ARCHIVUM MATHEMATICUM (BRNO)
Tomus 44 (2008), 403–418

MODULI SPACES OF LIE ALGEBROID CONNECTIONS

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Abstract. We shall prove that the moduli space of irreducible Lie algebroid connections over a connected compact manifold has a natural structure of a locally Hausdorff Hilbert manifold. This generalizes some known results for the moduli space of simple semi-connections on a complex vector bundle over a compact complex manifold.

Introduction

Moduli spaces have many applications in mathematics and physics. In geometry, they make it possible to construct invariants of manifolds, for example, Seiberg-Witten invariants, Gromow-Witten invariants and others, they are closely related to the subject of deformation theory. In particular, they form an indispensable tool in the study of four-manifolds (Donaldson theory, see [2]), Yang-Mills theory in physics, etc.

A basic motivation for the study of moduli spaces of flat Lie algebroid connections over a compact manifold is a description of the well known results for some moduli spaces of this type in a unified treatment. Two cornerstones of this general construction consist of the moduli space of holomorphic structures on a complex vector bundle over a compact complex manifold (more about this and the related Hitchin-Kobayashi correspondence can be found in [12], [11] and [13]) and the moduli space of Higgs bundles on a compact Riemann surface, see [9].

More recent examples of this phenomenon involve the moduli space of complex B-branes and the moduli space of symplectic A-branes, based on generalized complex geometry. They play a very important role in mirror symmetry and geometric Langlands program, see [10], [8].

In the paper, we study the space of Lie algebroid connections on a vector bundle on a compact manifold M and the action of the gauge group on this space. The purpose of the paper is to demonstrate that certain moduli spaces of Lie algebroid connections on real or complex vector bundles over compact manifolds carry natural structure of a locally Hausdorff Hilbert manifold.

Let L be a real or complex transitive Lie algebroid over a compact manifold M and E be a real or complex vector bundle over M. Let $\hat{A}(E, L)$ denote the space of

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2000 *Mathematics Subject Classification*: primary 32G13.

Key words and phrases: moduli space, connection, Lie algebroid.

The author of this article was supported by the grant GAUK 306-10/252442 of the Grant Agency of Charles University.

irreducible L-connections and $\mathrm{Gau}(E)$, resp. $\mathrm{Gau}(E)^{\mathrm{r}}$, the gauge group, resp. the reduced gauge group with natural left or right (according to conventions) action on $\hat{\mathcal{A}}(E,L)$. If we consider Sobolev completions $\hat{\mathcal{A}}(E,L)_\ell$, $\mathrm{Gau}(E)_{\ell+1}$ and $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$ of the corresponding spaces for $\ell > \frac{1}{2}\dim M$, we show that

$$\hat{\mathcal{B}}(E,L)_\ell = \hat{\mathcal{A}}(E,L)_\ell / \mathrm{Gau}(E)_{\ell+1}$$

is a locally Hausdorff Hilbert manifold and $\hat{p}\colon \hat{\mathcal{A}}(E,L)_\ell \to \hat{\mathcal{B}}(E,L)_\ell$ a principal $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$-bundle. It is a generalization of results in [12], [13] and [11] to the case of L-connections for a transitive Lie algebroid.

1. Lie algebroids and L-connections

In this section, we introduce notation for Lie algebroids and Lie algebroid connections. More about Lie algebroids and related generalized complex structures can be found in [7], [6].

Remark. We will use notation $\mathbb{K}$ for the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers.

