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INVARIANT PROLONGATION OF BGG-OPERATORS IN CONFORMAL GEOMETRY

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Abstract. BGG-operators form sequences of invariant differential operators and the first of these is overdetermined. Interesting equations in conformal geometry described by these operators are those for Einstein scales, conformal Killing forms and conformal Killing tensors. We present a deformation procedure of the tractor connection which yields an invariant prolongation of the first operator. The explicit calculation is presented in the case of conformal Killing forms.

1. Introduction: Geometric prolongation of overdetermined operators

A conformal structure on a manifold M is an equivalence class $[g]$ of (Pseudo-)Riemannian metrics, where two metrics g and $\hat{g}$ are equivalent iff there is a function $f \in \mathrm{C}^\infty(M)$ such that $\hat{g} = e^{2f} g$. The simplest way to explain what a conformally invariant operator is, is to give an example: regard the operator

$$\Theta^g \colon \mathrm{C}^\infty(M) \to S^2_0 T^* M, \tag{1}$$

$$\sigma \mapsto \big(DD\sigma + \sigma P\big)_0 \,. \tag{2}$$

Here D is the Levi-Civita connection of a metric g in the conformal class, $P = P_{ab}$ is the Schouten-tensor, which is a trace-modification of the Ricci tensor, and the subscript 0 takes the trace-free part. $S^2_0 T^* M$ denotes symmetric, trace-free bilinear forms on TM, which will also be written as $\mathcal{E}_{(ab)_0}$: Throughout the paper we are using Penrose's abstract index notation [16], with $\mathcal{E}^a = \Gamma(TM)$ denoting vector fields, $\mathcal{E}_a = \Gamma(T^*M)$ denoting 1-forms and multiple indices being tensor products. Square and round brackets around indices will indicate alternation resp. symmetrization.

Now Θ^g describes the equation governing Einstein scales: for $\sigma \in \mathrm{C}^\infty(M)$ one has $\Theta^g \sigma = 0$ iff $\sigma^{-2} g$ is Einstein. The operator Θ^g is conformally covariant between $\mathrm{C}^\infty(M)$ and $S^2_0 T^* M$: if one switches to another metric $\hat{g} = e^{2f} g$ in the conformal class, then

$$\Theta^{\hat{g}} \circ m(e^f) = m(e^f) \circ \Theta^g \,,$$

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where $m(e^f)$ is simply the multiplication operator with e^f. This yields a conformally invariant operator between the weighted bundles $\mathcal{H}_0 = \mathcal{E}[1]$ and $\mathcal{H}_1 = S^2_0 T^*M \otimes \mathcal{E}[1]$. Here $\mathcal{E}[w]$ is the bundle of conformal w-densities, which is a line bundle that is trivialized by every metric g in the conformal class; with $[\sigma]_g$ the trivialization of a section $\sigma \in \mathcal{E}[w]$, one has $[\sigma]_{\hat{g}} = e^{wf}[\sigma]_g$. Especially, $[g]$ gives rise to a well defined *conformal metric* $\boldsymbol{g} = \boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2]$.

In general, a conformally invariant operator is obtained by a universal formula in the Levi-Civita connection, the metric and the curvature, possibly followed by contractions, that gives a well-defined operator between natural bundles for the conformal structure.

The example of the operator for Einstein scales above has another interesting property: it is overdetermined, and thus one can wish to have a prolongation of the system: in classical terms, this means that one wants to introduce more dependent variables and derive differential consequences of the overdetermined system, such that one can write down a closed system of equations; i.e.: a system of first order PDEs in which all (first order) derivatives of the dependent variables are expressed in the dependent variables themselves.

1.1. **The standard tractor bundle of conformal geometry and the prolongation of the equation governing Einstein scales.**

The prolongation of (1) is well known, and is conformally invariant. We are going to describe this and the necessary background on conformal tractor bundles. Our notations are inspired by [12]. We note here that a reader who looks for an introduction to tractor calculus in conformal geometry and an explanation of related notational issues could for instance make use of the very careful and detailed exposition in the first part of [18].

With respect to a metric g in the conformal class the *standard tractor bundle* $\mathcal{S}$ of a conformal geometry is given by

$$[\mathcal{S}]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1] \tag{3}$$

and one writes elements $[s]_g = \sigma \oplus \varphi_a \oplus \rho \in [\mathcal{S}]_g$ as

$$[s]_g = \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix}. \tag{4}$$

We remark here that via the conformal metric $\boldsymbol{g}_{ab} \in \mathcal{E}_{(ab)}[2]$ and its inverse $\boldsymbol{g}^{ab} \in \mathcal{E}^{(ab)}[-2]$ one can move indices up and down, and thus we can also write $[\mathcal{S}]_g = \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$.

For $\hat{g} = \exp^{2f} g$ one has the transformation

$$[s]_{\hat{g}} = \begin{pmatrix} \hat{\rho} \\ \hat{\varphi}_a \\ \hat{\sigma} \end{pmatrix} = \begin{pmatrix} \rho - \Upsilon_a \varphi^a - \frac{1}{2}\sigma \Upsilon^b \Upsilon_b \\ \varphi_a + \sigma \Upsilon_a \\ \sigma \end{pmatrix} \tag{5}$$

where $\Upsilon = df$ and $\mathcal{S}$ is defined by the equivalence class of $[\mathcal{S}]_g$ for $g \in [g]$ with respect to this transformation ([1]). We see that we have a well defined semidirect composition series

$$\mathcal{S} = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1] \tag{6}$$

i.e.: $\mathcal{S}$ is filtered $\mathcal{S} = \mathcal{S}^{-1} \supset \mathcal{S}^0 \supset \mathcal{S}^1$, and with respect to a metric g in the conformal class this filtration splits according to (3).

Additionally, $[\mathcal{S}]_g$ is endowed with the connection

$$\nabla_c s = \nabla_c \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} D_c\rho - P_{cb}\varphi^b \\ D_c\varphi^a + \sigma P_c{}^a + \rho\delta_c{}^a \\ D_c\sigma - \varphi_c \end{pmatrix}, \tag{7}$$

which is invariant with respect to the transformation (5) and thus gives a well defined connection on $\mathcal{S}$, called the standard tractor connection.

We furthermore see from (5) that one has a well-defined projection Π to the 'lowest slot' $\mathcal{H}_0$ of $\mathcal{S}$. This projection splits via the differential operator $L : \mathcal{H}_0 \to \mathcal{S}$, which is again defined via a metric g:

$$\sigma \in \mathcal{E}[1] \mapsto \begin{pmatrix} -\frac{1}{n}(\triangle\sigma + P_a{}^a\sigma) \\ \nabla\sigma \\ \sigma \end{pmatrix}.$$

$(\mathcal{S}, \nabla, \Pi, L)$ is a *geometric prolongation* of $\Theta : \mathcal{H}_0 \to \mathcal{H}_1$: The maps Π and L restrict to inverse isomorphisms of the space of parallel sections of $\mathcal{S}$ with respect to ∇ and the space of Einstein scales in $\mathcal{H}_0$. This is well known. In the following Section 2 we will give an explanation of this fact in terms of the BGG-machinery and present a method to obtain more invariant geometric prolongations for other equations. In Section 3 we will give an explicit prolongation of conformal Killing-forms (28) via this method.

2. Conformal tractor bundles

The standard tractor bundle $\mathcal{S}$ and its tractor connection, introduced via a description with respect to metrics in the conformal class above, can alternatively be described as the associated bundle to the standard representation of the Cartan-group $SO(p+1, q+1)$ modelling conformal structures. More general (see [7],[6]): a tractor bundle comes about as the associated space to a $SO(p+1, q+1)$-representation and is canonically endowed with its tractor connection.

