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GLOBAL ASYMPTOTIC STABILITY FOR HALF-LINEAR DIFFERENTIAL SYSTEMS WITH COEFFICIENTS OF INDEFINITE SIGN

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Abstract. This paper is concerned with the global asymptotic stability of the zero solution of the half-linear differential system

$$x' = -\,e(t)x + f(t)\phi_{p^*}(y)\,, \quad y' = -\,g(t)\phi_p(x) - h(t)y\,,$$

where $p > 1$, $p^* > 1$ $(1/p + 1/p^* = 1)$, and $\phi_q(z) = |z|^{q-2}z$ for $q = p$ or $q = p^*$. The coefficients are not assumed to be positive. This system includes the linear differential system $\mathbf{x}' = A(t)\mathbf{x}$ with $A(t)$ being a 2×2 matrix as a special case. Our results are new even in the linear case $(p = p^* = 2)$. Our results also answer the question whether the zero solution of the linear system is asymptotically stable even when Coppel's condition does not hold and the real part of every eigenvalue of $A(t)$ is not always negative for t sufficiently large. Some suitable examples are included to illustrate our results.

1. Introduction

We consider a system of differential equations of the form

$$(S) \qquad \begin{aligned} x' &= -\,e(t)x + f(t)\phi_{p^*}(y)\,,\\ y' &= -\,g(t)\phi_p(x) - h(t)y\,, \end{aligned}$$

where the prime denotes d/dt; the variable coefficients $e(t)$, $f(t)$, $g(t)$, and $h(t)$ are continuous for $t \ge 0$; the two numbers p and p^* are positive and satisfy $(p-1)(p^*-1) = 1$; the function $\phi_q(z)$ is defined by

$$\phi_q(z) = |z|^{q-2}z$$

for some $q > 1$. System (S) has a close relation to the half-linear differential equation

$$(\phi_p(x'))' + h(t)\phi_p(x') + g(t)\phi_p(x) = 0\,, \tag{1.1}$$

because the substitution $y = \phi_p(x')$ transforms equation (1.1) into system (S) with $e(t) = 0$ and $f(t) = 1$. If $x(t)$ is a solution of (1.1), then the function $cx(t)$ is also a solution, where c is any constant. However, the sum of two solutions

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of (1.1) is not always a solution. This means that the solution space of (1.1) is homogeneous, but not additive. For this reason, equation (1.1) is often called *half-linear*. Note that if $\big(x(t), y(t)\big)$ is a solution of (S), then $(cx(t), \phi_p(c)y(t))$ is another solution. As for half-linear differential equations, we can refer the reader to [1, 2, 6, 8, 7, 9, 14, 16, 18, 21, 22] and the references cited therein.

We say that the zero solution of (S) is *globally asymptotically stable* if it is stable and if every solution $\big(x(t), y(t)\big)$ of (S) tends to $(0,0)$ as time t increases. Although many studies have been made on equation (1.1) and system (S) in the last four decades, there seems to be little research on the global asymptotic stability of the zero solution of (S). The first purpose of this paper is to give sufficient conditions for the zero solution of (S) to be globally asymptotically stable.

System (S) naturally includes the linear system

$$(L) \qquad \mathbf{x}' = A(t)\mathbf{x}\,,$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} -e(t) & f(t) \\ -g(t) & -h(t) \end{pmatrix}.$$

as a special case. Needless to say, the zero solution of (L) is globally asymptotically stable if and only if it is (locally) attractive; that is, every solution $\mathbf{x}(t)$ of (L) tends to $(0,0)$ as $t \to \infty$ whenever $\mathbf{x}(t_0)$ is near enough $(0,0)$, where $t_0 \geq 0$ is the initial time.

In the autonomous case $A(t) = A$, it is a well-known fact that the zero solution of (L) is asymptotically stable if and only if every eigenvalue of A has negative real part. However, this result does not hold for nonautonomous case. For example, we consider the matrices

$$B(t) = \begin{pmatrix} -1 - 9\cos^2 6t + 12\sin 6t\cos 6t & 12\cos^2 6t + 9\sin 6t\cos 6t \\ -12\sin^2 6t + 9\sin 6t\cos 6t & -1 - 9\sin^2 6t - 12\sin 6t\cos 6t \end{pmatrix}$$

and

$$C(t) = \begin{pmatrix} -1 + \dfrac{3}{2}\cos^2 t & 1 - \dfrac{3}{2}\sin t\cos t \\ -1 - \dfrac{3}{2}\sin t\cos t & -1 + \dfrac{3}{2}\sin^2 t \end{pmatrix}.$$

Although the eigenvalues of $B(t)$ and $C(t)$ are -1, -10 and $(-1+\sqrt{7}i)/4$, $(-1-\sqrt{7}i)/4$, respectively, the nonautonomous linear systems $\mathbf{x}' = B(t)\mathbf{x}$ and $\mathbf{x}' = C(t)\mathbf{x}$ have unbounded solutions

$$\mathbf{x}(t) = e^{2t}\begin{pmatrix} \cos 6t + 2\sin 6t \\ 2\cos 6t - \sin 6t \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = e^{t/2}\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix},$$

respectively. Hence, the zero solutions of the systems are not asymptotically stable (for more details, see [17, 23, 25]). For this reason, to show that the zero solution of (L) is asymptotically stable, we need some alternative condition to the assumption that every eigenvalue of $A(t)$ has negative real part for all $t \geq 0$. We can find such an condition in the book of Coppel [3]. Before we state his result, let us introduce some notations.

Let $\|\mathbf{x}\|$ be an arbitrary norm of a vector $\mathbf{x}$. For any matrix M, we define the induced norm of M to be

$$\|M\| = \sup_{\|\mathbf{x}\|=1} \|M\mathbf{x}\|$$

and denote by M^* the conjugate transpose of M. Let $\mu(M)$ be a measure defined by

$$\mu(M) = \lim_{h\to+0} \frac{\|E + hM\| - 1}{h},$$

where E is the unit matrix. If $\|\mathbf{x}\|$ denotes the Euclidean norm of $\mathbf{x}$, then the measure $\mu(M)$ is equal to the largest eigenvalue of the Hermitian matrix $H = (M + M^*)/2$. It is easy to show that if every eigenvalue of H has negative real part, then every eigenvalue of M also has negative real part.

Coppel [3, Chap. 3] have presented a simple sufficient condition for the zero solution of (L) to be asymptotically stable: if

$$\lim_{t\to\infty} \int^t \mu[A(s)]\, ds = -\infty, \tag{1.2}$$

then the zero solution of (L) is asymptotically stable. For example, we consider the symmetric matrix

$$A(t) = \begin{pmatrix} -3 + t\sin t & \cos t \\ \cos t & -3 + t\sin t \end{pmatrix}.$$

Since the eigenvalues of $A(t)$ are $-3 + t\sin t + |\cos t|$ and $-3 + t\sin t - |\cos t|$, both eigenvalues are not always negative for all $t \ge 0$. On the other hand, the matrix $A(t)$ satisfies condition (1.2). In fact, since $\mu[A(t)] = -3 + t\sin t + |\cos t| \le -2 + t\sin t$, we have

$$\begin{aligned}\int^t \mu[A(s)]\, ds &\le \int^t (-2 + s\sin s)\, ds \\ &= -2t - t\cos t + \sin t \le -t + 1,\end{aligned}$$

which tends to $-\infty$ as $t \to \infty$.

