ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 391–401

ON RIEMANNIAN GEOMETRY OF TANGENT SPHERE BUNDLES WITH ARBITRARY CONSTANT RADIUS

Oldřich Kowalski and Masami Sekizawa

Abstract. We shall survey our work on Riemannian geometry of tangent sphere bundles with arbitrary constant radius done since the year 2000.

Introduction

Let r be a positive real number. Then the *tangent sphere bundle of radius* r over a Riemannian manifold (M,g) is the hypersurface $T_rM = \{(x,u) \in TM \mid g_x(u,u) = r^2\}$ of the tangent bundle TM. Many papers have been written about the geometry of the *unit* tangent sphere bundle T_1M over a Riemannian manifold (M,g) with the metric $\tilde{g}^s$ induced by the Sasaki metric g^s on TM. The geometry of $(T_1M,\tilde{g}^s)$ is not so rigid as that of (TM,g^s) and more interesting results can be derived (see, for example, [4, 5, 6, 7, 8, 10, 24, 29]). We refer to E. Boeckx and L. Vanhecke [9] and G. Calvaruso [14] for surveys on the geometry of $(T_1M,\tilde{g}^s)$. More general metrics on T_1M have been treated by M. T. K. Abbassi and G. Calvaruso in [1]. They have studied properties of metrics on T_1M induced from g-natural metric on the tangent bundle TM. The present authors have published the original papers [18, 19, 21, 20] about this topic. Here we are going to survey our results.

1. Tangent sphere bundles with arbitrary constant radius

If $(\mathcal{U};\ x^1,x^2,\dots,x^n)$ is a system of local coordinates in the base manifold M, then a vector $u \in M_x$ is expressed as $u = \sum_{i=1}^n u^i(\partial/\partial x^i)_x$, and hence $(p^{-1}(\mathcal{U});\ x^1,x^2,\dots,x^n,u^1,u^2,\dots,u^n)$ is a system of local coordinates in the tangent bundle TM over M. The *canonical vertical vector field* on TM is a vector field $\boldsymbol{U}$ defined, in terms of local coordinates, by $\boldsymbol{U} = \sum_{i=1}^n u^i\partial/\partial u^i$. The vector field $\boldsymbol{U}$ does not depend on the choice of local coordinates and it is defined globally on TM. For a vector $u = \sum_{i=1}^n u^i(\partial/\partial x^i)_x \in M_x$, we see that $u^h_{(x,u)} = \sum_{i=1}^n u^i(\partial/\partial x^i)^h_{(x,u)}$ and $u^v_{(x,u)} = \sum_{i=1}^n u^i(\partial/\partial x^i)^v_{(x,u)} = \boldsymbol{U}_{(x,u)}$.

The canonical vertical vector field $\boldsymbol{U}$ is normal to T_rM in $(TM,\bar{g})$ at each point $(x,u) \in T_rM$. Also, $\bar{g}(\boldsymbol{U},\boldsymbol{U}) = r^2$ along T_rM. For any vector field X tangent to M,

2000 *Mathematics Subject Classification*: primary 53C07; secondary 53C30, 53C25.

Key words and phrases: Riemannian metric, tangent bundle, tangent sphere bundle, Riemannian curvature, scalar curvature.

The first author was supported by the grant GA ČR 201/05/2707 and by the project MSM 0021620839.

the horizontal lift X^h to TM is always tangent to T_rM at each point $(x,u) \in T_rM$. Yet, in general, the vertical lift X^v to TM is not tangent to T_rM at $(x,u) \in T_rM$. The *tangential lift* of X (see [10]) is a vector field X^t defined by

$$X^t = X^v - \frac{1}{r^2} g^s(X^v, \boldsymbol{U})\boldsymbol{U}\,,$$

which is tangent to T_rM at $(x,u) \in T_rM$.

From now on, simplifying the notations, we denote by $\bar{g}$ the Sasaki metric g^s on the tangent bundle TM and by $\tilde{g}$ the metric induced by $\bar{g}$ on the tangent sphere bundle T_rM of radius $r > 0$. Also we use the symbol $\langle \cdot, \cdot \rangle$ for the scalar product g_x on the tangent space M_x at $x \in M$. The Riemannian metric $\tilde{g}$ on the hypersurface $T_rM \subset (TM, \bar{g})$ induced by $\bar{g}$ on TM is uniquely determined by the formulas

$$\begin{aligned}
&\tilde{g}(X^h, Y^h) = \bar{g}(X^h, Y^h)\,,\\
&\tilde{g}(X^h, Y^t) = 0\,,\\
&\tilde{g}(X^t, Y^t) = \bar{g}(X^v, Y^v) - \frac{1}{r^2}\bar{g}(X^v, \boldsymbol{U})\bar{g}(Y^v, \boldsymbol{U})
\end{aligned}$$

for arbitrary vector fields X and Y on M.