Apart from spin representations, all tractor bundles appearing in conformal geometry appear as subbundles in tensorial powers of $\mathcal{S}$. Given $\mathcal{S}$ as in the previous section, i.e., written in terms of a Levi-Civita connection g, one has a natural inner product of signature $(p+1, q+1)$, which is given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{8}$$

Especially, one can identify $\Lambda^2\mathcal{S}$ with $\mathfrak{so}(\mathcal{S})$, which is the *adjoint* tractor bundle $\mathcal{A}M$ for conformal structures. Employing matrix notation, we will write its elements, or sections, as

$$\begin{pmatrix} -c & -\eta_b & 0 \\ \xi^a & C & \eta^b \\ 0 & -\xi_a & c \end{pmatrix}, \tag{9}$$

where $(c, C) \in \mathfrak{co}(p, q)$, $\xi^a \in \mathcal{E}^a$ and $\eta_a \in \mathcal{E}_a$.

One has a natural surjection (projection to ξ^a) of $\mathcal{A}M$ onto TM and an inclusion (inserting of η_b) of T^*M into $\mathcal{A}M$, while the inclusion via (9) of TM depends on the choice of g. Having fixed a metric g in the conformal class, the algebraic action $\bullet$ of $\mathcal{A}M$ on a tractor bundle T restricts to actions of TM and T^*M. Therefore, regarding TM and T^*M as (pointwise) abelian Lie algebras, we can thus introduce Lie algebra differentials on the the spaces $\mathcal{C}_k \colon = \mathcal{E}^k \otimes T$: we define $\partial \colon \mathcal{C}_k \to \mathcal{C}_{k+1}$,

$$\partial\varphi(\xi_0, \dots, \xi_k) = \sum_{j=0}^{k} (-1)^j \xi_j \bullet \varphi(\xi_0, \dots, \hat{\xi_j}, \dots, \xi_k) \tag{10}$$

and $\partial^* \colon \mathcal{C}_{k+1} \to \mathcal{C}_k$,

$$\partial^* Z_0 \wedge \dots \wedge Z_k \otimes V = \sum_{j=0}^{k} (-1)^{j+1} Z_0 \wedge \dots \wedge \hat{Z_j} \dots \wedge Z_k \otimes (Z_i \bullet V)\,. \tag{11}$$

It is straightforward to check that $\partial \circ \partial = \partial^* \circ \partial^* = 0$. It is a consequence of a general result by Kostant ([14]), that ∂ and ∂^* are naturally adjoint with respect to an (pointwise) inner product on the chain spaces $\mathcal{C}_k$. This gives a Hodge decomposition

$$\mathcal{C}_k = \text{im } \partial \oplus \ker \Box \oplus \text{im } \partial^* \tag{12}$$

with $\Box = \partial \circ \partial^* + \partial^* \circ \partial$.

Only ∂^*, but not ∂, is invariant with respect to a change in Levi-Civita connection in the conformal class. Thus we use ∂^* to define the spaces $\mathcal{Z}_k = \ker \partial^* \cap \mathcal{C}_k$,$\mathcal{B}_k = \text{im } \partial^* \cap \mathcal{C}_k$ and $\mathcal{H}_k = \mathcal{Z}_k / \mathcal{B}_k$. Using the Hodge decomposition, one can identify $\mathcal{H}_k$ with $\ker \Box \subset \mathcal{C}_k$ after choice of a metric in the conformal class.

As an $\mathfrak{so}(\mathcal{S})$-valued form $K \in \mathcal{E}_{[c_1 c_2]} \otimes \mathcal{A}M$, the curvature of the standard tractor connection is

$$K_{ab} = \begin{pmatrix} 0 & -A_{eab} & 0 \\ 0 & C_{ab}{}^{c}{}_{d} & A^e{}_{ab} \\ 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

Here C is the Weyl curvature and $A = A_{eab} = 2D_{[a}P_{b]e}$ is the Cotton-York tensor. We recall that both A and C are trace-free. Furthermore the skew-symmetrization over any 3 indices of C_{abcd} vanishes, as does the skew-symmetrization of A_{abc}. The Weyl curvature doesn't satisfy the differential Bianchi identity, however one has

$$D_{[a}C_{bc]de} = \boldsymbol{g}_{d[a}A_{|e|bc]} - \boldsymbol{g}_{e[a}A_{|d|bc]}.$$

3. Invariant geometric prolongation via the BGG-machinery

The BGG-machinery will associate to the tractor covariant derivative ∇ on T differential operators $\Theta_l \colon \mathcal{H}_l \to \mathcal{H}_{l+1}$. The core step in the construction of the BGG-operators is to find, in a natural way, a splitting of $\Pi_l \colon \mathcal{Z}_l \to \mathcal{H}_l$ adapted to ∇: For every $\sigma \in \mathcal{H}_l$ it can be shown that there is a unique lift $s \in \mathcal{Z}_l$ such that $\mathrm{d}^\nabla s \in \mathcal{Z}_{l+1}$. This defines the *BGG-splitting operators* $L_l \colon \mathcal{H}_l \to \mathcal{Z}_l$. By construction they give rise to the *BGG-operators*

$$\begin{aligned}\Theta_l &: \mathcal{H}_l \to \mathcal{H}_{l+1},\\ \Theta_k &= \Pi_{k+1} \circ \mathrm{d}^\nabla \circ L_k\,.\end{aligned}$$

Thus one obtains the BGG-sequence

$$0 \to \mathcal{H}_0 \xrightarrow{\Theta_0} \mathcal{H}_1 \to \cdots \xrightarrow{\Theta_{n-1}} \mathcal{H}_n \to 0\,.$$

We are interested in the first operator Θ_0, which gives an overdetermined system of equations. In [2] a prolongation method for operators $\Theta_0 + \eta$ for η a lower order, possibly nonlinear, differential operator, was developed, which did not however take into account the invariance respectively naturality of the operator Θ_0.

The construction of BGG-operators sketched above also works for more general connections $\tilde{\nabla} = \nabla + \Psi$, if $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$. Here $\mathfrak{gl}(T)^1$ denotes those endomorphisms of T which are homogeneous of degree ≥ 1 with respect to the filtration of T inherited from $\mathcal{S}$. More simply put: $\mathfrak{gl}(T)^1$ consists of upper triangular matrices if we use vector-notation as in (4) and (17).

3.1. Deformation of the tractor connection.

We would like to understand the solution space of $\Theta_0 \colon \mathcal{H}_0 \to \mathcal{H}_1$: Let $\sigma \in \mathcal{H}_0$. By definition $\Theta_0\sigma = \Pi_1\big(\nabla(L_0\sigma)\big)$, and thus

$$\Theta_0\sigma = 0 \quad \text{iff} \quad \nabla(L_0\sigma) \in \mathcal{B}_1 = \ker \Pi_1\,, \tag{14}$$

which shows that in general (T, ∇, Π_0, L_0) is not a prolongation, since the kernel of Θ_0 is not mapped into the space of parallel sections by L_0: while parallel sections of (T, ∇) always project into the kernel of Θ_0, a solution $\sigma \in \mathcal{H}_0$ of $\Theta_0\sigma = 0$ will, by definition of Θ_0, only have the property that $\nabla(L_0\sigma) \in \mathrm{im}\ \partial^*$. Our strategy is to deform ∇ to $\tilde{\nabla} = \nabla + \Psi$ by a map $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$ in suitable way, such that we obtain a 'better' connection on T which gives a geometric prolongation of Θ_0. I.e.: We want to find $\tilde{\nabla}$ such that $\Theta_0\sigma = 0$ implies $\tilde{\nabla}(L_0\sigma) = 0$ and conversely.