Note that if $A(t)$ is a constant matrix A, then condition (1.2) is stronger than the assumption that every eigenvalue of A has negative real part. From an easy calculation, we see that the eigenvalues of $(B(t) + B^*(t))/2$ are -13 and 2, and those of $(C(t) + C^*(t))/2$ are -1 and $1/2$. Hence, if $\|\mathbf{x}\|$ means the Euclidean norm of $\mathbf{x}$, then the measures $\mu[B(t)]$ and $\mu[C(t)]$ are 2 and $1/2$ for $t \ge 0$, respectively, and therefore, the matrices $B(t)$ and $C(t)$ do not satisfy condition (1.2). Other attempts were made to obtain an additional condition to the assumption that every eigenvalue of $A(t)$ has negative real part for all $t \ge 0$ (for example, see [4, 5]).

Hatvani [10] have discussed the asymptotic stability for system (L) from a different angle and obtained some theorems with applications. The following result is reached by one of his theorems.

Theorem A. *Suppose that* $e(t) = 0$, $f(t) > 0$, $g(t) > 0$, $h(t) \ge 0$, *and* $g(t)/f(t)$ *is continuously differentiable for* $t \ge 0$. *If*

(i) $\displaystyle\lim_{t\to\infty}\left(\log\sqrt{\frac{g(t)}{f(t)}}+\int^t h(s)ds\right)=\infty;$

(ii) *there exists a constant* $K>0$ *such that*

$$\frac{2}{K}\sqrt{f(t)g(t)}\le\frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)'+2h(t)\quad \textit{for}\ \ t\ge 0\,;$$

(iii) $\displaystyle\lim_{t\to\infty}\int^t f(s)\int^s g(\tau)\exp\left(-\int_\tau^s h(u)du\right)d\tau\,ds=\infty,$

then the zero solution of (L) *is asymptotically stable.*

Since $e(t)=0$, $f(t)>0$, $g(t)>0$, and $h(t)\ge 0$ for $t\ge 0$ in Theorem A, we see that $\mu[A(t)]\ge 0$ for $t\ge 0$. Hence, condition (1.2) is not satisfied. On the other hand, every eigenvalue of $A(t)$ has non-positive real part.

As to when condition (1.2) does not hold and the real part of every eigenvalue of $A(t)$ is not always non-positive for t sufficiently large, the following question then arises. What kind of condition will guarantee that the zero solution of (L) is asymptotically stable? The second purpose of this paper is to answer the question.

In Section 2, in order to accomplish our first purpose, we give our standard theorem which declares that the zero solution of (S) is globally asymptotically stable under the assumption that coefficients are bounded. A certain growth condition on the coefficients plays a major role in the proof of the theorem.

For illustration of our theorem, we take some concrete examples and draw a positive orbit of (S) (or (L)) in Section 3. Here, we call the projection of a positive semitrajectory of (S) onto the phase plane a *positive orbit*. The positive orbit in each figure exhibits complicated behavior and it ultimately tends to the origin. Also, we gain the second purpose through an example.

Finally in Section 4, we ease the assumption that coefficients are bounded and present the main theorem with an example.

2. The case in which coefficients are bounded

Let

$$E(t)=\int_0^t e(s)\,ds\,.$$

In this section, we assume that $f(t)$, $g(t)$, $h(t)$, and $E(t)$ are bounded and $g(t)/f(t)$ is continuously differentiable for $t\ge 0$, and

$$f(t)g(t)>0\quad\text{for}\ \ t\ge 0\qquad\text{and}\qquad \liminf_{t\to\infty}f(t)g(t)>0\,. \tag{2.1}$$

We will relax the boundedness of $g(t)$ and $E(t)$, and $\liminf_{t\to\infty}f(t)g(t)>0$ later. Although Theorem 2.1 below is a special case of the main result given in Section 4, the proof of the main result will be made clear if we first demonstrate Theorem 2.1.

For the sake of convenience, we write

$$\psi(t)=p^*h(t)-p\,e(t)+\frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)' \tag{2.2}$$

and define

$$V(t,x,y) = \exp(pE(t))\left(\frac{f(t)|y|^{p^*}}{p^*g(t)} + \frac{|x|^p}{p}\right).$$

From (2.1) and the boundedness of $f(t)$, $g(t)$, and $E(t)$, there exist positive constants $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$, and $\overline{\beta}$ such that

$$\underline{\alpha} \le \frac{f(t)}{g(t)} \le \overline{\alpha} \quad \text{and} \quad \underline{\beta} \le \exp(pE(t)) \le \overline{\beta} \tag{2.3}$$

for $t \ge 0$. Hence, we have

$$\underline{\beta}\left(\frac{\underline{\alpha}|y|^{p^*}}{p^*} + \frac{|x|^p}{p}\right) \le V(t,x,y) \le \overline{\beta}\left(\frac{\overline{\alpha}|y|^{p^*}}{p^*} + \frac{|x|^p}{p}\right)$$

for $t \ge 0$ and $(x,y) \in \mathbb{R}^2$, and therefore, $V(t,x,y)$ is positive definite and decrescent, and

$$V(t,x,y) \to \infty \quad \text{as} \quad |x| + |y| \to \infty \quad \text{uniformly for} \quad t \ge 0\,.$$

Taking account of the equalities

$$\frac{d}{dz}|z|^q = q\phi_q(z) \quad \text{and} \quad z\phi_q(z) = |z|^q$$

for $q = p$ or $q = p^*$, we have

$$\begin{aligned}
\dot{V}_{(S)}(t,x,y) &= pe(t)\exp(pE(t))\left(\frac{f(t)|y|^{p^*}}{p^*g(t)} + \frac{|x|^p}{p}\right) \\
&\quad + \exp(pE(t))\left\{\frac{f(t)}{g(t)}\phi_{p^*}(y)y' + \phi_p(x)x' + \left(\frac{f(t)}{g(t)}\right)'\frac{|y|^{p^*}}{p^*}\right\} \\
&= pe(t)\exp(pE(t))\left(\frac{f(t)|y|^{p^*}}{p^*g(t)} + \frac{|x|^p}{p}\right) \\
&\quad + \exp(pE(t))\Big(-f(t)\phi_p(x)\phi_{p^*}(y) - \frac{f(t)h(t)}{g(t)}y\phi_{p^*}(y)\Big) \\
&\quad + \exp(pE(t))\big(-e(t)x\phi_p(x) + f(t)\phi_p(x)\phi_{p^*}(y)\big) \\
&\quad + \exp(pE(t))\left(\frac{f(t)}{g(t)}\right)'\frac{|y|^{p^*}}{p^*} \\
&= \frac{\exp(pE(t))f(t)|y|^{p^*}}{p^*g(t)}\big(pe(t) - p^*h(t)\big) + \exp(pE(t))\left(\frac{f(t)}{g(t)}\right)'\frac{|y|^{p^*}}{p^*} \\
&= -\frac{\exp(pE(t))f(t)|y|^{p^*}}{p^*g(t)}\left\{p^*h(t) - pe(t) - \frac{g(t)}{f(t)}\left(\frac{f(t)}{g(t)}\right)'\right\}.
\end{aligned}$$

