leq \mu(V)$. This proves $\mu \in M_{(o,t)}(X, E)$. This proves the result. $\square$

If $\mu \in M_{(o,\tau)}(X, E)$, then it is easily seen that there is a smallest closed subset $Y \subset X$ such that $|\mu|(Y) = |\mu|(X)$. This Y is called the support of μ.

The following two theorems are well-knowm for scalar-valued measures ([20], [19]). We prove some extensions.

Theorem 7. *Let (X, d) be a metric space and E super Dekekind complete* ([14, p.78]) *and $\mu \in M_{(o,\tau)}(X, E^+)$. Then the support of μ is a separable subset of X.*

Proof. Let the support of μ be Y. Fix an $n \in N$ and let $\mathcal{A} = \big\{A \subset Y : d(x, y) \geq \frac{1}{n},\ \forall x \in A,\ \forall y \in A,\ x \neq y\big\}$. By Zorn's Lemma, $\mathcal{A}$ has a maximal element, say A_n. It is easily verified that that for any $x \in (Y \setminus A_n)$, there is a $y \in A_n$ such that $d(x, y) < \frac{1}{n}$. We claim that A_n is countable. Suppose not. Thus there is an uncountable collection $\big\{B(x, \frac{1}{2n}) : x \in A_n\big\}$ of mutually disjoint open subsets of Y and $\mu\big(B(x, \frac{1}{2n})\big) > 0$, $\forall x \in A_n$. Using τ-additivity of μ and the hypothesis that E is super Dekekind complete, we get, that except for countable $x \in A_n$, $\mu\big(B(x, \frac{1}{2n})\big) = 0$. Since Y is the support of μ, this is a contradiction. Thus A_n is countable and so $\cup A_n$ is dense in Y. This proves the result. $\square$

Theorem 8. *Let (X, d) be a complete metric space and E super Dekekind complete and also weakly σ-distributive* ([25])*. Then $M_{(o,\tau)}(X, E) = M_{(o,t)}(X, E)$.*

Proof. Take a $\mu \in M_{(o,\tau)}(X, E^+)$. By Theorem 7, we can assume X to be separable. Let Z be a compact metric space which is a compactification of X. It is well-known that X is a G_δ set in Z. Define $\bar{\mu}\colon \mathcal{B}(Z) \to E^+$, $\bar{\mu}(B) = \mu(B \cap X)$. It is obvious that $\bar{\mu} \in M_{(o)}(Z, E^+)$. It is Baire measure. Since E is weakly σ-distributive, $\bar{\mu}$ is inner regular by compact subset of Z. This means, since X is a Baire subset of Z, $\mu(X) = \sup\{\mu(C) : C$ compact and $C \subset X$. From this, it is a routine verification that $\mu \in M_{(o,t)}(X, E)$ (cf. [5]). $\square$

3. Representation theorem for $C(X)$, X completely regular

It is well-known that a linear map $\mu\colon C(X) \to R$, which maps order-bounded sets into bounded sets, gives a unique $\nu \in M_\sigma(X)$ such that $C(X) \subset L^1(\nu)$, $\mu(f) = \int f d\nu$, $\forall f \in C(X)$ and $\operatorname{supp}(\tilde{\nu}) \subset \upsilon X$ (the real-compactification of X) ([19, Theorem 23]). We will extend it to the vector case.

In this section $E = \big(C(S), \|\cdot\|\big)$, S being a Stone space and X completely regular T_1 space. We will prove a representation theorem for a positive linear map $\mu\colon C(X) \to E$. $B(X)$ denotes the space of all bounded real-valued functions. We will use the following results.

(**A**). Suppose F is a locally convex space whose topology is generated by the family $\big\{\|\cdot\|_p : p \in P\big\}$ of semi-norms, $M_\sigma(X, F)$ the space of all F-valued Baire measures on X, and $\mu\colon C(X) \to F$ be a linear map such that order-bounded subsets are mapped into relatively weakly compact subsets of F. Then:

(i) There is a unique $\nu \in M_\sigma(X, F)$ such that $C(X) \subset L^1(\nu)$ and $\mu(f) = \int f d\nu$, $\forall f \in C(X)$;

(ii) for every $p \in P$, there is compact $C \subset \upsilon X$ (the real-compactification of X), depending on p, such that $\bar{\tilde{\nu}}_p(\tilde{X} \setminus C) = 0$ ([9, Theorem 7]), $\bar{\tilde{\nu}}_p$ being the semi-variation of $\tilde{\nu}_p$.