We make the following observation: consider $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$ which has the property that

$$\Psi s \in \mathrm{im}\ \partial^* = \mathcal{B}_1 \quad \forall s \in T\,. \tag{15}$$

Then we can construct the BGG-splitting operators $\tilde{L}_0 \colon H_0 \to T$, $\tilde{L}_1 \colon H_1 \to \mathcal{C}_1$ and the first BGG-operator $\tilde{\Theta}_0 \colon \mathcal{H}_0 \to \mathcal{H}_1$ as above. But Since $(\tilde{\nabla} - \nabla) = \Psi$ has values in $\mathcal{B}_1$, we see that $\partial^* \circ \tilde{\nabla} \circ L_0 = \partial^* \circ \nabla \circ L_0 = 0$, which shows that we have $\tilde{L}_0 = L_0$; and since $\Pi_1(\mathcal{B}_1) = 0$, we have $\tilde{\Theta}_0 = \Theta_0$. Thus: maps $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$ which send T into $\mathcal{B}_1$ may be used to deform ∇ to another connection without changing the first BGG-operator. Thus the space of such Ψ gives us a freedom for suitable deformations of ∇.

Assume that we have managed to find such a Ψ for which the curvature R of $\tilde{\nabla} = \nabla + \Psi$ has the property that, for every $s \in \Gamma(T)$,

$$\partial^*(Rs) = 0\,. \tag{16}$$

Then we claim that $\tilde{\nabla} s = \tilde{L}_1 \Theta_0 \Pi_0(s)$. This means that for every $s \in \Gamma(T)$, one already has $\partial^*(\mathrm{d}^{\tilde{\nabla}}(\tilde{\nabla} s))$. But this expression equals $\partial^*(Rs)$, and thus we have the claimed commutativity.

But this is already enough: because now, if $\Theta_0 \sigma = 0$, we have that $\tilde{\nabla}(L_0\sigma) = \tilde{L}_1(0) = 0$. And on the other hand, for a parallel section s of T, one evidently has by construction of L_0 that $L_0(\Pi_0(s)) = s$. Thus, $\Pi_0 \colon T \to \mathcal{H}_0$ and $L_0 \colon \mathcal{H}_0 \to T$ restrict to inverse isomorphisms between parallel sections of T with respect to $\tilde{\nabla}$ and the kernel of Θ_0.

Therefore the whole problem lies in finding a deformation map $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$ which maps T into im ∂^* and which gives a $\tilde{\nabla} = \nabla + \Psi$ whose curvature R maps T into ker $\partial^* \subset \mathcal{E}_{[c_1 c_2]} \otimes T$. Existence and uniqueness of such a map can be shown using analogs of inductive normalization procedures well known in the realm of parabolic geometries and this prolongation method actually works in a more general situation. See also remark 4.6.

One constructs the new connection $\tilde{\nabla}$ in terms of a given metric g in the conformal class. By uniqueness of Ψ, this construction is however independent of the choice of g, i.e., the connection $\tilde{\nabla}$ is a well defined, conformally invariant object. In the following we are going to show how this construction of a deformation map Ψ works explicitly for a special and interesting case, that of conformal Killing forms.

4. Invariant prolongation of conformal Killing forms

Conformal Killing forms were first prolonged by U. Semmelmann [17], however the discussion there did not take into account conformal invariance of the equation. In [12] an invariant prolongation was calculated by ad hoc methods (see also [18]). The following is a completely conceptual derivation of an invariant geometric prolongation by describing conformal Killing forms as the kernel of a first BGG-operator and prolonging this operator via a deformation of the tractor connection as in Section 3.

We are going to proceed as follows: In 4.1 we describe the exterior powers of the standard tractor bundles, give explicit formulas for the Lie algebraic differentials on the first chain spaces and determine their $CO(p,q)$-decompositions. In 4.2 we describe explicitly how the operator governing conformal Killing k-forms comes about as first BGG-operator for the $k+1$-st exterior power of the standard tractor bundle. In 4.3 we obtain a geometric prolongation by constructing a deformation $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$ with the properties called for in Section 3. In 4.4 we show the conformal invariance of Ψ.

4.1. **The tractor bundle.**

In the following k will be ≥ 1. The tractor bundle $T = \Lambda^{k+1}\mathcal{S}$ decomposes (via a metric g in the conformal class) into

$$\begin{pmatrix} \mathcal{E}^{[a_1\dots a_k]}[-k-1] \\ \mathcal{E}^{[a_1\dots a_{k+1}]}[-k-1] \mid \mathcal{E}^{[a_1\dots a_{k-1}]}[-k+1] \\ \mathcal{E}^{[a_1\dots a_k]}[-k+1] \end{pmatrix} \tag{17}$$

and similarly as for (5), we have transformations

$$\begin{pmatrix} \hat{\rho}^{a_1\dots a_k} \\ \hat{\varphi}^{a_0\dots a_k} \mid \hat{\mu}^{a_2\dots a_k} \\ \hat{\sigma}^{a_1\dots a_k} \end{pmatrix} = \begin{pmatrix} \rho^{a_1\dots a_k} - \Upsilon_b\varphi^{ba_1\dots a_k} - k\Upsilon^{[a_1}\mu^{a_2\dots a_k]} - \frac{1}{2}\Upsilon^b\Upsilon_b\sigma^{a_1\dots a_k} \\ \varphi^a + (k+1)\Upsilon^{[a_0}\sigma^{a_1\dots a_k]} \mid \mu^{a_2\dots a_k} - \Upsilon_b\sigma^{ba_2\dots a_k} \\ \sigma^{a_1\dots a_k} \end{pmatrix}. \tag{18}$$

From (6), or directly from (18), we see that

$$T = \mathcal{E}^{[a_1\dots a_k]}[-k+1] \mathbin{⨭} (\mathcal{E}^{[a_1\dots a_{k+1}]}[-k-1] \oplus \mathcal{E}^{[a_1\dots a_{k-1}]}[-k+1]) \mathbin{⨭} \mathcal{E}^{[a_1\dots a_k]}[-k-1], \tag{19}$$

which splits into $T_{-1} \oplus T_0 \oplus T_1$ after choice of g in the conformal class.

The standard tractor connection (7) gives rise to the invariantly defined tractor connection ∇ on T:

$$\nabla_c \begin{pmatrix} \rho^{a_1\dots a_k} \\ \varphi^{a_0\dots a_k} \mid \mu^{a_2\dots a_k} \\ \sigma^{a_1\dots a_k} \end{pmatrix} = \begin{pmatrix} D_c\rho^{a_1\dots a_k} - P_{cp}\varphi^{pa_1\dots a_k} - kP_c^{[a_1}\mu^{a_2\dots a_k]} \\ \begin{pmatrix} D_c\varphi^{a_0\dots a_k} + (k+1)\delta_c^{[a_0}\rho^{a_1\dots a_k]} \\ +(k+1)P_c^{[a_0}\sigma^{a_1\dots a_k]} \end{pmatrix} \mid \begin{pmatrix} D_c\mu^{a_2\dots a_k} \\ -P_{cp}\sigma^{pa_2\dots a_k} + \rho_c{}^{a_2\dots a_k} \end{pmatrix} \\ D_c\sigma^{a_1\dots a_k} - \mathfrak{p}_c{}^{a_1\dots a_k} + k\delta_c^{[a_1}\mu^{a_2\dots a_k]} \end{pmatrix}. \tag{20}$$

4.1.1. *Description of the first homology groups.* $\partial^* : \mathcal{C}_1 \to \mathcal{C}_0 = T$ is given (see (11)) by $Z \otimes s \mapsto -Z \bullet s$ for $s \in \Gamma(T)$, $Z \in \mathcal{E}_a$. Thus $B_0 = \operatorname{im} \partial^* : \mathcal{C}_1 \to T$ is simply $\mathcal{E}_a \bullet T$, which is all of T^0. Thus $\mathcal{H}_0 = T/T^0$. By the Hodge decomposition (12) we can embed $\mathcal{H}_0$ as $T_{-1} = \ker \Box = \ker \partial \subset T$.