Definition 1. A real (complex) *Lie algebroid* $(L \xrightarrow{p} M, [\cdot,\cdot], a)$ is a real (complex) vector bundle $p\colon L \to M$ together with a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(M,L)$ and a homomorphism of vector bundles $a\colon L \to TM$ ($a\colon L \to TM_{\mathbb{C}}$) called the *anchor map* making the diagrams

$$\begin{tikzcd} L \arrow[r, "a"] \arrow[d, "p"'] & TM \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"'] & M \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} L \arrow[r, "a"] \arrow[d, "p"'] & TM_{\mathbb{C}} \arrow[d] \\ M \arrow[r, "\mathrm{id}_M"'] & M \end{tikzcd}$$

commutative. Moreover, the anchor map fulfills

i) $a([e_1,e_2]) = [a(e_1),a(e_2)]$ resp. $a([e_1,e_2]) = [a(e_1),a(e_2)]_{\mathbb{C}}$,

ii) $[e_1, fe_2] = f[e_1,e_2] + (a(e_1)f)e_2$

for all $e_1,e_2 \in \Gamma(M,L)$ and $f \in C^\infty(M,\mathbb{K})$.

Example.

(1) Every Lie algebra is a Lie algebroid over a point, $M = \{\mathrm{pt}\}$.

(2) The tangent bundle TM of a manifold M is a Lie algebroid for the Lie bracket of vector fields and the identity of TM as an anchor map.

(3) Every integrable subbundle of the tangent bundle – that is one whose sections are closed under the Lie bracket – also defines a Lie algebroid.

(4) Every bundle of Lie algebras over a manifold defines a Lie algebroid, where the Lie bracket is defined pointwise and the anchor map is equal to zero.

The space of sections $\Gamma(M, \Lambda^\bullet L^*) = \bigoplus_k \Gamma(M, \Lambda^k L^*)$ has a structure of a graded commutative algebra with respect to the exterior product

$$(1)\quad (\varphi \wedge \psi)(\xi_1, \ldots, \xi_{k+\ell}) = \frac{1}{k!\,\ell!} \sum_\sigma \operatorname{sign}\sigma \cdot \varphi(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)})\, \psi(\xi_{\sigma(k+1)}, \ldots, \xi_{\sigma(k+\ell)})$$

for $\varphi \in \Gamma(M, \Lambda^k L^*)$, $\psi \in \Gamma(M, \Lambda^\ell L^*)$ and $\xi_1, \ldots, \xi_{k+\ell} \in \Gamma(M, L)$. It is possible to define a graded derivation d_L of degree 1 on $\Gamma(M, \Lambda^\bullet L^*)$, which is a generalization of the *de Rham differential* on ordinary forms,

$$(2)\quad (d_L\varphi)(\xi_0, \ldots, \xi_k) = \sum_{i=0}^{k} (-1)^i a(\xi_i)\varphi(\xi_0, \ldots, \widehat{\xi_i}, \ldots, \xi_k) + \sum_{i<j} (-1)^{i+j} \varphi([\xi_i, \xi_j], \xi_0, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_k)\,,$$

where $\varphi \in \Gamma(M, \Lambda^k L^*)$ and $\xi_0, \ldots, \xi_k \in \Gamma(M, L)$. One can easily verify that $d_L^2 = 0$. By an analogy with ordinary differential forms, we can introduce the *insertion operator* and the *Lie derivative* for a Lie algebroid. The Lie derivative is given by

$$(3)\quad (\mathcal{L}^L_\xi \varphi)(\xi_1, \ldots, \xi_k) = a(\xi)\varphi(\xi_1, \ldots, \xi_k) - \sum_{i=1}^{k} \varphi(\xi_1, \ldots, [\xi, \xi_i], \ldots, \xi_k)\,,$$

where $\varphi \in \Gamma(M, \Lambda^k L^*)$, $\xi, \xi_1, \ldots, \xi_k \in \Gamma(M, L)$, and the insertion operator for a section $\xi \in \Gamma(M, L)$ is defined by

$$(4)\quad (i^L_\xi \varphi)(\xi_1, \ldots, \xi_{k-1}) = \varphi(\xi, \xi_1, \ldots, \xi_{k-1})\,,$$

where $\varphi \in \Gamma(M, \Lambda^k L^*)$ and $\xi_1, \ldots, \xi_{k-1} \in \Gamma(M, L)$. Note that i^L_ξ, resp. $\mathcal{L}^L_\xi$, is a graded derivation of $\Gamma(M, \Lambda^\bullet L^*)$ of degree -1, resp. 0.