(**B**). There is an order σ-continous positive linear map $\psi_1 : \beta_1(S) \to C(S)$ such that for every $f \in \beta_1(S)$, we get $f - \psi_1(f) = 0$ except on a meager set ([7, Lemma 2, p. 379]).

In the following theorem countable additivity is taken in the context of order convergence and integration and integrability in the sense of [21].

Theorem 9. *Suppose $\mu\colon C(X) \to E$ be a positive linear map. Then there is a unique E-valued positive Baire measure ν on X such that every $f \in C(X)$ is ν-integrable and $\mu(f) = \int f d\nu$, $\forall f \in C(X)$. Also the* $\operatorname{supp}(\tilde{\nu}) \subset \upsilon X$.

Proof. By taking the pointwise topology *pt* on $B(S)$ and noting that $C(S) \subset B(S)$, we have a positive linear map $\mu\colon C(X) \to \big(B(S), pt\big)$ with the property that order-bounded subsets of $C(X)$ are mapped into relatively weakly compact subsets of $\big(B(S), pt\big)$. By (**A**) there is a Baire measure $\lambda\colon \mathcal{B}_1(X) \to \big(B(S), pt\big)$ such that $C(X) \subset L_1(\lambda)$ ([10]) and $\mu(f) = \int f d\lambda$, $\forall f \in C(X)$. This measure is easily seen to be positive. Fix an $f \in C(X)$, $f \geq 0$ and let $f_n = f \wedge n$ $(n \in N)$. Put $h = \mu(f)$, $h_n = \mu(f_n)$. Since $f \in L_1(\lambda)$, $\lambda(f_n) \to \lambda(f)$ ([10]). From $\lambda^{-1}(\beta_1(S)) \supset C_b(X)$, we get $\lambda^{-1}(\beta_1(S) \supset \beta_1(X)$. Thus $\lambda\colon \mathcal{B}_1(X) \to \beta_1(S)$. Using (**B**) and defining $\nu = \psi_1 \circ \lambda$, we see that $\nu\colon \mathcal{B}_1(X) \to C(S)$ is countably additive in order convergence and $h_n = \mu(f_n) = \lambda(f_n) = \nu(f_n)$, $\forall n$. This means $h_n \uparrow h$ pointwise in $C(S)$ and so $o - \lim h_n = h$ in $C(S)$. By ([21, Prop. 3.3, p.113]) f is ν-integrable and $\int f d\nu = o - \lim \int f_n d\nu = o - \lim h_n = h = \lim h_n$ pointwise. This proves $\mu(f) = \int f d\nu$. This proves the result.

Uniqueness: If there is another E-valued positive Baire measure ν_0 on X having the above properties then $\mu(f) = \int f d\nu_0$, $\forall f \in C(X)$. Thus $\nu_0(f) = \nu(f)$, $\forall f \in C_b(X)$. Because of order countable additivity of ν_0 and ν, we get $\nu_0 = \nu$ on Baire subsets of X. This proves uniqueness.

Now we prove that $\mathrm{supp}(\tilde{\nu}) \subset \upsilon X$. Suppose $z \in \tilde{X} \setminus \upsilon X$ and $z \in (\mathrm{supp})(\tilde{\mu})$. Take an $f \geq 0, f \in C(X)$ with $\tilde{f}(z) = \infty$. Thus, for every n, $\tilde{\mu}(A_n) > 0$ where $A_n = \{x : \tilde{f}(x) > n\})$.

Suppose first that $\wedge_{n=1}^{\infty}(\tilde{\mu}(A_n)) = h > 0$ and put $f_n = f \wedge n$. Then $\tilde{f_n} = \tilde{f} \wedge n$. Now $\mu(f) \geq \mu(f_n) = \tilde{\mu}(\tilde{f} \wedge n) = \int(\tilde{f} \wedge n)d\tilde{\mu} \geq n\tilde{\mu}(A_n) \geq nh$. Since E is Archimedean, we get $h = 0$ which is a contradiction. Thus $h = 0$.