Also, $\mathcal{H}_i$ will be embedded into $\mathcal{C}_i$ as $\ker \Box = \ker (\partial\partial^* + \partial^*\partial)$ for $i = 1, 2$. The calculation of the $CO(p,q)$-decomposition of the spaces $\mathcal{H}_i$ is purely algorithmic using Kostant's version of the Bott-Borel-Weyl theorem [14]; the details of which are not important for us here. We just state the results for $\mathcal{H}_1$ and $\mathcal{H}_2$, which are all homologies we are going to need: We will write

$$\mathcal{C}_i = \begin{pmatrix} \mathcal{E}^i \otimes T_1 \\ \mathcal{E}^i \otimes T_0 \\ \mathcal{E}^i \otimes T_{-1} \end{pmatrix},$$

and speak of the top, middle and bottom slots.

$\mathcal{E}_c \otimes T_{-1}$ contains the highest weight part $\mathcal{E}^{\{c[a_1\dots a_k]\}_0}[-k-1]$, and this is all of $\mathcal{H}_1$. Explicitly, $\mathcal{E}^{\{c[a_1\dots a_k]\}_0}[-k-1]$ sits in $\mathcal{E}_c^{[a_1\dots a_k]}[-k+1]$ as those $\sigma = \sigma_c{}^{a_1\cdots a_k}$ which have both zero trace and vanishing alternation:

$$0 = \sigma_p{}^{pa_2\cdots a_k}, \quad 0 = \sigma^{[ca_2\cdots a_k]}.$$

If $k \geq 2$ then the analogous statement holds also for the second chain space: in this case $\mathcal{H}_2$ is exactly the highest weight part of $\mathcal{E}_{[c_1c_2]} \otimes T_{-1} = \mathcal{E}_{[c_1c_2]}{}^{[a_1\dots a_k]}[-k+1]$. i.e., $\mathcal{H}_2 = \mathcal{E}^{\{[c_1c_2][a_1\dots a_k]\}_0}[-k-3] \subset \mathcal{E}_{[c_1c_2]} \otimes T_{-1}$.

Especially, for $i = 0, 1$ we have that H_i lies in the lowest grading part of $\mathcal{C}_i$ and if $k \geq 2$ this also holds for $i = 2$:

$$\begin{pmatrix} T_1 \\ T_0 \\ \mathcal{H}_0 = T_{-1} \end{pmatrix} \xrightarrow{\partial} \begin{pmatrix} \mathcal{E}_c \otimes T_1 \\ \mathcal{E}_c \otimes T_0 \\ \mathcal{H}_1 \oplus \operatorname{im} \partial_{|T_0} \end{pmatrix} \xrightarrow{\partial} \begin{pmatrix} \mathcal{E}_{[c_1c_2]} \otimes T_1 \\ \mathcal{E}_{[c_1c_2]} \otimes T_0 \\ \mathcal{H}_2 \oplus \operatorname{im} \partial_{|\mathcal{E}_c \otimes T_0} \end{pmatrix}$$

Now we describe what ∂, ∂^* and $\Box$ do on the first few chain spaces $\mathcal{C}_0 = T$, $\mathcal{C}_1 = \mathcal{E}_c \otimes T$ and $\mathcal{C}_2 = \mathcal{E}_{[c_1c_2]} \otimes T$:

4.1.2. *Explicit formulas for ∂, ∂^* and $\Box$ on the first chain spaces.*

$$\partial_c \begin{pmatrix} \rho^{a_1\dots a_k} \\ \varphi^{a_0\dots a_k} \mid \mu^{a_2\dots a_k} \\ \sigma^{a_1\dots a_k} \end{pmatrix} = \begin{pmatrix} 0 \\ (k+1)\delta_c{}^{[a_0}\rho^{a_1\dots a_k]} \mid \rho_c{}^{a_2\dots a_k} \\ -\varphi_c{}^{a_1\dots a_k} + k\delta_c{}^{[a_1}\mu^{a_2\dots a_k]} \end{pmatrix} \tag{21}$$

$$\partial_{c_1} \begin{pmatrix} \rho_{c_2}{}^{a_1\dots a_k} \\ \varphi_{c_2}{}^{a_0\dots a_k} \mid \mu_{c_2}{}^{a_2\dots a_k} \\ \sigma_{c_2}{}^{a_1\dots a_k} \end{pmatrix} = \begin{pmatrix} 0 \\ 2(k+1)\delta_{[c_1}{}^{[a_0}\rho_{c_2]}{}^{a_1\dots a_k]} \mid -2\rho_{[c_1c_2]}{}^{a_2\dots a_k} \\ 2\varphi_{[c_1c_2]}{}^{a_1\dots a_k} + 2k\delta_{[c_1}{}^{[a_1}\mu_{c_2]}{}^{a_2\dots a_k]} \end{pmatrix}$$

$$\partial^* \begin{pmatrix} \rho_c{}^{a_1\dots a_k} \\ \varphi_c{}^{a_0\dots a_k} \mid \mu_c{}^{a_2\dots a_k} \\ \sigma_c{}^{a_1\dots a_k} \end{pmatrix} = \begin{pmatrix} \varphi_p{}^{pa_1\dots a_k} + k\mu^{[a_1\dots a_k]} \\ -(k+1)\sigma^{[a_0\dots a_k]} \mid \sigma_p{}^{pa_2\dots a_k} \\ 0 \end{pmatrix}$$

$$\partial^* \begin{pmatrix} \rho_{c_1c_2}{}^{a_1\dots a_k} \\ \varphi_{c_1c_2}{}^{a_0\dots a_k} \mid \mu_{c_1c_2}{}^{a_2\dots a_k} \\ \sigma_{c_1c_2}{}^{a_1\dots a_k} \end{pmatrix} = \begin{pmatrix} -2\varphi_{cp}{}^{pa_1\dots a_k} - 2k\mu_c{}^{[a_1\dots a_k]} \\ 2(k+1)\sigma_c{}^{[a_0\dots a_k]} \mid -2\sigma_{cp}{}^{pa_2\cdot a_k} \\ 0 \end{pmatrix}.$$

The image of ∂^* in $T = C_0$ is simply $T^0 = T_0 \oplus T_1$, and the Kostant Laplacian thus acts by positive real scalars on T_1 and the two components of T_0. It vanishes on T_{-1} by (12). Explicitly, $\Box$ is given on T by

$$\begin{pmatrix} n \\ (k+1) \mid (n-k+1) \\ 0 \end{pmatrix}. \tag{22}$$

The image of ∂^* in C_1 contains all of $\mathcal{E}_c \otimes T_1$ (since we have (12)). Now $\mathcal{E}_c \otimes T_1$ decomposes into three parts: the alternating maps, $\mathcal{E}^{k+1}[-k-1]$, the purely trace maps, $\mathcal{E}^{k-1}[-k-1]$, and finally those maps which have both trivial trace and trivial alternating part, $\mathcal{E}^{\{c[a_1\dots a_k]\}}[-k-1]$. We will denote the three irreducible components of $\mathcal{E}_c \otimes T_1$ by $(\mathcal{E}_c \otimes T_1)_{alt}$, $(\mathcal{E}_c \otimes T_1)_{\{\}_0}$ and $(\mathcal{E}_c \otimes T_1)_{tr}$. We will write this decomposition of $\mathcal{E}_c \otimes T_1 \cap \operatorname{im} \partial^*$

$$\begin{pmatrix} alt \\ \{\}_0 \\ tr \end{pmatrix}, \tag{23}$$

and in this picture the Kostant Laplacian $\Box$ acts by

$$\begin{pmatrix} 2(n+k-1) \\ 2(n-2) \\ 2(2n-k-1) \end{pmatrix}. \tag{24}$$

Now to the middle slot: We have

$$\mathcal{E}_c \otimes T_0 = \mathcal{E}_c^{[a_0 \dots a_k]}[-k-1] \oplus \mathcal{E}_c^{[a_2 \dots a_k]}[-k+1]$$

and both parts split into alternating, $\{\}_0$- and trace components. Both $\{\}_0$-components, the left alternating and the right trace component lie in the image of ∂^*. The only other component of $\operatorname{im} \partial^* \cap \mathcal{E}_c \otimes T_0$ is $\mathcal{E}^{[a_1 \dots a_k]}[-k-1]$, which embeds into $\mathcal{E}_c \otimes T_0$ via

$$\tau^{a_1 \dots a_k} \mapsto \begin{pmatrix} 0 \\ -k(k+1)\delta_c{}^{[a_0}\tau^{a_1 \dots a_k} \mid (n-k)\tau_c{}^{a_2 \dots a_k} \\ 0 \end{pmatrix}.$$