Since

$$-\frac{g(t)}{f(t)}\left(\frac{f(t)}{g(t)}\right)' = \frac{f(t)}{g(t)}\left(\frac{g(t)}{f(t)}\right)',$$

we obtain

$$\dot{V}_{(S)}(t,x,y) = -\frac{\exp(pE(t))f(t)|y|^{p^*}\psi(t)}{p^*g(t)}\,.$$

If $\psi(t) \geq 0$ for $t \geq 0$, then by (2.1) we have

$$\dot{V}_{(S)}(t,x,y) \leq 0$$

for $t \geq 0$ and $(x,y) \in \mathbb{R}^2$. We therefore conclude that the zero solution of (S) is uniformly stable and all solutions of (S) are uniformly bounded by using Liapunov-type theorems (for example, see [15, 20, 24]). Thus, it is enough to give sufficient conditions for every solution $\big(x(t),y(t)\big)$ of (S) to tend to $(0,0)$ as $t \to \infty$.

It is well-known that the condition

$$\int_0^\infty \psi(t)\,dt = \infty \tag{2.4}$$

is insufficient for the zero solution of (S) to be attractive. For example, consider system (L) with

$$A(t) = \begin{pmatrix} 0 & k \\ -k & -h(t) \end{pmatrix}, \tag{2.5}$$

where k is a positive number and $h(t)$ is an on-off function defined by

$$h(t) = \begin{cases} 1 & \text{if } (2n-1)\pi - \dfrac{1}{n} \leq t \leq (2n-1)\pi, \quad n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi(t) = 2h(t)$ and condition (2.4) is satisfied. However, the zero solution is not asymptotically stable (for the proof, see [11, 12, 13, 19]).

For the reason above, we have to impose a stronger condition on $\psi(t)$ than (2.4) for the attractivity of the zero solution of (S). We adopt the following concept given by Hatvani and Totik [13] (see also [10]). The function $\psi(t)$ is said to be *weakly integrally positive* if

$$\int_I \psi(s)\,ds = \infty$$

holds on every set $I = \displaystyle\bigcup_{n=1}^\infty [\tau_n, \sigma_n]$ such that $\tau_n + \delta \leq \sigma_n < \tau_{n+1} \leq \sigma_n + \Delta$ for some $\delta > 0$ and $\Delta > 0$. For example, $\psi(t) = \sin^2 t/(1+t)$ is weakly integrally positive (refer to [10]). If $\psi(t)$ is weakly integrally positive, then condition (2.4) is naturally satisfied.

We are now ready to state Theorem 2.1.

Theorem 2.1. *Suppose that $E(t)$, $f(t)$, $g(t)$, and $h(t)$ are bounded and that $g(t)/f(t)$ is continuously differentiable for $t \geq 0$. Suppose also that condition* (2.1) *holds. If $\psi(t)$ is nonnegative for $t \geq 0$ and it is weakly integrally positive, where $\psi(t)$ is the function given by* (2.2), *then the zero solution of (S) is globally asymptotically stable.*

Proof. Let

$$V(t,x,y) = \exp(pE(t))\left(\frac{f(t)|y|^{p^*}}{p^* g(t)} + \frac{|x|^p}{p}\right).$$

Then, as mentioned above, it follows from classical Lyapunov's direct methods that the zero solution of (S) is stable and all solutions of (S) are uniformly bounded. The global existence of solutions of (S) are guaranteed as a matter of course. For this reason, we have only to prove that every solution of (S) approaches the origin.

Let $(x(t), y(t))$ be a solution of (S) with the initial time $t_0 \geq 0$ and let $v(t) = V(t, x(t), y(t))$. From (2.3), we see that

$$\underline{\beta}\left(\frac{\underline{\alpha}|y(t)|^{p^*}}{p^*} + \frac{|x(t)|^p}{p}\right) \leq v(t) \leq \overline{\beta}\left(\frac{\overline{\alpha}|y(t)|^{p^*}}{p^*} + \frac{|x(t)|^p}{p}\right) \tag{2.6}$$

for $t \geq t_0$. Since

$$v'(t) = -\frac{\exp(pE(t))f(t)|y(t)|^{p^*}\psi(t)}{p^* g(t)} \leq 0\,, \tag{2.7}$$

$v(t)$ is nonincreasing for $t \geq t_0$, and therefore, $v(t)$ has a limiting value $v_0 \geq 0$. For the case in which $v_0 = 0$, it follows from (2.6) that the solution $\big(x(t), y(t)\big)$ tends to $(0, 0)$ as $t \to \infty$, as required. Hereafter, we will show that the other case does not occur.

Suppose that $v_0 > 0$. Let

$$z(t) = \frac{\exp(pE(t))f(t)|y(t)|^{p^*}}{p^* g(t)}\,.$$

Then, using (2.3) again, we have

$$\frac{\underline{\alpha}\underline{\beta}|y(t)|^{p^*}}{p^*} \leq z(t) \leq \frac{\overline{\alpha}\overline{\beta}|y(t)|^{p^*}}{p^*} \tag{2.8}$$

for $t \geq t_0$. Since $y(t)$ is bounded, $z(t)$ is also bounded. Hence, $z(t)$ has the inferior limit and the superior limit.

Claim 1: $\liminf_{t\to\infty} z(t) = 0$. By way of contradiction, we suppose that there exist an $\varepsilon_0 > 0$ and a $T_1 \geq t_0$ such that $z(t) > \varepsilon_0$ for $t \geq T_1$. From (2.7) and the fact that $v(t) \geq v_0$ for $t \geq t_0$, it turns out that

$$v(t_0) - v_0 \geq v(t_0) - v(t) = -\int_{t_0}^{t} v'(s)\,ds = \int_{t_0}^{t} z(s)\psi(s)\,ds\,.$$

Since $\psi(t)$ is nonnegative for $t \geq 0$ and weakly integrally positive, we obtain

$$v(t_0) > \int_{t_0}^{\infty} z(s)\psi(s)\,ds > \varepsilon_0 \int_{T_1}^{\infty} \psi(s)\,ds = \infty\,,$$

which is a contradiction. We therefore conclude that $\liminf_{t\to\infty} z(t) = 0$.