Since $\tilde{\mu}(A_n) > 0$ for every n and $h = 0$, select a strictly increasing sequence $\{a_k\}$ of positive integers such that $a_{k+1} - a_k > 4 \;\; \forall k$ and $h_k = \tilde{\mu}\big(\{x : a_{k+1} < \tilde{f}(x) < a_{k+2}\}\big) > 0, \;\; \forall k$. Let $p_k = \|h_k\| > 0$. Putting $B_k = f^{-1}\big([a_{k+1}, a_{k+2}]\big)$, $C_k = f^{-1}\big((a_{k+1} - 1, a_{k+2} + 1)\big)$, we see that B_k and C_k' are two disjoint zero subsets of X. Define a $g_k \in C_b(X)$, $g_k \geq 0$, $g_k \equiv 0$ on C_k' and $g_k \equiv k\frac{1}{p_k}$ on B_k. It is a routine verification that $g = \sum_{k=1}^{\infty} g_k \in C(X)$.

For $A \subset \tilde{X}$, $\overline{A}$ will denote its closure in $\tilde{X}$. Now $B_k \supset V \cap X$, where $V = \big\{x : a_{k+1} < \tilde{f}(x) < a_{k+2}\big\}$ is an open non-void subset of $\tilde{X}$. Since X is dense in $\tilde{X}$, $\overline{V \cap X} \supset V$ and so $\overline{B_k} \supset V$. Also $g_k \equiv k\frac{1}{p_k}$ on B_k implies $\tilde{g_k} \equiv k\frac{1}{p_k}$ on $\overline{B_k}$. So we get

$$\tilde{\mu}(\tilde{g_k}) \geq \int_{\overline{B_k}} \tilde{g_k} d\tilde{\mu} \geq k\frac{1}{p_k}\tilde{\mu}(V) = kh_k\frac{1}{p_k}\,.$$

We have, for every $n \in N$, $\mu(g) \geq \sum_{k=1}^{n} \mu(g_k) = \sum_{k=1}^{n} \tilde{\mu}(\tilde{g_k}) \geq \sum_{k=1}^{n} kh_k\frac{1}{p_k}$. Now $\big\|kh_k\frac{1}{p_k}\big\| = k$ and so $\|\mu(g)\| = \infty$ (note E is an AM space) which is a contradiction. This proves that $\mathrm{supp}(\tilde{\nu}) \subset \upsilon X$. □

Corollary 10. *Suppose $\mu\colon C(X) \to E$ be an order-bounded linear map* ([13, p.24]). *Then there is a unique E-valued Baire measure ν on X such that every $f \in C(X)$ is ν-integrabe and $\mu(f) = \int f d\nu, \;\; \forall f \in C(X)$ and* $\mathrm{supp}(\tilde{\mu}) \subset \upsilon X$.

Proof. By [13, Theorem 1.3.2, p.24], $\mu = \mu^+ - \mu^-$. Now μ^+ and μ^- are positive linear maps. Applying Theorem 9 to μ^+ and μ^- we get an E-valued Baire measure ν on X such that every $f \in C(X)$ is ν-integrabe and $\mu(f) = \int f d\nu$, $\forall f \in C(X)$. As in Theorem 9, the uniqueness of ν and $\mathrm{supp}(\tilde{\mu}) \subset \upsilon X$ can be proved.

4. The case of E with points separated by E_n^*

For the order complete vector lattice E, let E^* be its order dual and E_n^* its continuous order dual. In this section we assume that E_n^* separates the points of E. It is known that E_n^* is a band in E^* and order intervals in E_n^* are $\sigma(E_n^*, E)$-compact and convex ([14], [13]). $o(E, E_n)$ will denote the locally convex topology on E, of uniform convergence on the order intervals of E_n^*; in this topology the lattice operations are continuous and so the positive cone is closed and convex. Since this topology is compatible with the duality $< E, E_n^* >$, E_+ is also closed in $\sigma(E, E_n^*)$. $\square$

The following theorem is well-known. We include a new proof.

Theorem 11 ([16, Theorem 3])**.** *Suppose $\mathcal{A}$ be a σ-algebra of subsets of a set X and $\mu\colon \mathcal{A} \to E$ a finitely additive measure. Then μ is countably additive in order convergence iff μ is countably in the locally convex topology $\sigma(E, E_n^*)$.*

Proof. Obviously countably additivity in order convergence implies countably additivity in $\sigma(E, E_n^*)$. Assume that μ is countably in $\sigma(E, E_n^*)$; this means μ is countably additive in $o(E, E_n)$. We first prove that μ^+ countably additive in order convergence.