We will write the decomposition of $\mathcal{E}_c \otimes T_0 \cap \operatorname{im} \partial^* \subset \mathcal{E}_c \otimes T_0$

$$\begin{pmatrix} alt & | & * \\ \{\}_0 & | & \{\}_0 \\ tr & | & tr \end{pmatrix}, \tag{25}$$

and the Kostant Laplacian is seen to act by the scalars

$$\begin{pmatrix} 4(k+1) & | & * \\ 2k & | & 2(n-k) \\ 2n & | & 2(n-k-1) \end{pmatrix}. \tag{26}$$

4.2. **The first BGG-operator $\Theta_0 \colon \mathcal{H}_0 \to \mathcal{H}_1$ and conformal Killing forms.** Using (20), (21) and (22), we compute that the first BGG-splitting operator $L_0 \colon \mathcal{H}_0 \to T$ is, up to first homogeneity, given by

$$\sigma \mapsto \begin{pmatrix} * \\ D^{[a_0}\sigma^{a_1 \dots a_k]} \mid -\frac{1}{n-k+1} D_p \sigma^{p a_2 \dots a_k} \\ \sigma \end{pmatrix}. \tag{27}$$

In 4.1.1 we saw that (using $\boldsymbol{g} \in \mathcal{E}^{(ab)}[2]$),

$$\begin{aligned} \mathcal{H}_0 &= \mathcal{E}^k[k+1]\,, \\ \mathcal{H}_1 &= \mathcal{E}^{\{c[a_1 \dots a_k]\}_0}[k+1]\,, \\ \mathcal{H}_2 &= \mathcal{E}^{\{[c_1 c_2][a_1 \dots a_k]\}_0}[k+1]\,. \end{aligned}$$

Thus we immediately see using (20) that

$$\Theta_0 : \mathcal{H}_0 \to \mathcal{H}_1 \,, \qquad \Theta_0 = \Pi_1 \circ \nabla \circ L$$

is given by ${\Theta_0}_c \sigma_{a_1\ldots a_k} = D_{\{c}\sigma_{a_1\ldots a_k\}_0}$. But this is exactly the conformal Killing operator. Thus our prolongation procedure will yield an isomorphism between the space conformal Killing forms

$$\sigma_{a_1\ldots a_k} \in \mathcal{E}^k[k+1]\,,\ D_{\{c}\sigma_{a_1\ldots a_k\}_0} = 0 \tag{28}$$

and the space of parallel sections of a natural connection on T.

For $k = 1$ we get exactly the operator describing conformal Killing fields, i.e., infinitesimal automorphisms of the conformal structure; see also remark 4.1. This case has been treated in detail in [10]. The main result of this text, an explicit geometric prolongation, will also work for $k = 1$. We only need $k \geq 2$ for obtaining an algebraic obstruction tensor which is described in subsection 4.3.1.

4.3. **The deformation of the tractor connection.**

We are now going to construct a $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$ with the properties called for in Section 3.

The calculations will be made more readable by providing beforehand the mappings which will appear: We will make use of the vector bundle maps $L_i\colon \mathcal{E}^{[a_1\cdots a_k]}[-k+1] \to \mathcal{E}^{[a_1\ldots a_{k+1}]}[-k-1]$, $i = 1,2$, and $R_i\colon \mathcal{E}^{[a_1\cdots a_k]}[-k+1] \to \mathcal{E}^{[a_1\ldots a_{k-1}]}[-k+1]$, $i = 1,2$, of homogeneity 1:

$$\begin{aligned} L_1(\sigma) &= C_{cp}{}^{[a_0 a_1}\sigma^{|p|a_2\ldots a_k]} & L_2(\sigma) &= \delta_c^{[a_0} C_{pq}{}^{a_0 a_1}\sigma^{|pq|a_3\ldots a_k]} \\ R_1(\sigma) &= C_c{}^{[a_2}{}_{pq}\sigma^{|pq|a_3\ldots a_k]} & R_2(\sigma) &= C^{[a_2 a_3}{}_{pq}\sigma_c{}^{|pq|a_4\ldots a_k]}\,. \end{aligned} \tag{29}$$

In homogeneity 2 we will need the maps F_i, G_i from $\mathcal{E}^{[a_1\cdots a_k]}[-k+1]$ to $\mathcal{E}^{[a_1\cdots a_k]}TM[-k-1]$, the maps $E_i\colon \mathcal{E}^{[a_1\ldots a_{k+1}]}[-k-1] \to \mathcal{E}^{[a_1\ldots a_k]}TM[-k-1]$ and the maps T_i : $\mathcal{E}^{[a_1\ldots a_{k-1}]}[-k+1] \to \mathcal{E}^{[a_1\ldots a_k]}TM[-k-1]$.

$$\begin{aligned} E_1(\varphi) &= C_c{}^{[a_1}{}_{pq}\varphi^{|pq|a_2\ldots a_k]} & E_2(\varphi) &= C^{[a_1 a_2}{}_{pq}\varphi_c{}^{|pq|a_3\ldots a_k]} \\ T_1(\mu) &= C_{cp}{}^{[a_1 a_2}\mu^{|p|a_3\ldots a_k]} & T_2(\mu) &= \delta_c^{[a_1} C^{a_2 a_3}{}_{pq}\mu^{|pq|a_3\ldots a_k]} \\ F_1(\sigma) &= A^{[a_1}{}_{cp}\sigma^{|p|a_2\ldots a_k]} & F_2(\sigma) &= A_{pc}{}^{[a_1}\sigma^{|p|a_2\ldots a_k]} \\ F_3(\sigma) &= A_p{}^{[a_1 a_2}\sigma_c{}^{|p|a_3\ldots \mathfrak{a}_k]} & F_4(\sigma) &= \delta_c^{[a_1} A^{a_2}{}_{pq}\sigma^{|pq|a_3\ldots \mathfrak{a}_k]} \\ G_1(\sigma) &= (D_c C^{[a_1 a_2}{}_{pq})\sigma^{|pq|a_3\ldots a_k]} & G_2(\sigma) &= (D_p C_{cq}{}^{[a_1 a_2})\sigma^{|pq|a_3\ldots a_k]} \\ G_3(\sigma) &= (D^{[a_1} C_c{}^{a_2}{}_{pq})\sigma^{|pq|a_3\ldots a_k]}\,. \end{aligned} \tag{30}$$

With respect to the $CO(p,q)$-decompositions (25) and (23) a more natural basis for the linear space formed by these maps into of $\mathrm{gr}(\mathcal{C}_1)_1$ and $\mathrm{gr}(\mathcal{C}_1)_2$ is formed by