Claim 2: $\limsup_{t\to\infty} z(t) = 0$. We suppose that $\limsup_{t\to\infty} z(t) > 0$. From (2.1) and the boundedness of $g(t)$ and $h(t)$, we can choose two positive numbers $\underline{g}$ and $\overline{h}$ such that

$$\underline{g} \leq |g(t)| \quad \text{and} \quad |h(t)| \leq \overline{h} \tag{2.9}$$

for $t \geq 0$. Let $\varepsilon > 0$ be so small that

$$\overline{h}\left(\frac{p^*\varepsilon}{\underline{\alpha}\underline{\beta}}\right)^{1/p^*} < \underline{g}\left(\frac{p(v_0 - \varepsilon)}{\overline{\beta}}\right)^{1/p^*}. \tag{2.10}$$

Taking account of Claim 1, we can find two sequences $\{\tau_n\}$ and $\{\sigma_n\}$ with $t_0 < \tau_n < \sigma_n < \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ such that $z(\tau_n) = z(\sigma_n) = \varepsilon$ and

$$z(t) \geq \varepsilon \quad \text{for} \;\; \tau_n < t < \sigma_n\,,$$
$$0 \leq z(t) \leq \varepsilon \quad \text{for} \;\; \sigma_n < t < \tau_{n+1}\,.$$

From (2.3), we have

$$\frac{|x(t)|^p}{p} = \exp(-pE(t))(v(t) - z(t)) \geq \frac{v_0 - \varepsilon}{\overline{\beta}} > 0$$

for $\sigma_n \leq t \leq \tau_{n+1}$. This inequality is rewritten as

$$\big|\phi_p(x(t))\big| \geq \left(\frac{p(v_0 - \varepsilon)}{\overline{\beta}}\right)^{1/p^*} \quad \text{for} \quad \sigma_n \leq t \leq \tau_{n+1}\,. \tag{2.11}$$

It also follows from (2.8) that

$$|y(t)| \leq \left(\frac{p^* z(t)}{\alpha\underline{\beta}}\right)^{1/p^*} \leq \left(\frac{p^*\varepsilon}{\alpha\underline{\beta}}\right)^{1/p^*} \tag{2.12}$$

for $\sigma_n \leq t \leq \tau_{n+1}$. Using the second equation in system (S), we get

$$\big|g(t)\phi_p(x(t))\big| \leq |y'(t)| + |h(t)y(t)| \quad \text{for} \quad t \geq t_0\,. \tag{2.13}$$

Hence, by (2.9)–(2.12), we have

$$\begin{aligned} |y'(t)| &\geq \big|g(t)\phi_p(x(t))\big| - |h(t)y(t)| \\ &\geq \underline{g}\left(\frac{p(v_0 - \varepsilon)}{\overline{\beta}}\right)^{1/p^*} - \overline{h}\left(\frac{p^*\varepsilon}{\alpha\underline{\beta}}\right)^{1/p^*} > 0 \end{aligned}$$

for $\sigma_n \leq t \leq \tau_{n+1}$. Let

$$\lambda = \underline{g}\left(\frac{p(v_0 - \varepsilon)}{\overline{\beta}}\right)^{1/p^*} - \overline{h}\left(\frac{p^*\varepsilon}{\alpha\underline{\beta}}\right)^{1/p^*},$$

which is independent of n. Then we obtain

$$\begin{aligned} |y(\tau_{n+1})| + |y(\sigma_n)| &\geq |y(\tau_{n+1}) - y(\sigma_n)| \\ &= \left|\int_{\sigma_n}^{\tau_{n+1}} y'(s)\,ds\right| = \int_{\sigma_n}^{\tau_{n+1}} |y'(s)|\,ds \geq \lambda\,(\tau_{n+1} - \sigma_n)\,. \end{aligned}$$

Since $y(t)$ is bounded for $t \geq t_0$, there exists a constant $\Delta > 0$ such that

$$\tau_{n+1} \leq \sigma_n + \Delta \quad \text{for} \quad n \in \mathbb{N}\,. \tag{2.14}$$

Again, from (2.7), we see that

$$\varepsilon\int_{\tau_n}^{\sigma_n} \psi(s)\,ds \leq \int_{\tau_n}^{\sigma_n} z(s)\psi(s)\,ds = -\int_{\tau_n}^{\sigma_n} v'(s)\,ds = v(\tau_n) - v(\sigma_n)$$

for each $n \in \mathbb{N}$. Since $v(t)$ positive and nonincreasing for $t \geq t_0$, we have

$$\varepsilon\int_I \psi(s)\,ds \leq v(\tau_1) < \infty\,,$$

where $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$. Hence, from (2.14) and the weak integral positivity of $\psi(t)$, we see that

$$\liminf_{n\to\infty} (\sigma_n - \tau_n) = 0\,. \tag{2.15}$$

Now, let $\nu = \limsup_{t\to\infty} z(t) > 0$. Then, by means of Claim 1 again, we can select two sequences $\{t_i\}$ and $\{s_i\}$ with $t_0 < t_i < s_i < t_{i+1}$ and $t_i \to \infty$ as $i \to \infty$ such that $z(t_i) = \nu/2$, $z(s_i) = 3\nu/4$ and

$$\frac{\nu}{2} < z(t) < \frac{3\nu}{4} \quad \text{for} \quad t_i < t < s_i\,.$$

Judging from (2.10), we may regard ε as being small enough. Hence, for any $i \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $[t_i, s_i] \subset [\tau_n, \sigma_n]$. From (2.15), we see that

$$\liminf_{i\to\infty} (s_i - t_i) = 0\,. \tag{2.16}$$

Since $x(t)$, $y(t)$, and $f(t)$ are bounded for $t \ge t_0$, there exists an $L > 0$ such that

$$|f(t)\phi_p(x(t))\phi_{p^*}(y(t))| \le L \quad \text{for} \ \ t \ge t_0\,.$$

Hence, by (2.3) and (2.7), we have

$$\begin{aligned}
z'(t) &= v'(t) - \left(\frac{\exp(pE(t))|x(t)|^p}{p}\right)' \\
&= v'(t) - \exp(pE(t))\phi_p(x(t))\big\{e(t)x(t) + x'(t)\big\} \\
&\le f(t)\exp\big(pE(t)\big)\phi_p\big(x(t)\big)\phi_{p^*}(y(t)) \\
&\le \exp\big(pE(t)\big)\big|f(t)\phi_p\big(x(t)\big)\phi_{p^*}\big(y(t)\big)\big| \le \overline{\beta}L
\end{aligned}$$

for $t \ge t_0$. Integrate this inequality from t_i to s_i to obtain

$$\frac{\nu}{4} = z(s_i) - z(t_i) \le \overline{\beta}L(s_i - t_i)$$

for each $i \in \mathbb{N}$. This contradicts (2.16). Thus, we conclude that $\nu = 0$. Claim 2 is now proved.