1.1. **Sectional curvature.**

It is obvious that each tangent two-plane $\tilde{P} \subset (T_rM)_{(x,u)}$ is spanned by an orthonormal basis of the form $\{{X_1}^h + {Y_1}^t,\, {X_2}^h + {Y_2}^t\}$. For such a basis we have $\|X_i\|^2 + \|Y_i\|^2 = 1$, $i = 1,2$, and $\langle X_1,\, X_2\rangle + \langle Y_1,\, Y_2\rangle = 0$. Moreover, we can assume $\langle X_1,\, X_2\rangle = \langle Y_1,\, Y_2\rangle = 0$. This can be reached easily by a convenient rotation of the given basis. As usual, Y_1 and Y_2 are supposed to be orthogonal to u. Then the tangential lifts ${Y_1}^t$ and ${Y_2}^t$ coincide with the vertical lifts ${Y_1}^v$ and ${Y_2}^v$, respectively. From the formulas for the curvature operators one obtains as in [18] the following formula for the sectional curvature of the two-plane $\tilde{P}$:

$$\begin{aligned}
\tilde{K}(\tilde{P}) = {} & \langle R_x(X_1, X_2)X_2,\, X_1\rangle + 3\,\langle R_x(X_1, X_2)Y_2,\, Y_1\rangle + \frac{1}{r^2}\|Y_1\|^2\|Y_2\|^2 \\
& - \frac{3}{4}\|R_x(X_1, X_2)u\|^2 + \frac{1}{4}\|R_x(u, Y_2)X_1\|^2 + \frac{1}{4}\|R_x(u, Y_1)X_2\|^2 \\
& + \frac{1}{2}\,\langle R_x(u, Y_1)X_2,\, R_x(u, Y_2)X_1\rangle - \langle R_x(u, Y_1)X_1,\, R_x(u, Y_2)X_2\rangle \\
& + \langle (\nabla_{X_1} R)_x(u, Y_2)X_2,\, X_1\rangle + \langle (\nabla_{X_2} R)_x(u, Y_1)X_1,\, X_2\rangle\,.
\end{aligned} \tag{1.1}$$

We start with study on the sign of the sectional curvature, and its slight generalization.

Theorem 1.1 ([18]). *Let (M, g) be either locally symmetric with positive sectional curvature or locally flat, $n = \dim M \geq 2$. Then, for each sufficiently small positive number r, the tangent sphere bundle $(T_rM, \tilde{g})$ is a space of nonnegative sectional curvature.*

Sketch of the proof. We choose an orthonormal basis $\{{X_1}^h + {Y_1}^t,\, {X_2}^h + {Y_2}^t\} = \{{X_1}^h + {Y_1}^v,\, {X_2}^h + {Y_2}^v\}$ for the tangent two-plane $\tilde{P}$ of T_rM at $(x,u) \in T_rM$ as above. Then there are orthonormal pairs $\{\hat{X}_1, \hat{X}_2\}$ and $\{\hat{Y}_1, \hat{Y}_2\}$, and angles $\alpha, \beta \in [0, \pi/2]$ such that

$$X_1 = \cos\alpha\, \hat{X}_1\,, \quad Y_1 = \sin\alpha\, \hat{Y}_1\,;$$

$$X_2 = \cos\beta\, \hat{X}_2\,, \quad Y_2 = \sin\beta\, \hat{Y}_2\,.$$

Also there are positive numbers L_1 and L_2 such that

$$\big|\langle R_x(\hat{X}_1, \hat{X}_2)\hat{Y}_2\hat{Y}_1\rangle\big| < L_1\,, \quad \big|\langle R_x(\hat{Z}, \hat{Y}_1)\hat{X}_1, R_x(\hat{Z}, \hat{Y}_2)\hat{X}_2\big|\rangle < L_2\,.$$

Estimating (1.1) from below, we obtain for sufficiently small $r > 0$ that

$$\tilde{K}(\tilde{P}) \geq \Big(\varepsilon A - \frac{B}{r}\Big)^2 + 2AB\Big(\frac{\varepsilon}{r} - L\Big)\,, \tag{1.2}$$

where $A = \cos\alpha\cos\beta$, $B = \sin\alpha\sin\beta$, $L = 3(2L_1 + L_2)/4$ and ε is a positive constant. The right-hand side of (1.2) becomes nonnegative for all sufficiently small positive numbers r. □

This result is closely connected with those by A. A. Borisenko and A. L. Yampolsky [12, 11, 28, 29]. Its equivalent was claimed to be proved for the first time in [13] using a special criterion (see [13, Theorem 3.6], [12, Theorem 1] and also [11]). Yet, the proof is not completely rigorous. Our new proof is rigorous and different from that given by A. A. Borisenko and A. L. Yampolsky. As is well-known, every locally symmetric space with strictly positive sectional curvature is locally isometric to a rank one symmetric space of compact type. This gives the link between Theorem 1.1 and the result claimed in [13, p.79].