Remark. For the sake of simplicity and to follow an analogy with ordinary differential forms, the space of sections $\Gamma(M, \Lambda^k L^*)$ will be denoted by $\Omega^k_L(M)$ and the graded commutative algebra $\Gamma(M, \Lambda^\bullet L^*)$ by $\Omega^\bullet_L(M)$. Furthermore, the space of sections $\Gamma(M, L)$ will be denoted by $\mathfrak{X}_L(M)$.

Definition 2. Let $L \xrightarrow{a} TM$ ($L \xrightarrow{a} TM_{\mathbb{C}}$) be a real (complex) Lie algebroid and E a real (complex) vector bundle. We denote the space of sections $\Gamma(M, \Lambda^k L^* \otimes E)$ by $\Omega^k_L(M, E)$. Their sections will be called *Lie algebroid k-forms*, or simply k-forms, with values in E. A *linear L-connection* on E is a $\mathbb{K}$-linear map

$$(5)\quad \nabla \colon \Omega^0_L(M, E) \to \Omega^1_L(M, E)$$

satisfying Leibniz rule: $\nabla(fs) = d_L f \otimes s + f\nabla s$ for any $f \in C^\infty(M, \mathbb{K})$ and $s \in \Omega^0_L(M, E)$. Such ∇ is called a *generalized connection* or *Lie algebroid connection.*

Remark. For any $\xi \in \mathfrak{X}_L(M)$, we have the *covariant derivative* $\nabla_\xi s = i^L_\xi(\nabla s) \in \Omega^0_L(M, E)$ of s in the direction ξ and with the following property

$$(6)\quad \nabla_\xi(fs) = (\mathcal{L}^L_\xi f)s + f\nabla_\xi s\,.$$

The map $\nabla_\xi \colon \Omega^0_L(M,E) \to \Omega^0_L(M,E)$ is $\mathbb{K}$-linear for any $\xi \in \mathfrak{X}_L(M)$.

Lemma 1. *Any L-connection ∇ on E has a natural extension to an operator*

$$d^\nabla \colon \Omega^\bullet_L(M,E) \to \Omega^\bullet_L(M,E) \tag{7}$$

uniquely determined by:

(1) $d^\nabla\big(\Omega^k_L(M,E)\big) \subset \Omega^{k+1}_L(M,E)$,

(2) $d^\nabla{}_{|\Omega^0_L(M,E)} = \nabla$,

(3) *the graded Leibniz rule:* $d^\nabla(\alpha \wedge \omega) = d_L\alpha \wedge \omega + (-1)^k \alpha \wedge d^\nabla \omega$ *for all* $\alpha \in \Omega^k_L(M)$ *and* $\omega \in \Omega^\ell_L(M,E)$.

The operator d^∇ (called the covariant exterior derivative*) is given by the following formula*

$$\begin{aligned}(d^\nabla\omega)(\xi_0,\xi_1,\dots,\xi_k) = &\sum_{i=0}^k (-1)^i \nabla_{\xi_i}\omega(\xi_0,\dots,\hat{\xi}_i,\dots,\xi_k)\\ &+ \sum_{i<j}^k (-1)^{i+j}\omega([\xi_i,\xi_j],\xi_0,\dots,\hat{\xi}_i,\dots,\hat{\xi}_j,\dots,\xi_k)\end{aligned} \tag{8}$$

for $\xi_0,\dots,\xi_k \in \mathfrak{X}_L(M)$.