From (2.8) and Claim 2 it turns out that $y(t)$ tends to zero as $t \to \infty$. Since $\lim_{t\to\infty} v(t) = v_0$ and $\lim_{t\to\infty} z(t) = 0$, it follows that

$$\lim_{t\to\infty} \frac{\exp(pE(t))|x(t)|^p}{p} = v_0\,.$$

Hence, together with (2.3) and (2.9), we get

$$\begin{aligned}
\liminf_{t\to\infty} \big|g(t)\phi_p(x(t))\big| &\ge \liminf_{t\to\infty} \frac{\underline{g}\exp((p-1)E(t))|\phi_p(x(t))|}{\overline{\beta}^{1/p^*}} \\
&= \underline{g}\left(\frac{pv_0}{\overline{\beta}}\right)^{1/p^*} > 0\,.
\end{aligned}$$

Since $h(t)$ is bounded for $t \ge 0$ and $y(t)$ tends to zero as $t \to \infty$, we see that

$$\lim_{t\to\infty} |h(t)y(t)| = 0\,.$$

Hence, using (2.13) again, we obtain

$$\liminf_{t\to\infty} |y'(t)| > 0\,.$$

We therefore conclude that there exist numbers $T_2 > t_0$ and an $M > 0$ such that $|y'(t)| \ge M$ for $t \ge T_2$. Integrating this inequality from T_2 to $t \ge T_2$, we have

$$|y(t) - y(T_2)| = \left|\int_{T_2}^{t} y'(s)\,ds\right| = \int_{T_2}^{t} |y'(s)|\,ds \ge M(t - T_2)\,,$$

which tends to ∞ as $t \to \infty$. This contradicts the fact that $y(t)$ tends to zero as $t \to \infty$. Thus, the case of $v_0 > 0$ does not happen.

The proof of Theorem 2.1 is now complete. □

3. Illustrations

As mentioned in Section 1, the zero solution of (L) with

$$A(t) = A = \begin{pmatrix} -e & f \\ -g & -h \end{pmatrix},$$

where e, f, g, and h are constants, is asymptotically stable if and only if every eigenvalue of A has negative real part. It is clear that the condition

$$-(e + h) = \operatorname{tr} A < 0 < \det A = eh + fg$$

is necessary and sufficient for every eigenvalue of A to have negative real part. Hence, to show that the zero solution is asymptotically stable, we need to assume that either e or h is positive. For this reason, it might be natural to consider system (L) (or system (S)) under the assumption that the variable coefficients $e(t)$ and $h(t)$ are nonnegative for $t \ge 0$ (for example, see Theorem A). However, in order to cover many practical cases, we did not assume the nonnegativity of $e(t)$ and $h(t)$ in Theorem 2.1.

To illustrate Theorem 2.1, we give two examples: one is used for system (S) and the other is used for system (L). Since systems (S) and (L) are nonautonomous, positive orbits have various shapes even if those initial times are the same. Consequently, if we describe several positive orbits in the same figure, then we cannot find the essential feature of positive orbits. For this reason, we sketch only one positive orbit in each figure.

Example 3.1. Consider system (S) with $p = 3$,

$$e(t) = \frac{3}{2}\sin t\,, \quad f(t) = \frac{1}{7}\,, \quad g(t) = e^{-3\cos t} \quad \text{and} \quad h(t) = \sin t + \frac{2\sin^2 t}{3(1+t)}\,. \tag{3.1}$$

Then the zero solution of (S) is globally asymptotically stable.

It is clear that $f(t)$, $g(t)$, $h(t)$ and $E(t)$ are bounded and $g(t)/f(t)$ is continuously differentiable for $t \ge 0$. Since $f(t)g(t) = 7e^{-3\cos t} \ge 7/e^3$ for $t \ge 0$, condition (2.1) is satisfied. Also, we have

$$\psi(t) = \frac{3}{2}\left(\sin t + \frac{2\sin^2 t}{3(1+t)}\right) - \frac{9}{2}\sin t + \frac{e^{3\cos t}}{7}\left(\frac{7}{e^{3\cos t}}\right)'$$
$$= \frac{3}{2}\left(\sin t + \frac{2\sin^2 t}{3(1+t)}\right) - \frac{9}{2}\sin t + 3\sin t = \frac{\sin^2 t}{1+t} \geq 0$$

for $t \geq 0$. Hence, $\psi(t)$ is weakly integrally positive. Thus, by Theorem 2.1 the zero solution of (S) is globally asymptotically stable.

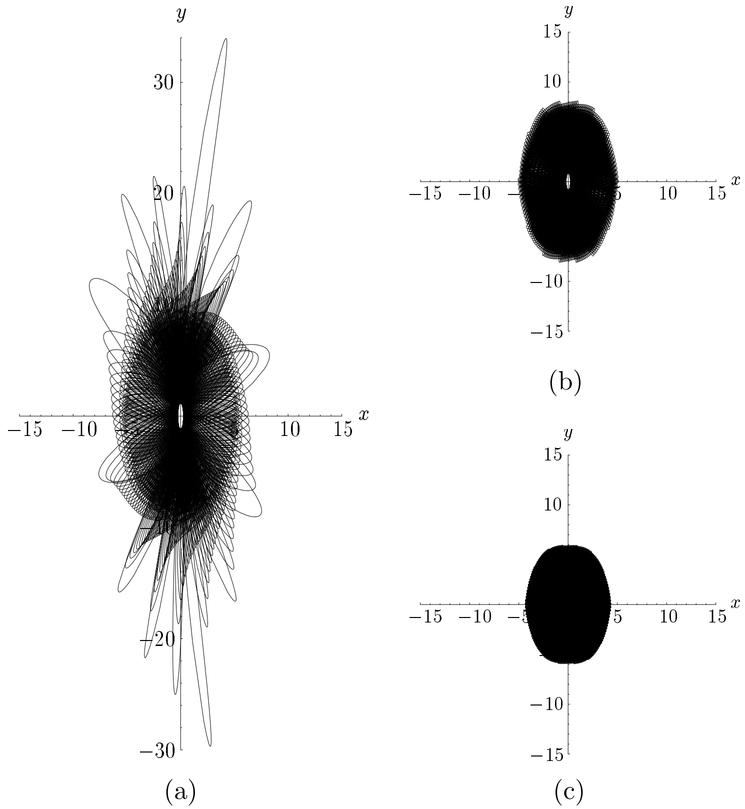

FIG. 1: A positive orbit of (S) with $p = 3$ and (3.1).

In Figure 1, we give a positive orbit of (S) with $p = 3$ and (3.1). The curves in Figure 1 indicate the same positive orbit starting from the point $(-6, 0)$ at the initial time $t_0 = 1$. Figures 1(a), 1(b) and 1(c) show three shapes of the positive orbit for $1 \leq t \leq 5000$, for $5000 \leq t \leq 20000$ and for $t \geq 20000$, respectively. First the movement of the positive orbit fluctuates in unpredictable ways (see Figure 1(a)), and then steadies down slightly (see Figures 1(b) and (c)). The positive orbit is inextricably intertwined with itself in Figures 1(b) and (c). The region drawn by the positive orbit in Figure 1(c) is smaller than that in Figure 1(b). This means that the positive orbit approaches the origin very slowly as it moves about in confusion.