Theorem 1.2 ([20])**.** *Let (M, g) be an n-dimensional Riemannian locally symmetric space with nonnegative sectional curvature, $n \geq 3$. Then, for each sufficiently small positive number $r > 0$, the tangent sphere bundle $(T_rM, \tilde{g})$ is a space of nonnegative sectional curvature.*

Sketch of the proof. Because the statement of the Theorem is purely local, we can assume that (M, g) itself is globally symmetric and simply connected. Then we have the de Rham decomposition:

$$(M, g) = (M_0, g_0) \times (M_1, g_1) \times \cdots \times (M_s, g_s)\,,$$

where (M_0, g_0) is the Euclidean part and all (M_i, g_i) for $i = 1, 2, \ldots, s$ are irreducible symmetric spaces of compact type. We take a two-plane $\tilde{P}$ as in the proof of Theorem 1.1. If both $\hat{X}_1$ and $\hat{X}_2$ are tangent to M_0, then, from the formula (1.1), we see at once that $\tilde{K}(\tilde{P}) \geq 0$. If $\hat{X}_1$ and $\hat{X}_2$ are tangent to an irreducible factor M_i, $i = 1, 2, \ldots, s$, then we can use the same argument as in the proof of Theorem 1.1 to show that $\tilde{K}(\tilde{P}) \geq 0$ holds for every choice of an orthonormal triplet $\{\hat{Y}_1, \hat{Y}_2, \hat{u}\}$ in M_x and for all radii $r > 0$ depends only on the geometry of (M, g). Finally, let $\hat{X}_1$ and $\hat{X}_2$ be tangent to two different components M_i and M_j, $i \neq j$. Then $R_x(\hat{X}_1, \hat{X}_2) = 0$. So we can easily obtain the assertion of the Theorem. □

Under the hypothesis of Theorem 1.2, we can see easily from Theorem 1.9 below that $(T_rM, \tilde{g})$ is never a space of strictly positive sectional curvature. On the other hand, if (M, g) is a two-dimensional standard sphere, then $(T_rM, \tilde{g})$ is a space of positive sectional curvature according to the criterion by A. L. Yampolsky in [28].

The natural problem now is the question whether the conclusion of Theorem 1.2 may also hold for Riemannian manifolds which are not locally symmetric. We have not definitely solved this problem but some new evidence was given that the converse of Theorem 1.2 might hold, too. The first step in this direction has been made in [18]:

Theorem 1.3 ([18])**.** *There exist arbitrarily small perturbations of a spherical cap of the standard four-sphere with the following property: if (M, g) is such a perturbation, then $(T_rM, \tilde{g})$ admits negative sectional curvatures for every positive number r.*

Sketch of the proof. Let $\mathcal{B} \subset \mathbb{R}^4[u^1, \dots, u^4]$ be the open ball with center at the origin o and with radius $1 - \varepsilon$, where ε is a small positive number. Let $\phi \colon \mathcal{B} \longrightarrow \mathbb{R}^5$ be the map given by the formula

$$\phi(u) = \left(u^1, u^2, u^3, u^4, \sqrt{1 - \sum (u^i)^2 - F(u)}\,\right), \tag{1.3}$$

where $F(u) = \varepsilon_1 u^2 u^4 + \varepsilon_2 (u^1)^2 u^3$. Obviously, the smooth graph $M = \phi(\mathcal{B})$ is well-defined if $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are small enough. Now the idea of the proof is that, for arbitrary small radius $r > 0$, we show the existence of a tangent two-plane in T_rM over the origin $\phi(o) \in M$ such that its sectional curvature is negative. First we make a special choice $(\phi(o), u_0) \in T_rM$ and a special choice of a two-plane in the corresponding tangent space. Next we express the sectional curvature through certain trigonometric functions and finally we show that this expression becomes negative asymptotically. See more details in the proof of Theorem 1.5 below. The software *Mathematica* 3.0 is used here for deriving some more advanced general formula. □

To find an algebraic modification of Theorem 1.3, we have proved first the following Lemma:

Lemma 1.4 ([20])**.** *Let x be a fixed point of a Riemannian manifold (M, g). Then either there is an orthonormal triplet $\{X, Y, Z\}$ of M_x such that $\langle (\nabla_X R)_x(X, Y)Y, Z\rangle \neq 0$ or $(\nabla R)_x = 0$ identically.*

Now we have

Theorem 1.5 ([20])**.** *Let (M, g) be an n-dimensional Riemannian manifold, $n \geq 3$, and let x be a spherical point of M, i.e., such that all sectional curvatures at x are constant. Moreover, let the covariant derivative $(\nabla R)_x$ of the Riemannian curvature tensor R be nonzero. Then, for every $r > 0$, there is a vector $u \in M_x$, $\|u\| = r$, such that the tangent space $(T_rM)_{(x,u)}$ contains a two-plane with negative sectional curvature.*