Proof. The proof goes along the same line as the proof of Lemma 1 for an affine connection. □

Lemma 2. *Denote by $\mathcal{A}(E,L)$ the set of all L-connections on a vector bundle E. Then $\mathcal{A}(E,L)$ is an affine space modeled on the vector space $\Omega^1_L\big(M,\operatorname{End}(E)\big)$.*

Proof. We first prove that $\mathcal{A}(E,L)$ is non-empty. To see this, consider a vector bundle atlas $(U_\alpha,\psi_\alpha)_{\alpha\in\mathcal{A}}$ and take a smooth partition of unity $(g_\alpha)_{\alpha\in\mathcal{A}}$ subordinate to $(U_\alpha)_{\alpha\in\mathcal{A}}$.[1] If $\hat\nabla$ denotes the trivial L-connection on the trivial vector bundle $V_M = M\times V$, where V is the standard fiber of E, given by $\hat\nabla s = \hat\nabla(f\otimes v) = d_Lf\otimes v$, where $f\in C^\infty(M,\mathbb{K})$, $v\in V$ and $s = f\otimes v \in \Gamma(M,V_M)$.

Furthermore, observe that for every $\alpha\in\mathcal{A}$ and $s\in\Omega^0_L(M,E)$ the section $g_\alpha s$ has its support in U_α, hence $\psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big) \in \Gamma(U_\alpha,V_M)$ has the support in U_α, as well. If we extend $\psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big)$ by 0 outside U_α, we obtain smooth section $\psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big) \in \Gamma(M,V_M)$. Using locality of the operator $\hat\nabla$, i.e. $\operatorname{supp}\hat\nabla u \subset \operatorname{supp} u$ for all $u\in\Gamma(M,V_M)$, we have

$$\operatorname{supp}\ \big(\mathrm{id}_{L^*{}_{|U_\alpha}}\otimes\psi_\alpha^{-1}\big)\circ\big(\hat\nabla\big(\psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big)\big)\big)_{|U_\alpha} \subset U_\alpha\,.$$

Therefore we can extend this section to a global section of $L^*\otimes E$. A natural definition

$$\nabla s = \sum_{\alpha\in\mathcal{A}} \big(\mathrm{id}_{L^*{}_{|U_\alpha}}\otimes\psi_\alpha^{-1}\big)\circ\big(\hat\nabla\big(\psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big)\big)\big)_{|U_\alpha},$$

[1] We do not require compact support of g_α, hence we can find a smooth partition of unity subordinated to $(U_\alpha)_{\alpha\in\mathcal{A}}$.

yields an L-connection on E, because for given $f \in C^\infty(M,\mathbb{K})$, we get

$$\begin{aligned}\nabla(fs) &= \sum_{\alpha\in\mathcal{A}} \big(\mathrm{id}_{L^*|U_\alpha} \otimes \psi_\alpha^{-1}\big) \circ \big(\hat\nabla\big(\psi_\alpha\big((g_\alpha f s)_{|U_\alpha}\big)\big)\big)_{|U_\alpha} \\ &= \sum_{\alpha\in\mathcal{A}} \big(\mathrm{id}_{L^*|U_\alpha} \otimes \psi_\alpha^{-1}\big) \circ \big(\hat\nabla\big(f\psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big)\big)\big)_{|U_\alpha} \\ &= \sum_{\alpha\in\mathcal{A}} \big(\mathrm{id}_{L^*|U_\alpha} \otimes \psi_\alpha^{-1}\big) \circ \big(d_L f \otimes \psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big) + f\hat\nabla\big(\psi_\alpha\big((g_\alpha s)_{|U_\alpha}\big)\big)\big)_{|U_\alpha} \\ &= \sum_{\alpha\in\mathcal{A}} d_L f \otimes g_\alpha s + f\big(\mathrm{id}_{L^*|U_\alpha} \otimes \psi_\alpha^{-1}\big) \circ \big(\hat\nabla\big(\psi_\alpha\big((g_\alpha f s)_{|U_\alpha}\big)\big)\big)_{|U_\alpha} \\ &= d_L f \otimes s + f\nabla s.\end{aligned}$$