Recall our earlier question whether the zero solution of (L) is asymptotically stable even if Coppel's condition (1.2) does not hold and the real part of every eigenvalue of $A(t)$ is not always non-positive for t sufficiently large. To settle the question, we give the following example to which Theorem 2.1 is applied.

Example 3.2. Consider system (L) with

$$A(t) = \begin{pmatrix} -2\sin t & \dfrac{1}{7} \\ -e^{-2\cos t} & -\sin t - \dfrac{\sin^2 t}{2(1+t)} \end{pmatrix}. \tag{3.2}$$

Then the zero solution of (L) is asymptotically stable.

It is easy to confirm that all assumptions of Theorem 2.1 are satisfied. We omit the details.

In Figure 2 below, we draw four shapes of the positive orbit of (L) with (3.2) starting from the point $(-6, 0)$ at the initial time $t_0 = 1$. The positive orbit runs for $1 \le t \le 5000$ in Figure 2(a), for $5000 \le t \le 10000$ in Figure 2(b), for $10000 \le t \le 20000$ in Figure 2(c) and for $t \ge 20000$ in Figure 2(d). Although the positive orbit displays intricate behavior, it approaches the origin ultimately.

In Example 3.2, the characteristic equation is

$$\det(\lambda E - A(t)) = \lambda^2 + \left(3\sin t + \frac{\sin^2 t}{2(1+t)}\right)\lambda + \left(2\sin^2 t + \frac{\sin^3 t}{1+t} + \frac{1}{7e^{2\cos t}}\right) = 0$$

and its roots $\lambda_+(t)$ and $\lambda_-(t)$ are given by

$$\lambda_\pm(t) = -\frac{3}{2}\sin t - \frac{\sin^2 t}{4(1+t)} \pm \frac{1}{2}\sqrt{\sin^2 t\left(1 - \frac{\sin t}{2(1+t)}\right)^2 - \frac{4}{7e^{2\cos t}}}\,.$$

The real parts of $\lambda_+(t)$ and $\lambda_-(t)$ are not always non-positive for $t \ge 0$. To show this, we let $t_n = (2n - 2/3)\pi$ for $n \in \mathbb{N}$. Then

$$\begin{aligned} \sin^2 t_n\left(1 - \frac{\sin t_n}{2(1+t_n)}\right)^2 - \frac{4}{7e^{2\cos t_n}} &= \frac{3}{4}\left(1 + \frac{\sqrt{3}}{4(1+t_n)}\right)^2 - \frac{4e}{7} \\ &< \frac{3}{4}\left(1 + \frac{1}{8}\right)^2 - \frac{4e}{7} < 0 \end{aligned}$$

for each $n \in \mathbb{N}$. Hence, we see that

$$\mathrm{Re}\lambda_\pm(t_n) = -\frac{3}{2}\sin t_n - \frac{\sin^2 t_n}{4(1+t_n)} > \frac{3\sqrt{3}}{4} - \frac{3}{16} > 0$$

for all $n \in \mathbb{N}$. Let $s_n = (2n - 1/2)\pi$ for $n \in \mathbb{N}$. Then, we have

$$\sin^2 s_n\left(1 - \frac{\sin s_n}{2(1+s_n)}\right)^2 - \frac{4}{7e^{2\cos s_n}} = \left(1 + \frac{1}{2(1+s_n)}\right)^2 - \frac{4}{7} > 1 - \frac{4}{7} > 0$$

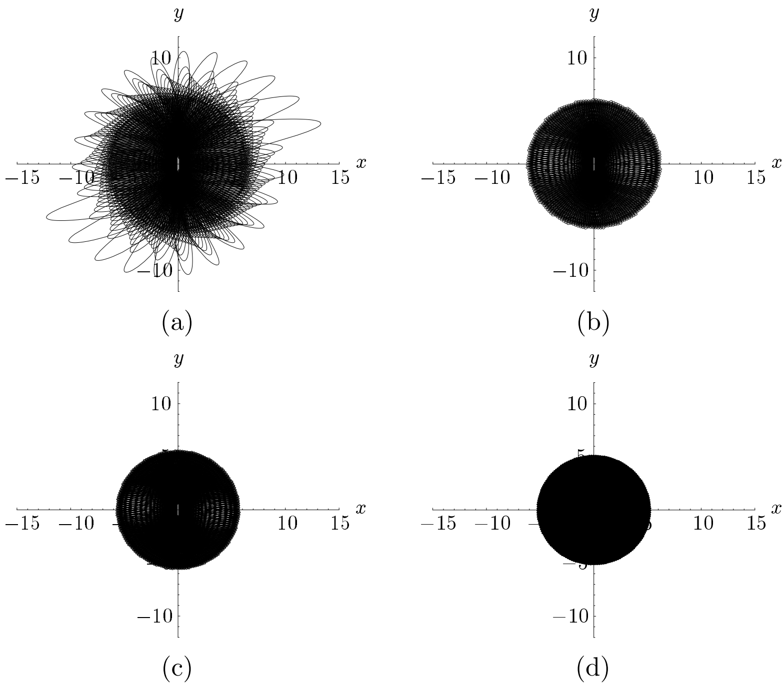

FIG. 2: A positive orbit of (L) with (3.2).

for each $n \in \mathbb{N}$, and therefore, we get

$$\begin{aligned}
&\mathrm{Re}\lambda_+(s_n) > \mathrm{Re}\lambda_-(s_n)\\
&\qquad = -\frac{3}{2}\sin s_n - \frac{\sin^2 s_n}{4(1+s_n)} - \frac{1}{2}\sqrt{\sin^2 s_n\left(1 - \frac{\sin s_n}{2(1+s_n)}\right)^2 - \frac{4}{7e^{2\cos s_n}}}\\
&\qquad \geq \frac{3}{2} - \frac{1}{4} - \frac{1}{2}\sqrt{\left(1+\frac{1}{2}\right)^2 - \frac{4}{7}} > 0
\end{aligned}$$

for all $n \in \mathbb{N}$.