Sketch of the proof. Because $(\nabla R)_x$ is nonzero, then, according to the Lemma 1.4, there is an orthonormal triplet $\{Z_1, Z_2, Z_3\}$ in the tangent space M_x such that

$b = \langle (\nabla_{Z_1} R)_x (Z_2, Z_3) Z_2,\, Z_1 \rangle > 0$. We put

$$X_1 = Z_1\,, \quad Y_1 = 0\,, \quad X_2 = \cos\beta\, Z_2\,, \quad Y_2 = -\sin\beta\, Z_3\,, \quad u = r Z_2\,,$$

where $r > 0$ and $\beta \in (0, \pi/2)$. Further, we put $c = K(Z_1 \wedge Z_2) > 0$. Finally, let $\tilde{P}$ be the tangent two-plane spanned by ${X_1}^h$ and ${X_2}^h + {Y_2}^t$ in $(T_r M)_{(x,u)}$. Since $x \in M$ is a spherical point, we have $\|R_x(X_1, X_2)u\| = cr\cos\beta$ and $R_x(u, Y_2)X_1 = 0$. Thus, from (1.1), we obtain

$$\tilde{K}(\tilde{P}) = \cos\beta\Big(c\cos\beta - \frac{3}{4}c^2 r^2 \cos\beta - br\,\sin\beta\Big)\,,$$

which becomes negative for $\beta \in (0,\, \pi/2)$ tending to $\pi/2$. □

Corollary 1.6 ([20])**.** *Let (M, g) be a Riemannian manifold such that the covariant derivative ∇R of the Riemannian curvature tensor R is nonzero everywhere. If, for some radius $r > 0$, the tangent sphere bundle $(T_r M, \tilde{g})$ has nonnegative sectional curvature, then (M, g) has no spherical points.*

We have also proved, with the exception $\dim M = 8$, that the tangent sphere bundles are never spaces of strictly positive curvature. We proceed as follows:

Proposition 1.7 ([21, 29])**.** *Let (M, g) be an n-dimensional Riemannian manifold such that $n \geq 3$, $n \neq 4,\ 8$. Then, at every point $x \in M$, there are unit vectors X, Y and Z in the tangent space M_x such that $\langle X,\, Y\rangle = 0$ and $R_x(X, Y)Z = 0$.*

Sketch of the proof due to A. L. Yampolsky [29]. Suppose that there is a point $x \in M$ such that $R_x(X, Y)Z \neq 0$ holds for every triplet $\{X, Y, Z\}$ of unit vectors satisfying $\langle X, Y\rangle = 0$. Let $\{E_1, E_2, \dots, E_n\}$ be an orthonormal basis of $(M_x, \langle\cdot,\,\cdot\rangle)$. Then the vector $(\mathcal{V}_i)_Z = T_Z(R_x(E_i, E_n)Z) \neq 0$ is always tangent to the unit sphere $S_x \subset M_x$ at the end-point of Z, where $T_Z : M_x \longrightarrow (TM)_Z$ is a canonical isomorphism given by $T_Z(W) = \left.\dfrac{\mathrm{d}}{\mathrm{d}t}\right|_0 (Z + tW)$ for all $W \in M_x$. Now the vector fields $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_{n-1}$ on the sphere S_x are linearly independent. Hence S_x is parallelizable. Thus, from the well-known theorem by J. F. Adams [2], we see that $n = 2, 4$ and 8. □

Proposition 1.8 ([21])**.** *Let $n = 4$ and suppose, in addition, that (M, g) is a space of positive sectional curvature. Then the conclusion of* Proposition 1.7 *still holds.*

Sketch of the proof. We prove the existence of at least one solution of the equation $R_x(X, Y)Z = 0$ such that $X \wedge Y \neq 0$ and $Z \neq 0$. Using the so-called Chern basis in the tangent space, we reduce the number of the curvature components. Then we show that the wanted property follows from the fact that a certain homogeneous quadratic polynomial Q of three variables (whose coefficients are functions of the curvature components) is never positive definite or negative definite. Here a careful discussion of several cases must be done, and the positivity of the sectional curvature of (M, g) as well as the continuity argument are applied. □

We can not remove the assumption about the positive sectional curvature in Proposition 1.8. In fact, S. Ivanov and I. Petrova have found in [15], among spaces with sign-changing sectional curvature, an example on which there do not exist nontrivial solutions of the equation $R_x(X, Y)Z = 0$.