(31)

$$
\begin{aligned}
L_{\mathrm{tr}} &= -\frac{k-1}{n-k}L_2 & L &= L_1 - L_{\mathrm{tr}}\\
R_{\mathrm{alt}} &= \frac{2}{k}R_1 + \frac{k-2}{k}R_2 & R &= \frac{k-2}{k}(R_1 - R_2)\\
E_{\mathrm{alt}} &= \frac{2}{k+1}E_1 + \frac{k-1}{k+1}E_2 & E &= \frac{k-1}{k+1}(E_1 - E_2)\\
T_{\mathrm{tr}} &= -\frac{k-2}{n-k+1}T_2 & T &= T_1 - T_{tr}\\
F_{\mathrm{tr}} &= \frac{k}{n-k+1}F_4 & F_{\mathrm{alt}} &= \frac{2}{k+1}F_2 - \frac{k-1}{k+1}F_3\\
F_i &= F_1 - \frac{1}{k+1}F_2 + \frac{k-1}{2(k+1)}F_3 - \frac{k-1}{k}F_{\mathrm{tr}} & F_{ii} &= \frac{k-1}{k+1}(F_2+F_3) - \frac{k-1}{2k}F_{\mathrm{tr}}\\
G_i &= G_1 + 2F_{\mathrm{alt}} - \frac{2}{k}(n-k-1)F_{tr} & G_{ii} &= G_2 - \frac{2(k-2)}{k}F_{\mathrm{tr}}\\
G_{iii} &= G_3 - 2F_{\mathrm{alt}} - \frac{n-3}{k}F_{\mathrm{tr}}\,.
\end{aligned}
$$

L_{tr}, T_{tr} and F_{tr} are purely trace, R_{alt}, E_{alt} and F_{alt} are alternating and all other maps have both vanishing alternation and trace.

The maps of (29) and (30) can be expressed as

$$
\begin{aligned}
&(32) & L_1 &= L + L_{\mathrm{tr}} & L_2 &= -\frac{n-k}{k-1}L_{\mathrm{tr}} & R_1 &= R + R_{\mathrm{alt}}\\
&& R_2 &= -\frac{2}{k-2}R + R_{\mathrm{alt}} & E_1 &= E + E_{\mathrm{alt}} & E_2 &= -\frac{2}{k-1}E + E_{\mathrm{alt}}\\
&& T_1 &= T + T_{\mathrm{tr}} & T_2 &= -\frac{n-k+1}{k-2}T_{\mathrm{tr}}
\end{aligned}
$$

and

$$
\begin{aligned}
F_1 &= F_i + \frac{1}{2}F_{\mathrm{alt}} + \frac{k-1}{k}F_{\mathrm{tr}} & F_2 &= F_{ii} + F_{alt} + \frac{k-1}{2k}F_{tr}\\
F_3 &= \frac{2}{k-1}F_{ii} - F_{\mathrm{alt}} + \frac{1}{k}F_{\mathrm{tr}} & F_4 &= \frac{n-k+1}{k}F_{\mathrm{tr}}\\
G_1 &= G_i - 2F_{\mathrm{alt}} + \frac{2}{k}(n-k-1)F_{\mathrm{tr}} & G_2 &= G_{ii} + \frac{2(k-2)}{k}F_{\mathrm{tr}}\\
G_3 &= G_{iii} + 2F_{\mathrm{alt}} + \frac{n-3}{k}F_{\mathrm{tr}}\,.
\end{aligned}
$$

For $s = \begin{pmatrix} \rho^{a_1\dots a_k} \\ \varphi^{a_0\dots a_k} \mid \mu^{a_2\dots a_k} \\ \sigma^{a_1\dots a_k} \end{pmatrix}$ we have

$$(K\bullet s) = \\ \begin{pmatrix} C_{c_1c_2}{}^{[a_1}{}_p\rho^{pa_2\dots a_k]} - kA^{[a_1}{}_{c_1c_2}\mu^{a_2\dots a_k} - A_{pc_1c_2}\varphi^{|p|a_1\dots a_k} \\ C_{c_1c_2}{}^{[a_0}{}_p\varphi^{pa_1\dots a_k]} + A^{[a_0}{}_{c_1c_2}\sigma^{a_1\dots a_k]} \mid C_{c_1c_2}{}^{[a_2}{}_p\mu^{pa_3\dots a_k]} - A_{pc_1c_2}\sigma^{|p|a_2\dots a_k} \\ C_{c_1c_2}{}^{[a_1}{}_p\sigma^{pa_2\dots a_k]} \end{pmatrix}.$$

We calculate

$$\partial^*(K\bullet s) = \begin{pmatrix} 2kF_1 + 2kF_2 - kE_1 + k(k-1)T_1 \\ -k(k+1)L_1 \mid -(k-1)R_1 \\ 0 \end{pmatrix} \tag{33}$$

and thus have that the lowest homogeneous component of $\partial^*(K\bullet s)$, which is of homogeneity 1, is given by $(-k(k+1)L_1 \mid -(k-1)R_1)$. Now we use (32), (22) and (31) to apply $-\Box^{-1}$ to this expression, which yields

$$\Psi_1 := \begin{pmatrix} 0 \\ \lambda_1 L_1 + \lambda_2 L_2 \mid \rho_1 R_1 + \rho_2 R_2 \\ 0 \end{pmatrix} \tag{34}$$

where

$$\lambda_1 = \frac{1+k}{2} \qquad \lambda_2 = \frac{(k-1)(k+1)}{2n}$$
$$\rho_1 = \frac{(k-1)(n-2)}{2(n-k)n} \qquad \rho_2 = \frac{2-3k+k^2}{2(k-n)n}.$$

Now the curvature of the deformed connection $\nabla + \Psi_1$ is

$$R = K\bullet + \mathrm{d}^\nabla\Psi_1 + [\Psi_1, \Psi_1],$$

but $[\Psi_1, \Psi_1]$ obviously vanishes. Let us calculate R: The only term which demands our attention is $\mathrm{d}^\nabla\Psi_1$; Take any $s = \begin{pmatrix} \rho \\ \varphi \mid \mu \\ \sigma \end{pmatrix} \in \Gamma(T)$.

Then for $\xi_1, \xi_2 \in \mathfrak{X}(M)$, we have, since Ψ_1 is a 1-form on M with values in $\mathfrak{gl}(T)$,

$$\begin{aligned} (\mathrm{d}^\nabla\Psi_1)s(\xi_1,\xi_2) &= \\ &= \nabla_{\xi_1}(\Psi_1(\xi_2)s) - \Psi_1(\xi_2)(\nabla_{\xi_1}s) - \nabla_{\xi_2}(\Psi_1(\xi_1)s) + \Psi_1(\xi_1)(\nabla_{\xi_2}s) \\ &\qquad - \Psi_1([\xi_1,\xi_2])s. \end{aligned} \tag{35}$$

We may expand (35) and write

$$(\mathrm{d}^\nabla\Psi_1)s = \tag{36}$$
$$\begin{pmatrix} \overset{*}{\begin{pmatrix} D_{\xi_1}\big(\Psi_1(\xi_2)\sigma\big) - \Psi_1(\xi_2)\big(D_{\xi_1}\sigma\big) - D_{\xi_2}\big(\Psi_1(\xi_1)\sigma\big) + \Psi_1(\xi_1)\big(D_{\xi_2}\sigma\big) \\ -\Psi_1([\xi_1,\xi_2])\sigma \\ -\Psi_1(\xi_2)\partial_{\xi_1}\varphi + \Psi_1(\xi_1)\partial_{\xi_2}\varphi - \Psi_1(\xi_2)\partial_{\xi_1}\mu + \Psi_1(\xi_1)\partial_{\xi_2}\mu \end{pmatrix}} \\ \partial_{\xi_1}\Psi_1(\xi_2)\sigma - \partial_{\xi_2}\Psi_1(\xi_1)\sigma \end{pmatrix},$$

where we don't care about the top component since it vanishes after an application of ∂^*. The lowest component is simply $\partial(\Psi_1\sigma) = -\partial\Box^{-1}\partial^*(K\bullet\sigma)$. Thus $\partial^*(Rs)$ lies in the top slot (i.e., in homogeneity 1). So our first deformation had the effect of moving the expression $\partial^* \circ R$ one slot higher. This can be repeated: it is a straightforward calculation using the expression in the middle component of (36) and, in that order, (33), (32),(24) and (31) to find, with $\phi := -\Box^{-1} \circ \partial^* \circ R$,