The matrix $A(t)$ in Example 3.2 does not satisfy Coppel's condition (1.2). Since

$$A^*(t) = \begin{pmatrix} -2\sin t & -e^{-2\cos t} \\ \dfrac{1}{7} & -\sin t - \dfrac{\sin^2 t}{2(1+t)} \end{pmatrix},$$

the largest eigenvalue of the Hermitian matrix $H(t) = (A(t) + A^*(t))/2$, namely, $\mu[A(t)]$ is

$$-\frac{3}{2}\sin t - \frac{\sin^2 t}{4(1+t)} + \frac{1}{2}\sqrt{\left(\sin t - \frac{\sin^2 t}{2(1+t)}\right)^2 + \left(\frac{1}{e^{2\cos t}} - \frac{1}{7}\right)^2}$$

for $t \geq 0$. Hence, we have

$$\mu[A(t)] \geq -\frac{3}{2}\sin t - \frac{\sin^2 t}{4(1+t)} + \frac{1}{2}|\sin t|\left(1 - \frac{\sin t}{2(1+t)}\right)$$
$$= \begin{cases} -\sin t - \dfrac{\sin^2 t}{2(1+t)} & \text{if } \sin t \geq 0 \\ -2\sin t & \text{if } \sin t < 0\,. \end{cases}$$

From this inequality, we can estimate that

$$\lim_{t\to\infty}\int^t \mu[A(s)]\,ds = \infty\,.$$

Hence, Coppel's result mentioned in Section 1 is of no use to Example 3.2.

Incidentally, Theorem A is also inapplicable to Example 3.2. In fact, since

$$e(t) = 2\sin t \quad \text{and} \quad h(t) = \sin t + \frac{\sin^2 t}{2(1+t)}\,,$$

$e(t)$ and $h(t)$ change sign.

4. Extensions

Having achieved our two purposes which were stated in Section 1, we may now proceed to a generalization of Theorem 2.1. Although we assume the boundedness of $g(t)$ and $E(t)$ in Theorem 2.1, we can relax the assumption by changing variables.

Theorem 4.1. *Suppose that*

(i) $E(t)$*is bounded from below;*

(ii) $g(t)/f(t)$ *is continuously differentiable for* $t \geq 0$;

(iii) $f(t)\exp(E(t))$ *and* $g(t)/\exp((p-1)E(t))$ *are bounded, and*

$$f(t)g(t) > 0 \quad \textit{for} \;\; t \geq 0 \qquad \textit{and} \qquad \liminf_{t\to\infty}\frac{f(t)g(t)}{\exp((p-2)E(t))} > 0\,;$$

(iv) $h(t)$ *is bounded.*

If $\psi(t)$ *is nonnegative for* $t \geq 0$ *and it is weakly integrally positive, where* $\psi(t)$ *is the function given by* (2.2)*, then the zero solution of* (S) *is globally asymptotically stable.*

Proof. Let $w = x\exp\big(E(t)\big)$. Then we can transform system (S) into the system

$$(\widetilde{S}) \qquad \begin{aligned} w' &= \tilde{f}(t)\phi_{p^*}(y)\,, \\ y' &= -\,\tilde{g}(t)\phi_p(w) - h(t)y\,, \end{aligned}$$

where $\tilde{f}(t) = f(t)\exp(E(t))$ and $\tilde{g}(t) = g(t)/\exp((p-1)E(t))$. We will confirm that Theorem 2.1 can be applied to system $(\widetilde{S})$.

It follows from assumption (i) that if $w(t)$ tends to zero as $t \to \infty$, then $x(t)$ also tends to zero as $t \to \infty$. By assumptions (iii) and (iv), the coefficients $\tilde{f}(t)$,

$\tilde{g}(t)$ and $h(t)$ in system $(\widetilde{S})$ are bounded for $t \geq 0$. From assumption (iii) again, it turns out that

$$\tilde{f}(t)\tilde{g}(t) > 0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \liminf_{t\to\infty} \tilde{f}(t)\tilde{g}(t) > 0\,.$$

From assumption (ii), we see that $\tilde{g}(t)/\tilde{f}(t)$ is continuously differentiable for $t \geq 0$. Let

$$\tilde{\psi}(t) = p^* h(t) + \frac{\tilde{f}(t)}{\tilde{g}(t)} \left(\frac{\tilde{g}(t)}{\tilde{f}(t)}\right)'.$$

Then we have

$$\begin{aligned}\tilde{\psi}(t) &= p^* h(t) + \frac{f(t)\exp(pE(t))}{g(t)} \left(\frac{g(t)}{f(t)\exp(pE(t))}\right)' \\ &= p^* h(t) - p\,e(t) + \frac{f(t)}{g(t)} \left(\frac{g(t)}{f(t)}\right)' = \psi(t)\,.\end{aligned}$$

Hence, it follows from the assumption on $\psi(t)$ that $\tilde{\psi}(t)$ is nonnegative for $t \geq 0$ and weakly integrally positive. Thus, all conditions of Theorem 2.1 are satisfied, and therefore, the zero solution of (S) is globally asymptotically stable. This completes the proof of Theorem 4.1. □

Theorem 4.1 is applicable to the following example, but Theorem 2.1 is inapplicable.

Example 4.2. Let $p > 1$ be arbitrary. Consider system (S) with

$$e(t) = 1+2\sin t\,, \quad f(t) = e^{-t}\,, \quad g(t) = e^{(p-1)t} \quad \text{and} \quad h(t) = 2(p-1)\sin t + \frac{\sin^2 t}{1+t}\,.$$

Then the zero solution of (S) is globally asymptotically stable.

It is clear that the assumptions in Theorem 4.1 are satisfied. We have

$$\begin{aligned}\psi(t) &= p^* h(t) - p\,e(t) + \frac{f(t)}{g(t)} \left(\frac{g(t)}{f(t)}\right)' \\ &= \frac{p^*\sin^2 t}{1+t} + 2p\sin t - p(1+2\sin t) + p \\ &= \frac{p^*\sin^2 t}{1+t} \geq 0\end{aligned}$$

for $t \geq 0$. Hence, $\psi(t)$ is weakly integrally positive. Thus, by Theorem 4.1 the zero solution of (S) is globally asymptotically stable. Since $E(t) = t - 2\cos t$ and $g(t) = e^{(p-1)t}$, both functions tend to ∞ as $t \to \infty$. If $p < 2$, then $f(t)g(t) = e^{(p-2)t} \to 0$ as $t \to \infty$. Hence, we cannot apply Theorem 2.1 to Example 4.2.

Finally, we give other expressions for Theorem 4.1. For this purpose, we need the following lemma.

Lemma 4.3. *Let $a(t)$ and $b(t)$ be piecewise continuous functions on $(0,\infty)$ satisfying*

$$a(t)b(t) > 0 \quad \textit{for} \quad t \geq 0\,. \tag{4.1}$$

Then the following conditions are equivalent:

(i) $a(t)$ *and* $b(t)$ *are bounded, and* $\liminf_{t\to\infty} a(t)b(t) > 0$;
(ii) $a(t)$ *and* $1/b(t)$ *are bounded, and* $\liminf_{t\to\infty} a(t)/b(t) > 0$;
(iii) $1/a(t)$ *and* $b(t)$ *are bounded, and* $\liminf_{t\to\infty} b(t)/a(t) > 0$;
(iv) $1/a(t)$ *and* $1/b(t)$ *are bounded, and* $\liminf_{t\to\infty} 1/(a(t)b(t)) > 0$.