Theorem 1.9 ([21]). *Let $(T_rM, \tilde{g})$ be a tangent sphere bundle over an n-dimensional Riemannian manifold (M, g) such that $n \geq 3$, $n \neq 8$. Then $(T_rM, \tilde{g})$ is never a space of positive sectional curvature.*

Sketch of the proof. Suppose that $(T_rM, \tilde{g})$ with arbitrary fixed $r > 0$ has positive sectional curvature $\tilde{K}(\tilde{P})$. Putting $Y_1 = Y_2 = 0$ in the formula (1.1) we see at once that (M, g) is a space of positive sectional curvature. Hence, by the above two Propositions, there are unit vectors $X, Y, Z \in M_x$ such that $\langle X, Y\rangle = 0$ and $R_x(X, Y)Z = 0$. From the general formula (1.1), in which we take $u = X$, we obtain $\tilde{K}(\mathrm{span}\{Y^t, Z^h\}) = 0$, which is a contradiction. □

It remains an open problem if Theorem 1.9 and Proposition 1.7 still hold in dimension $n = 8$.

The following result shows that the conclusion of Theorem 1.2 is the best possible for $n \geq 3$.

Theorem 1.10 ([21]). *Let $(T_rM, \tilde{g})$ be a tangent sphere bundle over an n-dimensional locally symmetric Riemannian manifold (M, g), $n \geq 3$, and r be an arbitrary positive number. Then $(T_rM, \tilde{g})$ is never a space of positive sectional curvature.*

Sketch of the proof. If $n = 3$, then the result follows from Theorem 1.9 (or it can be proved directly in an easy way). Suppose now that $n \geq 4$. Then, recalling a theorem by J. A. Wolf [27, Theorem 1], we see that there exists a rank one symmetric space $N \subset M$ of compact type which is a totally geodesic submanifold of dimension four. Now Proposition 1.8 is valid for N and hence it is valid also for M. The rest of the proof is the same as that for Theorem 1.9. □

Now we look for the converse to Theorem 1.2.

Proposition 1.11 ([20]). *Let (M, g) be an n-dimensional Riemannian manifold with nonnegative sectional curvature, $n \geq 3$, and let $x \in M$ be a point such that the covariant derivative $(\nabla R)_x$ of the Riemannian curvature tensor R is nonzero. Then, for every sufficiently large $r > 0$, there is a vector $u \in M_x$, $\|u\| = r$, such that the tangent space $(T_rM)_{(x,u)}$ contains a two-plane with negative sectional curvature.*

Sketch of the proof. We write $R_x(Z_1, Z_2)Z_2 = c\,Z_1 + W$, where $W \in M_x$ is orthogonal to Z_1. Hence, putting $C = \|R_x(Z_1, Z_2)Z_2\|$, we get $C \geq c > 0$. Put $D = \|R_x(Z_2, Z_3)Z_1\| \geq 0$. Now, from (1.1), we obtain, for the two-plane $\tilde{P}$ as in the the proof of Theorem 1.5, that

$$\tilde{K}(\tilde{P}) = r\,\sin\beta\Big(\frac{1}{4}rD^2\sin\beta - b\cos\beta\Big) + \cos^2\beta\Big(c - \frac{3}{4}C^2r^2\Big)\,.$$

The second term is zero for $C = 0$ and every $r > 0$; and it is nonpositive for $C > 0$ and for every $r \geq 2\sqrt{c}/(\sqrt{3}\,C)$. Let us fix a number $r > 0$ for which this second term is nonpositive. The first term is then negative for all $\beta \in (0, \pi/2)$ such that $\mathrm{ctg}\,\beta > rD^2/(4b)$. Thus a two-plane at $(x, u) \in T_rM$ with negative sectional curvature exists. □

Thus, we obtain easily the following "nonstandard" converse of Theorem 1.2.

Theorem 1.12 ([20]). *Let (M, g) be an n-dimensional Riemannian manifold, $n \geq 3$, such that, for all sufficiently large radii $r > 0$, the tangent sphere bundles $(T_rM, \tilde{g})$ over (M, g) are spaces of nonnegative sectional curvature. Then the space (M, g) is locally symmetric.*

In the rest of this section we assume that the conformal Weyl tensor W vanishes. This assumption reads that either $\dim M = 3$, or $\dim M > 3$ and (M, g) is conformally flat.

Lemma 1.13 ([20]). *Let (M, g), $\dim M \geq 3$, be a Riemannian manifold such that the conformal Weyl tensor W vanishes. Let $\{E_1, E_2, \ldots, E_n\}$ be a basis of M_x which diagonalizes the Ricci tensor Ric_x. Then $R_x(E_i, E_j)E_k = 0$ for every triplet of distinct indices $\{i, j, k\}$.*

Lemma 1.14 ([20]). *Let x be a fixed point of a Riemannian manifold (M, g), $\dim M \geq 3$, such that the conformal Weyl tensor W vanishes and let $\langle (\nabla_X R)_x (X, Z)Y, Z\rangle = 0$ holds whenever $\{X, Y, Z\}$ is an orthonormal triplet in M_x such that $R_x(X, Y)Z = 0$. Then $(\nabla R)_x = 0$ identically.*