The rest of the proof is very simple. We need to verify that if ∇_1 and ∇_0 are two L-connections, then $\nabla_1 - \nabla_0$ is a tensor. We have $(\nabla_1 - \nabla_0)(fs) = d_L f \otimes s + f\nabla_1 s - d_L f \otimes s - f\nabla_0 s = f(\nabla_1 - \nabla_0)s$, hence $\nabla_1 - \nabla_0 \in \Omega^1_L(M, \mathrm{End}(E))$. $\square$

Thus, if we fix some ∇_0 in $\mathcal{A}(E,L)$, we may write

$$\mathcal{A}(E,L) = \{\nabla_0 + \alpha;\ \alpha \in \Omega^1_L(M,\mathrm{End}(E))\}\,. \tag{9}$$

This description will permit us to define various Sobolev completions of $\mathcal{A}(E,L)$.

Tensorial operations on vector bundles may be extended naturally to vector bundles with L-connections. More precisely, if E_1 and E_2 are two bundles with L-connections ∇^{E_1} and ∇^{E_2}, then there is a naturally induced connection $\nabla^{E_1\otimes E_2}$ on $E_1 \otimes E_2$ uniquely determined by the formula

$$\nabla^{E_1\otimes E_2}(s_1 \otimes s_2) = \nabla^{E_1} s_1 \otimes s_2 + s_1 \otimes \nabla^{E_2} s_2\,. \tag{10}$$

Consider a vector bundle E with an L-connection ∇^E. Then the dual bundle E^* of E has a natural connection ∇^{E^*} defined by the identity

$$\mathcal{L}^L_\xi\langle s,t\rangle = \langle \nabla^E_\xi s, t\rangle + \langle s, \nabla^{E^*}_\xi t\rangle \tag{11}$$

for all $\xi \in \mathfrak{X}_L(M)$, $s \in \Omega^0_L(M,E)$ and $t \in \Omega^0_L(M,E^*)$, where $\langle\cdot,\cdot\rangle\colon \Omega^0_L(M,E) \times \Omega^0_L(M,E^*) \to C^\infty(M,\mathbb{K})$ is the natural pairing. In particular, any L-connection ∇^E on a vector bundle E induces a connection $\nabla^{\mathrm{End}(E)}$ on $\mathrm{End}(E) \cong E^* \otimes E$ by the rule

$$(\nabla^{\mathrm{End}(E)}T)s = \nabla^E(Ts) - T(\nabla^E s) = [\nabla^E, T]s \tag{12}$$

for all $T \in \Omega^0_L(M,\mathrm{End}(E))$ and $s \in \Omega^0_L(M,E)$.

The graded vector spaces $\Omega^\bullet_L(M,\mathrm{End}(E))$ has a natural structure of a graded associative algebra via

$$\begin{aligned}&(\omega\wedge\tau)(\xi_1,\dots,\xi_{p+q}) \\ &\qquad = \frac{1}{p!\,q!}\sum_\sigma \mathrm{sign}(\sigma)\cdot\omega(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)})\,\tau(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)})\end{aligned} \tag{13}$$

and a natural structure of a graded Lie algebra via

$$
\begin{aligned}
(14)\quad & [\omega,\tau](\xi_1,\dots,\xi_{p+q}) \\
&= \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}(\sigma)\cdot\big[\omega(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)}),\tau(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)})\big]\,,
\end{aligned}
$$

where $\omega\in\Omega^p_L(M,\operatorname{End}(E))$, $\tau\in\Omega^q_L(M,\operatorname{End}(E))$ and $\xi_1,\dots,\xi_{p+q}\in\mathfrak{X}_L(M)$. In fact, the graded vector space $\Omega^\bullet_L(M,\operatorname{End}(E))$ is a differential graded Lie algebra[2] with the bracket $[\cdot,\cdot]$ and with the differential $d^{\nabla^{\operatorname{End}(E)}}$, since

$$
(15)\qquad d^{\nabla^{\operatorname{End}(E)}}[\omega,\tau]=\big[d^{\nabla^{\operatorname{End}(E)}}\omega,\tau\big]+(-1)^{\deg(\omega)}\big[\omega,d^{\nabla^{\operatorname{End}(E)}}\tau\big]\,.
$$