Proof. We first prove (i) $\Rightarrow$ (ii). From (4.1) and (i), we can find positive numbers m_1 and m_2 such that

$$a(t)b(t) \geq m_1 \quad \text{and} \quad |a(t)| \leq m_2 \quad \text{for} \quad t \geq 0\,.$$

Hence, we have

$$\frac{1}{|b(t)|} \leq \frac{|a(t)|}{m_1} \leq \frac{m_2}{m_1} \quad \text{for} \quad t \geq 0\,,$$

namely, $1/b(t)$ is bounded. Since $b(t)$ is bounded, there exists an $m_3 > 0$ such that

$$b^2(t) \leq m_3 \quad \text{for} \quad t \geq 0\,.$$

From this, we get

$$\frac{a(t)}{b(t)} = \frac{a(t)b(t)}{b^2(t)} \geq \frac{m_1}{m_3} > 0 \quad \text{for} \quad t \geq 0\,.$$

We next prove (ii) $\Rightarrow$ (i). Let $\tilde{b}(t) = 1/b(t)$. Then, as in the proof of part (i) $\Rightarrow$ (ii), we can show that $1/\tilde{b}(t)$ is bounded and $\liminf_{t\to\infty} a(t)/\tilde{b}(t) > 0$. Hence, $b(t)$ is bounded and $\liminf_{t\to\infty} a(t)b(t) > 0$.

Similarly, we can prove (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv). We omit the details. $\square$

By means of Lemma 4.3, we can replace assumption (iii) in Theorem 4.1 with one of the following conditions:

(v) $f(t)\exp(E(t))$ and $\exp((p-1)E(t))/g(t)$ are bounded, and

$$f(t)g(t) > 0 \quad \text{for} \quad t \geq 0 \qquad \text{and} \qquad \liminf_{t\to\infty} \frac{f(t)\exp(pE(t))}{g(t)} > 0\,;$$

(vi) $1/(f(t)\exp(E(t)))$ and $g(t)/\exp((p-1)E(t))$ are bounded, and

$$f(t)g(t) > 0 \quad \text{for} \quad t \geq 0 \qquad \text{and} \qquad \liminf_{t\to\infty} \frac{g(t)}{f(t)\exp(pE(t))} > 0\,;$$

(vii) $1/(f(t)\exp(E(t)))$ and $\exp((p-1)E(t))/g(t)$ are bounded, and

$$f(t)g(t) > 0 \quad \text{for} \quad t \geq 0 \qquad \text{and} \qquad \liminf_{t\to\infty} \frac{\exp((p-2)E(t))}{f(t)g(t)} > 0\,.$$

References

[1] Agarwal, R. P., Grace, S. R., O'Regan, D., *Oscillation Theory for Second Order Linear, Half-linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Academic Publishers, Dordrecht-Boston-London, 2002.

[2] Bihari, I., *On the second order half-linear differential equation*, Studia Sci. Math. Hungar. **3** (1968), 411–437.

[3] Coppel, W. A., *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.

[4] Desoer, C. A., *Slowly varying system $\dot{x} = A(t)x$*, IEEE Trans. Automat. Control **AC-14** (1969), 780–781.

[5] Dickerson, J. R., *Stability of linear systems with parametric excitation*, Trans. ASME Ser. E J. Appl. Mech. **37** (1970), 228–230.

[6] Došlý, O., Řehák, P., *Half-linear Differential Equations*, North-Holland Mathematics Studies 202, Elsevier, Amsterdam, 2005.

[7] Elbert, Á., *Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations*, Ordinary and Partial Differential Equations (Dundee, 1982), Lecture Notes in Math. 964, Springer-Verlag, Berlin-Heidelberg-New York.

[8] Elbert, Á., *A half-linear second order differential equation*, Qualitative Theory of Differential Equations, Vol I, II (Szeged, 1979) (Farkas, M., ed.), Colloq. Math. Soc. János Bolyai 30, North-Holland, Amsterdam-New York, 1981, pp. 153–180.

[9] Elbert, Á., *Asymptotic behaviour of autonomous half-linear differential systems on the plane*, Studia Sci. Math. Hungar. **19** (1984), 447–464.

[10] Hatvani, L., *On the asymptotic stability for a two-dimensional linear nonautonomous differential system*, Nonlinear Anal. **25** (1995), 991–1002.

[11] Hatvani, L., *Integral conditions on the asymptotic stability for the damped linear oscillator with small damping*, Proc. Amer. Math. Soc. **124** (1996), 415–422.

[12] Hatvani, L., Krisztin, T., Totik, V., *A necessary and sufficient condition for the asymptotic stability of the damped oscillator*, J. Differential Equations **119** (1995), 209–223.

[13] Hatvani, L., Totik, V., *Asymptotic stability for the equilibrium of the damped oscillator*, Differential Integral Equations **6** (1993), 835–848.

[14] Jaroš, J., Kusano, T., Tanigawa, T., *Nonoscillation theory for second order half-linear differential equations in the framework of regular variation*, Results Math. **43** (2003), 129–149.

[15] LaSalle, J. P., Lefschetz, S., *Stability by Liapunov's Direct Method, with Applications*, Mathematics in Science and Engineering 4, Academic Press, New-York-London, 1961.

[16] Li, H.-J., Yeh, C.-C., *Sturmian comparison theorem for half-linear second-order differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), 1193–1204.

[17] Markus, L., Yamabe, H., *Global stability criteria for differential systems*, Osaka Math. J. **12** (1960), 305–317.

[18] Mirzov, J. D., *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl. **53** (1976), 418–425.

[19] Pucci, P., Serrin, J., *Asymptotic stability for intermittently controlled nonlinear oscillators*, SIAM J. Math. Anal. **25** (1994), 815–835.

[20] Rouche, N., Habets, P., Laloy, M., *Stability Theory by Liapunov's Direct Method*, Applied Mathematical Sciences 22, Springer-Verlag, New York-Heidelberg-Berlin, 1977.

[21] Sugie, J., Onitsuka, M., Yamaguchi, A., *Asymptotic behavior of solutions of nonautonomous half-linear differential systems*, Studia Sci. Math. Hungar. **44** (2007), 159–189.

[22] Sugie, J., Yamaoka, N., *Comparison theorems for oscillation of second-order half-linear differential equations*, Acta Math. Hungar. **111** (2006), 165–179.

[23] Vinograd, R. E., *On a criterion of instability in the sense of Lyapunov of the solutions of a linear system of ordinary differential equations*, Dokl. Akad. Nauk **84** (1952), 201–204.

[24] Yoshizawa, T., *Stability Theory by Liapunov's Second Method*, Math. Society Japan, Tokyo (1966).

[25] Zubov, V. I., *Mathematical Methods for the Study of Automatic Control Systems*, Pergamon Press, New-York-Oxford-London-Paris, 1962.

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