Theorem 1.15 ([20]). *Let (M, g) be a Riemannian manifold such that the conformal Weyl tensor W vanishes* (*in particular, let* $\dim M = 3$). *If the tangent sphere bundle $(T_rM, \tilde{g})$ is a space of nonnegative sectional curvature for some radius $r > 0$, then (M, g) is locally symmetric.*

Sketch of the proof. Let us suppose that the space (M, g) is not locally symmetric. Then, at some point $x \in M$ we have $(\nabla R)_x \neq 0$. According to Lemma 1.14, there is an orthonormal triplet $\{Z_1, Z_2, Z_3\}$ in M_x such that $\langle (\nabla_{Z_1} R)_x (Z_1, Z_2)Z_2,\ Z_3)\rangle > 0$ and, at the same time, $R_x(Z_1, Z_2)Z_3 = 0$. Then, using the same procedure as in the proof of Theorem 1.5, we find for every $r > 0$ a tangent two-plane of T_rM with negative sectional curvature, which is a contradiction. $\square$

From this theorem we have deduced the following

Corollary 1.16 ([20]). *Let (M, g) be a Riemannian manifold of dimension n such that the conformal Weyl tensor W vanishes* (*in particular, let* $\dim M = 3$). *Then the tangent sphere bundle $(T_rM, \tilde{g})$ is a space of nonnegative sectional curvature for all sufficiently small radii $r > 0$ if and only if (M, g) is locally isometric to one of the following spaces*:

$$\mathbb{R}^n\,, \quad S^n(c)\,, \quad or \quad S^{n-1}(c) \times \mathbb{R}^1\,,$$

where $\mathbb{R}^n$ is the Euclidean n-space and $S^n(c)$ is the n-sphere of radius $1/\sqrt{c}$.

Sketch of the proof. If $(T_rM, \tilde{g})$ is a space of nonnegative sectional curvature for every sufficiently small radius $r > 0$, then, by Theorem 1.15, (M, g) is locally symmetric and hence locally isometric to a symmetric space, which is globally homogeneous. Hence, for $n > 3$, the result follows from the Theorem by H. Takagi in [26]. For $n = 3$, the only simply connected symmetric spaces with nonnegative sectional curvature are $\mathbb{R}^3$, $S^3(c)$ and $S^2(c) \times \mathbb{R}^1$. The "only if" part follows from Theorem 1.2. $\square$

1.2. **Ricci curvature.**

According to Theorem 1.9, a tangent sphere bundle equipped with the induced Sasaki metric can hardly have a strictly positive sectional curvature. In the present section we show that the situation is different for the Ricci curvature.

Proposition 1.17 ([18]). *The Ricci tensor* $\widetilde{\mathrm{Ric}}$ *of* $(T_rM, \tilde{g})$ *is given, at each fixed point* $(x,u) \in T_rM$, *by*

$$\begin{aligned} &\widetilde{\mathrm{Ric}}_{(x,u)}(X^h + Y^t, X^h + Y^t) \\ &= \mathrm{Ric}_x(X,X) + r\big((\nabla_{\hat{u}} \mathrm{Ric})_x(Y,X) - (\nabla_Y \mathrm{Ric})_x(\hat{u},X)\big) \\ &\quad + r^2\Big[\frac{1}{4}\sum_i \|R_x(\hat{u},Y)E_i\|^2 - \frac{1}{2}\sum_i \|R_x(\hat{u},E_i)X\|^2\Big] + \frac{n-2}{r^2}\|Y\|^2\,, \end{aligned} \tag{1.4}$$

for any $X \in M_x$ *and any* $Y \in M_x$ *orthogonal to* u *such that* $\tilde{g}_{(x,u)}(X^h + Y^t, X^h + Y^t) = 1$, *where* Ric *is the Ricci tensor of* (M,g) *and we put* $\hat{u} = u/r$.

Theorem 1.18 ([18]). *Let* (M,g) *be an* n*-dimensional compact Riemannian manifold with positive Ricci curvature,* $n \geq 3$. *Then, for each sufficiently small positive number* r, *the tangent sphere bundle* $(T_rM, \tilde{g})$ *is a space of positive Ricci curvature.*

Sketch of the proof. First we see that the coefficients of r and r^2 in the formula (1.4) are bounded. Then we see that $\mathrm{Ric}(X,X) + ((n-2)/(r^2))\|Y\|^2$ is positive for sufficiently small positive number r. □

It is worth mentioning that our specific and explicit result is very closely related to the paper by J. Nash [23] and to that by W. Poor [25], where some general existence results are proved for Riemannian submersions.

1.3. **Scalar curvature.**

The scalar curvature of tangent sphere bundle $(T_rM, \tilde{g})$ with an arbitrary constant radius is of particular interest. Namely, we have seen in [18] that it can take, under some additional assumptions, positive values for small radii and negative values for large radii. First we show the Proposition, which is a generalization for an arbitrary radius of the formula given by E. Boeckx and L. Vanhecke in [10].