$$\Psi = \Psi_2 = \Psi_1 - \phi = \begin{pmatrix} \begin{pmatrix} \varepsilon_1 E_1 + \varepsilon_2 E_2 + \tau_1 T_1 + \tau_2 T_2 \\ +\phi_1 F_1 + \phi_2 F_2 + \phi_3 F_3 + \phi_4 F_4 \\ +\gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3 \end{pmatrix} \\ \lambda_1 L_1 + \lambda_2 L_2 |\ \rho_1 R_1 + \rho_2 R_2 \\ 0 \end{pmatrix} \tag{37}$$

where

$$\varepsilon_1 = \frac{k-1}{2(n-k)} \qquad \varepsilon_2 = \frac{(k-1)k}{2(k-n)n}$$

$$\tau_1 = \frac{(k-1)(n(n-k+1)-2k)}{2(k-n)n} \qquad \tau_2 = -\frac{(k-2)(k-1)}{2n}$$

$$\phi_1 = -\frac{n+k-3}{n-2} \qquad \phi_2 = \frac{1-k}{n}$$

$$\phi_3 = \frac{(k-1)(n+k)}{2(k-n)n} \qquad \phi_4 = \frac{(k-1)(2+k-2n)}{2(k-n)(n-2)}$$

$$\gamma_1 = -\frac{k-1}{2(n-2)n} \qquad \gamma_2 = \frac{k-1}{2(n-2)}$$

$$\gamma_3 = \frac{(k-1)k}{2(k-n)n}\,.$$

Now the curvature R' of $\nabla + \Psi = \nabla + \Psi_1 + \phi$ is given by

$$R + \mathrm{d}^\nabla \phi + [\Psi_1, \phi] + [\phi, \Psi_1] + [\phi, \phi]\,.$$

One sees that for every $s \in \Gamma(T)$, $([\Psi_1, \phi] + [\phi, \Psi_1] + [\phi, \phi])s$ has only values in the top component and we may therefore forget about this term when calculating $\partial^*(R's)$. As in the calculation (36), we see that $(\mathrm{d}^\nabla \phi)s$ has only values in the middle and top slots and the middle slot is given by $2\partial_{[c_1}\phi_{c_2]}s$. Therefore, by construction of ϕ, we see that $\partial^*(Rs)$ vanishes, and thus, via the considerations of Section 3, we have solved the prolongation problem for conformal Killing forms.

We have already remarked there that this solution must already be conformally invariant by virtue of uniqueness, which is not difficult to see, but to see what is going on we are going to check independence of the choice of metric by hand in 4.4.

Remark 4.1. For $k = 1$, we have $T = \Lambda^2\mathcal{S} = \mathfrak{so}(\mathcal{S}) = \mathcal{A}M$. Thus $\nabla + \Psi$ prolongs the first BGG-operator for the adjoint tractor connection in this case. But in [4] it was shown that for a parabolic geometry which is either 1-graded or torsion free and which has the property that the first homology of the adjoint tractor bundle $\mathcal{H}_1$ is concentrated in lowest homogeneity, the corresponding first BGG-operator describes infinitesimal automorphisms of the structure - in our case conformal

Killing fields - and is prolonged by the connection $\tilde{\nabla} s = \nabla s + i_s K$ which maps T into $\operatorname{im} \partial^*$ and satisfies (16). Thus uniqueness implies that $\Psi s = i_s K$, which can also be read off (37) directly.

Remark 4.2. The invariant connection prolonging the conformal Killing equation (28) which was constructed in [12] differs from our result Ψ as defined in (37) since it can be checked to have nontrivial intersection with $\operatorname{im} \partial$; Recall that our solution obeys the normalization conditions (15) and (16), but the first of these conditions implies that it has trivial intersection with $\operatorname{im} \partial$ since one has the Hodge decomposition (12).

If one wants to translate the solution (37) into the notation used in [12] one has to use the automorphism

$$\begin{pmatrix} \rho^{a_1\dots a_k} \\ \varphi^{a_0\dots a_k} \mid \mu^{a_2\dots a_k} \\ \sigma^{a_1\dots a_k} \end{pmatrix} \mapsto \begin{pmatrix} (k+1)\rho^{a_1\dots a_k} \\ \varphi^{a_0\dots a_k} \mid -k(k+1)\mu^{a_2\dots a_k} \\ (k+1)\sigma^{a_1\dots a_k} \end{pmatrix}$$

which transforms an element of

$$\mathcal{E}^{[a_1\dots a_k]}[-k+1] \lplus (\mathcal{E}^{[a_1\dots a_{k+1}]}[-k-1] \oplus \mathcal{E}^{[a_1\dots a_{k-1}]}[-k+1]) \lplus \mathcal{E}^{[a_1\dots a_k]}[-k-1]$$

in our notation to the equivalent element in the notation of [12]. Then Ψ as defined in (37) has the following form with respect to the notations of Gover-Šilhan:

$$\Psi_c \begin{pmatrix} \rho^{a_1\dots a_k} \\ \varphi^{a_0\dots a_k} \mid \mu^{a_2 \cdot s a_k} \\ \sigma^{a_1\dots a_k} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} (k+1)(\varepsilon_1 E_1 + \varepsilon_2 E_2)\varphi - \frac{1}{k}(\tau_1 T_1 + \tau_2 T_2)\mu \\ +(\phi_1 F_1 + \phi_2 F_2 + \phi_3 F_3 + \phi_4 F_4 \\ +\gamma_1 G_1 + \gamma_2 G_2 + \gamma_3 G_3)\sigma \end{pmatrix} \\ \frac{1}{k+1}(\lambda_1 L_1 + \lambda_2 L_2)\sigma \mid -k(\rho_1 R_1 + \rho_2 R_2)\sigma \\ 0 \end{pmatrix}.$$

4.3.1. *The curvature of the deformed connection.* Since $\Psi \in \mathcal{E}_c \otimes \mathfrak{gl}(T)^1$ we know that the curvature $R \in \mathcal{E}_{[c_1 c_2]}(\mathfrak{gl}(T))$ of $\tilde{\nabla} = \nabla + \Psi$ agrees with $K\bullet$ in homogeneity 0. But if $\sigma_{a_1\dots a_k} \in \mathcal{E}^k[k+1]$ is a conformal Killing k-form, then $L_0\sigma$ is given by (27); and thus $0 = \mathrm{d}^{\tilde{\nabla}} \tilde{\nabla} s = Rs$ agrees with $K \bullet L_0 \sigma$ in $\mathcal{E}_{[c_1 c_2]} \otimes T_{-1}$. But by (13) this is simply (minus) $C_{c_1 c_2}{}^{p}{}_{[a_1} \sigma_{|p| a_2 \dots a_k]}$. For $k \geq 2$ we have $\mathcal{H}_2 = \mathcal{E}_{\{[c_1 c_2][a_1\dots a_k]\}_0}[k+1]$ and projecting the previous expression to this space gives the conformally invariant algebraic map

$$\Phi : \sigma \mapsto C_{\{c_1 c_2}{}^{p}{}_{[a_1} \sigma_{|p| a_2 \cdots a_k]\}_0}$$

which vanishes on conformal Killing k-forms. This obstruction has also been constructed in [13], [17] and [12]. Our derivation is completely conceptual: the map is simply the composition of the first two BGG-operators for the deformed connection $\tilde{\nabla}$: $\Phi = \tilde{\Theta}_1 \circ \Theta_0$. This evidently explains both conformal invariance of Φ and why it vanishes on conformal Killing forms. That Φ is algebraic has the cohomological reason that $\mathcal{H}_2$ is concentrated in lowest homogeneity.