Fix a sequence $B_n \downarrow \emptyset$ in $\mathcal{A}$. Take a $C \subset X, C \in \mathcal{A}$. From $\mu(C - C \cap B_n) = \mu(B_n \cup C - B_n)$, we get $\mu(C) - \mu(C \cap B_n) \leq \mu^+(X) - \mu^+(B_n)$. Let $0 \leq z = \inf_n(\mu^+(B_n))$. Thus $z \leq \mu(C \cap B_n) + \mu^+(X) - \mu(C)$. Since $\mu(C \cap B_n) \to 0$ in $\sigma(E, E_n^*)$, we get, for every $f \in (E_n^*)_+$, $\langle f, z\rangle \leq \langle f, \mu(C \cap B_n)\rangle + \langle f, \mu^+(X) - \mu(C)\rangle$; using the fact $\mu(C \cap B_n) \to 0$ in $\sigma(E, E_n^*)$, this gives $\langle f, z\rangle \leq \langle f, \mu^+(X) - \mu(C)\rangle$ for every $f \in (E_n^*)_+$. Thus $z \leq \mu^+(X) - \mu(C)$ for every $C \in \mathcal{A}$. Taking inf of the right hand side as C varies in $\mathcal{A}$, we get $z = 0$. This proves μ^+ is countably additive in order convergence. Similarly μ^- is countably additive in order convergence and so μ is countably additive in order convergence. This proves the theorem. $\square$

The next theorem extends the well-known Alexanderov's theorem ([19], p. 195) about the convergent sequence of real-valued measures to our setting.

Theorem 12. *Suppose X is a completely regular T_1 space, E is a boundedly order-complete vector-lattice, E^* its order dual and E_n^* its continuos order dual. Assume that E_n^* separates the points of E. Let $\{\mu_n\} \subset M_{(o,\sigma)}(X, E)$ be a uniformly order-bounded sequence such that, in order convergence, $\mu(g) = \lim \mu_n(g)$ exists for every $g \in C_b(X)$. Then the order-bounded $\mu : C_b(X) \to E$ is generated by E-valued order-bounded Baire measure on X.*

Proof. Since the $\{\mu_n\}$ is uniformly order-bounded, we can assume that E has an order unit. By taking the order unit norm ([13, p.8]), we assume $E = C(S)$ for some hyperstonian space S. Thus $F = E_n^*$ is a band in E' and $E = F'$. Note the locally convex space $\big(E, \tau(E, E_n^*)\big) = \big(F', \tau(F', F)\big)$ is complete (Grothendieck completeness theorem ([15, Theorem 6.2, p.148])).

For every $g \in E_n^*$, $g \circ \mu_n \to g \circ \mu$, pointwise on $C_b(X)$ and $g \circ \mu_n \in M_\sigma(X)$, $\forall n$. Fix a $g \in E_n^*$ and take a sequence $\{f_m\} \subset C_b(X)$, $f_m \downarrow 0$. By ([19, p.195]), $g \circ \mu_n(f_m) \to g \circ \mu(f_m)$ as $n \to \infty$, uniformly in m. Thus $g \circ \mu(f_m) \to 0$. By ([20, Corollary 11.16]), $g \circ \mu\colon (C_b(X), \beta_\sigma) \to R$ is continuous, β_σ being the strict topology ([20]). Thus the weakly compact map $\mu\colon (C_b(X), \beta_\sigma) \to (E, \tau(E, E_n^*))$ is continuous in the weak topology $\sigma(E, E_n^*)$ on E ($\tau(E, E_n^*)$ is the Mackey topology in the duality $\langle E, E_n^*\rangle$); since the topology β_σ is Mackey ([20]), it is continuous. Since $\big(E, \tau(E, E_n^*)\big)$ is complete, by ([9, Theorem 2]), μ can be extended to an E-valued Baire measure which is countably additive in $\tau(E, E_n^*)$. This implies that

μ is countably additive in $\sigma(E, E_n^*)$. By Theorem 11, μ is countably additive in order convergence. $\square$

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