Proposition 1.19 ([18]). *The scalar curvature* $\widetilde{\mathrm{Sc}}(\tilde{g})$ *of* $(T_rM, \tilde{g})$ *at each fixed point* $(x,u) \in T_rM$ *is given by*

$$\widetilde{\mathrm{Sc}}(\tilde{g})_{(x,u)} = \frac{(n-1)(n-2)}{r^2} + \mathrm{Sc}(g)_x - \frac{1}{4}r^2\xi_x(\hat{u},\hat{u})\,, \tag{1.5}$$

where $\hat{u} = u/r$, $\mathrm{Sc}(g)$ *is the scalar curvature of* (M,g) *and* ξ *is a tensor field on* M *given by*

$$\xi(X,Y) = \sum_{i,j}\langle R(X,E_i)E_j, R(Y,E_i)E_j\rangle$$

for all vector fields X *and* Y *on* M *and any (local) orthonormal frame* $\{E_1, E_2, \ldots, E_n\}$ *on* M.

We should also mention that the formula (1.5) can be generalized to any Riemannian submersion with totally geodesic fibers where the metric of the total space is subjected to the so-called canonical variation (see [3, Proposition 9.70]). In our case, the canonical variation corresponds to the "variation" of the constant radius $r > 0$ starting from the initial value $r = 1$.

Theorem 1.20 ([18])**.** *Let (M, g) be an n-dimensional Riemannian manifold with bounded sectional curvature (or, in particular, let (M, g) be compact), $n \geq 3$. Then, for each sufficiently small positive number r, the tangent sphere bundle $(T_r M, \tilde{g})$ is a space of positive scalar curvature.*

Sketch of the proof. We see first that the scalar curvature $\mathrm{Sc}(g)$ and the function $\xi_x(\hat{u}, \hat{u})$ are bounded on M. The result follows from Proposition 1.19. □

Let us recall notions we need in the following. A Riemannian manifold (M, g) is called *δ-pinched* if there are positive numbers $\delta \leq 1$ and A such that $A\delta \leq K \leq A$ holds for its sectional curvature K. The *index of nullity* at a point $x \in M$ is defined as the dimension of the subspace $\{X \in M_x \mid R_x(X, Y) = 0 \text{ for all } Y \in M_x\}$. (See, for example, [16].)

Theorem 1.21 ([18])**.** *Let (M, g) be an n-dimensional δ-pinched Riemannian manifold (or, alternatively, let (M, g) be compact and such that its index of nullity is zero at every point), $n \geq 2$. Then, for each sufficiently large positive number r, the tangent sphere bundle $(T_r M, \tilde{g})$ is a space of negative scalar curvature.*

Sketch of the proof. Let first (M, g) be δ-pinched. Then the scalar curvature $\mathrm{Sc}(g)$ is bounded on M and $\xi_x(\hat{u}, \hat{u})$ is nonnegative on M for every $(x, u) \in T_r M$, where we put $\hat{u} = u/r$. It is sufficient to prove that $\xi_x(\hat{u}, \hat{u}) > \delta'$ for all $(x, u) \in T_r M$ and for some $\delta' > 0$ which is independent of r. But if we choose an orthonormal basis $\{E_1, E_2, \dots, E_n\}$ such that $E_n = \hat{u}$, we get

$$\begin{aligned}\xi_x(\hat{u}, \hat{u}) = &\sum_{i,j} \|R_x(E_n, E_i)E_j\|^2 \geq \sum_{i=1}^{n-1} \|R_x(E_n, E_i)E_i\|^2 \\ \geq &\sum_{i=1}^{n-1} (K_x(E_n \wedge E_i))^2 \geq (n-1)A^2\delta^2 .\end{aligned}$$

Now the result is obvious from (1.5).

Alternatively, if (M, g) is compact and such that its index of nullity is zero everywhere, we see first that $\mathrm{Sc}(g)$ is bounded on M and $\xi_x(\hat{u}, \hat{u})$ is nonzero and hence positive for all $(x, u) \in T_r M$. Because $T_r M$ is compact, we have again $\xi_x(\hat{u}, \hat{u}) > \delta'$ for some positive number δ' independent of r. □

References

[1] Abbassi, M. T. K., Calvaruso, G., *g-natural contact metrics on unit tangent sphere bundles*, Monatsh. Math. **151** (2) (2007), 89–109.

[2] Adams, J. F., *On the non-existence of elements of Hopf invariant one*, Ann. Math. **72** (1960), 20–104.

[3] Besse, A. L., *Einstein Manifolds*, Springer-Verlag, Berlin–Heidelberg–New York, 1987.

[4] Blair, D., *When is the tangent sphere bundle locally symmetric?*, Geom. Topol., World Sci. Publishing, Singapore (1989), 15–30.

[5] Boeckx, E., Vanhecke, L., *Characteristic reflections on unit tangent sphere bundles*, Houston J. Math. **23** (1997), 427–448.