Remark 4.3. Apart from the (trivial) cases of Einstein scales and twistor spinors where one doesn't need any deformation and automatically has (16), the case of conformal Killing forms is the simplest situation in which to explicitly compute the prolongation. The next interesting case to treat will be conformal Killing tensors, for which, as far as we know, there has not yet been given any prolongation, and which can be treated similarly as the form case. There the situation becomes more complicated however, since the modelling representations $S^k\mathfrak{g}$ are $2k+1$-graded. This case could have interesting relations to symmetries of the Laplacian ([9]).

Remark 4.4. The holonomy $\mathrm{Hol}(\tilde{\nabla})$ of the thus obtained prolongation connection $\tilde{\nabla}$ describes the solution space of the operator Θ_0. In the case of the standard tractor bundle and the spinor tractor bundle one has $\nabla = \tilde{\nabla}$ (see Section 3) and thus the solution space is governed by the conformal holonomy of the structure, i.e., existence of Einstein scales and twistor spinors correspond to reductions of the conformal holonomy. In general, the existence of non-trivial solutions of Θ_0 doesn't imply a holonomy reduction: for instance, full conformal holonomy doesn't obstruct the existence of conformal Killing fields or conformal Killing forms.

Because of (14), parallel sections of a tractor bundle give special solutions to Θ_0. In the case of conformal Killing Forms, those coming from parallel sections were called *normal* conformal Killing forms in [15]. This notion of *normal* solutions of first BGG-operators makes sense for every tractor bundle and they correspond to reductions in conformal holonomy.

Remark 4.5. Using the tractor approach above for describing Einstein scales as parallel sections, R. Gover and P. Nurowski [11] used the curvature R of the standard tractor connection and its derivatives to obtain (under a genericity condition on the Weyl curvature) a conformally invariant system of tensors which provides a sharp obstruction against the existence of Einstein scales. For a general tractor bundle and R the curvature of the prolongation connection, one can similarly build natural systems of obstruction tensors, but it is not known whether these will be sharp.

4.4. **Conformal invariance of Ψ.**

For this calculation we need some transformation formulas. We will denote by $\hat{D}$ the Levi-Civita connection of the metric rescaled by e^{2f}. More generally, we will denote by a hatted symbol the corresponding quantity calculated with respect to the metric $\hat{g}$. With $\Upsilon = df$ we have

$$\begin{aligned}
\hat{D}_u C_{abcd} &= D_u C_{abcd} - 2\Upsilon_u C_{abcd} - 2\Upsilon_{[a} C_{|u|b]cd} - 2\Upsilon_{[c} C_{|u|d]ab} \\
&\quad + 2(n-3) g_{u[a} A_{b]cd} + 2(n-3) g_{u[c} A_{d]ab} \\
\hat{A}_{abc} &= A_{abc} + \Upsilon^d C_{dabc}\,.
\end{aligned}$$

In the calculation the following transformation-maps

$$H_i \colon \mathcal{E}^{[a_1\dots a_k]}[-k+1] \to \mathcal{E}^{[a_1\dots a_k]}[-k-1]$$

will appear:

$$H_1(\sigma)=\Upsilon_c C^{[a_1a_2}{}_{pq}\sigma^{|pq|a_3\dots a_k]} \qquad H_6(\sigma)=\delta_c{}^{[a_1}\Upsilon_u C^{a_2a_3}{}_{pq}\sigma^{|upq|a_4\dots a_k]}$$
$$H_2(\sigma)=\Upsilon_c C^{[a_1a_2}{}_{pq}\sigma^{|pq|a_3\dots a_k]} \qquad H_7(\sigma)=\Upsilon^d C_{dp}{}^{[a_1a_2}\sigma_c{}^{|p|a_3\dots a_k]}$$
$$H_3(\sigma)=\Upsilon_p C_{cq}{}^{[a_1a_2}\sigma^{|pq|a_3\dots a_k]} \qquad H_8(\sigma)=\Upsilon^d C_d{}^{[a_1}{}_{cp}\sigma^{|p|a_2\dots a_k]}$$

$$H_4(\sigma)=\delta_c{}^{[a_1}\Upsilon^{|d|}C_d{}^{a_2}{}_{pq}\sigma^{|pq|a_3\dots a_k]} \qquad H_9(\sigma)=\Upsilon^d C_{dpc}{}^{[a_1}\sigma^{|p|a_2\dots a_k]}$$
$$H_5(\sigma)=\Upsilon^{[a_1}C^{a_2a_3}{}_{pq}\sigma_c{}^{|pq|a_4\dots a_k]}\,.$$

The maps (29) of homogeneity 1 are invariant with respect to the choice of $g\in[g]$ since the Weyl curvature is conformally invariant. It is straightforward to calculate that the individual maps (30) transform like

$$\hat{E}_1=E_1+2H_9-(k-1)H_2 \qquad \hat{E}_2=E_2+H_1-2H_7-(k-2)H_5$$
$$\hat{G}_1=G_1-2H_1-2H_2-2H_3+2H_4+2H_7$$
$$\hat{G}_2=G_2-H_1-H_2-H_3+H_7+2H_8$$
$$\hat{G}_3=G_3-H_1-H_2-H_3+H_4+2H_9\,,$$

and

$$\hat{F}_1=F_1+H_8 \qquad \hat{F}_2=F_2+H_9 \qquad \hat{F}_3=F_3+H_7$$
$$\hat{F}_4=F_4+H_4 \qquad \hat{T}_1=T_1-H_3 \qquad \hat{T}_2=T_2-H_6\,.$$

Thus, if we switch to another metric $\hat{g}$ respectively the corresponding linear connection $\hat{D}$ and then calculate $\hat{\Psi}$ using (37), the result differs from Ψ only in the top slot of C_1 and it does so by

$$\begin{aligned}(38)\quad &(\varepsilon_2-2\gamma_1-\gamma_2-\gamma_3)H_1-\tau_2H_6-((k-1)\varepsilon_1-2\gamma_1-\gamma_2-\gamma_3)H_2\\ &+(-2\varepsilon_2+\phi_3+2\gamma_1+\gamma_2)H_7-(\tau_1-2\gamma_1-\gamma_2-\gamma_3)H_3+(\phi_1+2\gamma_2)H_8\\ &+(\phi_4+2\gamma_1+\gamma_3)H_4+(2\varepsilon_1+\phi_2+2\gamma_3)H_9-(k-2)\varepsilon_2H_5\,.\end{aligned}$$

On the other hand, if we calculate Ψ with respect to g and then transform the expression via $\hat{\rho}=\rho-\Upsilon_d\varphi^{da_1\dots a_k}-k\Upsilon^{[a_1}\mu^{a_2\dots a_k]}$, the difference to Ψ also lies in homogeneity two and is

$$\begin{aligned}(39)\quad &-\lambda_2\frac{1}{k+1}H_1-\rho_1H_2+\frac{k-1}{k+1}\lambda_1H_3+\\ &\qquad\frac{2}{k+1}\lambda_2H_4-k\rho_2H_5+\frac{k-1}{k+1}\lambda_2H_6-\frac{2}{k+1}\lambda_1H_8\,.\end{aligned}$$

Now it is straightforward to check that the expressions (38) and (39) coincide. Thus Ψ is seen not to depend on the choice of the metric in the conformal class used to construct it. As we already remarked this is in fact a consequence of the uniqueness property of Ψ stated at the end of Section 3.

Remark 4.6. The prolongation method of above works more generally: the construction of the BGG-sequence works for arbitrary tractor bundles over regular k-graded parabolic geometries ([8],[3]), and again the first operator in this sequence is overdetermined, and we ask for a natural prolongation. The analog of a choice of metric in the conformal case is the choice of a Weyl structure of the parabolic geometry ([7]). The homogeneity conditions become a bit more subtle, but the basic principle of finding a natural deformation of the tractor connection yielding a prolongation is the same as presented in Section 3. This is the subject of a forthcoming joint paper with J. Šilhan, P. Somberg and V. Souček.

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