[6] Boeckx, E., Vanhecke, L., *Geometry of the tangent sphere bundle*, Proceedings of the Workshop on Recent Topics in Differential Geometry (Cordero, L. A., García-Río, E., eds.), Santiago de Compostela, 1997, pp. 5–17.

[7] Boeckx, E., Vanhecke, L., *Curvature homogeneous unit tangent sphere bundles*, Publ. Math. Debrecen **35** (1998), 389–413.

[8] Boeckx, E., Vanhecke, L., *Unit tangent sphere bundles and two-point homogeneous spaces*, Period. Math. Hungar. **36** (1998), 79–95.

[9] Boeckx, E., Vanhecke, L., *Harmonic and minimal vector fields on tangent and unit tangent bundles*, Differential Geom. Appl. **13** (2000), 77–93.

[10] Boeckx, E., Vanhecke, L., *Unit tangent sphere bundles with constant scalar curvature*, Czechoslovak Math. J. **51** (126) (2001), 523–544.

[11] Borisenko, A. A., Yampolsky, A. L., *On the Sasaki metric of the tangent and the normal bundles*, Sov. Math., Dokl. **35** (1987), 479–482.

[12] Borisenko, A. A., Yampolsky, A. L., *The sectional curvature of the Sasaki metric of* T_rM^n, Ukrain. Geom. Sb. **30** (1987), 10–17.

[13] Borisenko, A. A., Yampolsky, A. L., *Riemannian geometry of fiber bundles*, Russian Math. Surveys **46** (6) (1991), 55–106.

[14] Calvaruso, G., *Contact metric geometry of the unit tangent sphere bundle*, Complex, contact and symmetric manifolds. In honor of L. Vanhecke (Kowalski, O. et al, ed.), vol. 234, Progress in Mathematics, 2005, pp. 41–57.

[15] Ivanov, S., Petrova, I., *Riemannian manifold in which the skew-symmetric curvature operator has pointwise constant eigenvalues*, Geom. Dedicata **70** (1998), 269–282.

[16] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry II*, Interscience Publishers, New York–London–Sydney, 1969.

[17] Kowalski, O., Sekizawa, M., *Geometry of tangent sphere bundles with arbitrary constant radius*, Proceedings of the Symposium Contemporary Mathematics (Bokan, N., ed.), Faculty of Mathematics, University of Belgrade, 2000, pp. 219–228.

[18] Kowalski, O., Sekizawa, M., *On tangent sphere bundles with small or large constant radius*, Ann. Global Anal. Geom. **18** (2000), 207–219.

[19] Kowalski, O., Sekizawa, M., *On the scalar curvature of tangent sphere bundles with arbitrary constant radius*, Bull. Greek Math. Soc. **44** (2000), 17–30.

[20] Kowalski, O., Sekizawa, M., *On Riemannian manifolds whose tangent sphere bundles can have nonnegative sectional curvature*, Univ. Jagellon. Acta Math. **40** (2002), 245–256.

[21] Kowalski, O., Sekizawa, M., Vlášek, Z., *Can tangent sphere bundles over Riemannian manifolds have strictly positive sectional curvature?*, Global Differential Geometry: The Mathematical Legacy of Alfred Gray (Fernandez, M. and Wolf, J. A., eds.), Contemp. Math. **288** (2001), 110–118.

[22] Nagy, P. T., *Geodesics on the tangent sphere bundle of a Riemannian manifold*, Geom. Dedicata **7** (1978), 233–243.

[23] Nash, J., *Positive Ricci curvature on fiber bundles*, J. Differential Geom. **14** (1979), 241–254.

[24] Podestà, F., *Isometries of tangent sphere bundles*, Boll. Un. Mat. Ital. A(7) **5** (1991), 207–214.

[25] Poor, W., *Some exotic spheres with positive Ricci curvature*, Math. Ann. **216** (1975), 245–252.

[26] Takagi, H., *Conformally flat Riemannian manifolds admitting a transitive group of isometries*, Tôhoku Math. J. **27** (1975), 103–110.

[27] Wolf, J. A., *Elliptic spaces in Grassmann manifolds*, Illinois J. Math. **7** (1963), 447–462.

[28] Yampolsky, A. L., *On the geometry of tangent sphere bundles of Riemannian manifolds*, Ukrain. Geom. Sb **24** (1981), 129–132, in Russian.

[29] Yampolsky, A. L., *On Sasaki metric of tangent and normal bundle*, Ph.D. thesis, Odessa, 1986, (Russian).

Faculty of Mathematics and Physics
Charles University in Prague
Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: kowalski@karlin.mff.cuni.cz

Department of Mathematics
Tokyo Gakugei University
Koganei-shi Nukuikita-machi 4-1-1, Tokyo 184-8501, Japan
E-mail: sekizawa@u-gakugei.ac.jp