2. Geometry of L-connections

Consider a real or complex vector bundle $E\xrightarrow{p}M$. Then a *vector bundle endomorphism* of E is a vector bundle morphism $\varphi\colon E\to E$, i.e. a smooth mapping $\varphi\colon E\to E$ such that there exists a (smooth) mapping $\underline{\varphi}\colon M\to M$, the diagram

$$
\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[d, "p"'] & E \arrow[d, "p"] \\
M \arrow[r, "\underline{\varphi}"'] & M
\end{tikzcd}
$$

commutes and for each $x\in M$ the mapping $\varphi_x=\varphi_{|E_x}\colon E_x\to E_{\underline{\varphi}(x)}$ is $\mathbb{K}$-linear. In fact, the vector bundle endomorphism can be written as a pair $(\underline{\varphi},\varphi)$. We say that φ covers $\underline{\varphi}$.

A composition of vector bundle endomorphisms is defined in the obvious manner and a vector bundle endomorphism with an inverse vector bundle endomorphism is called a *vector bundle automorphism.* The set of all vector bundle automorphisms of E forms a group with a multiplication given by $(\underline{\varphi_1},\varphi_1)\cdot(\underline{\varphi_2},\varphi_2)=(\underline{\varphi_1}\circ\underline{\varphi_2},\varphi_1\circ\varphi_2)$ denoted by $\operatorname{Aut}(E)$. The subgroup $\operatorname{Gau}(E)$ of all vector bundle automorphisms of E covering identity on M is called *gauge group*, its elements are called *gauge transformations.*

Denote by $\operatorname{Diff}(M)$ the group of all diffeomorphisms of M (a multiplication in the group is given by $f_1\cdot f_2=f_1\circ f_2$), then we obtain an exact sequence

$$
(16)\qquad \{e\}\to\operatorname{Gau}(E)\xrightarrow{i}\operatorname{Aut}(E)\xrightarrow{p}\operatorname{Diff}(M)
$$

of groups, where $i\colon\operatorname{Gau}(E)\to\operatorname{Aut}(E)$ is an inclusion map only and $p\colon\operatorname{Aut}(E)\to\operatorname{Diff}(M)$ is a projection map, i.e. $p((\underline{\varphi},\varphi))=\underline{\varphi}$.

Consider now the natural left action of the gauge group $\operatorname{Gau}(E)$ on the space $\mathcal{A}(E,L)$ of L-connections defined by

$$
(17)\qquad (\varphi,\nabla)\mapsto\varphi\cdot\nabla\equiv\nabla^\varphi=(\mathrm{id}_{L^*}\otimes\varphi)\circ\nabla\circ\varphi^{-1},
$$

[2]Sometimes, in different contexts, it is called a differential graded Lie superalgebra. The notation is not unified.

or equivalently

$$(\varphi, \nabla_\xi) \mapsto (\varphi \cdot \nabla)_\xi \equiv (\nabla^\varphi)_\xi = \varphi \circ \nabla_\xi \circ \varphi^{-1} \tag{18}$$

for the covariant derivative, where $\xi \in \mathfrak{X}_L(M)$. Obviously, $\nabla^\varphi \colon \Omega^0_L(M,E) \to \Omega^1_L(M,E)$ is a $\mathbb{K}$-linear and satisfies

$$\begin{aligned}\nabla^\varphi(fs) &= (\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla\big(\varphi^{-1}(fs)\big) = (\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla\big(f\varphi^{-1}(s)\big)\\ &= (\mathrm{id}_{L^*} \otimes \varphi) \circ \big(d_L f \otimes \varphi^{-1}(s) + f\nabla\varphi^{-1}(s)\big)\\ &= d_L f \otimes s + f(\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla\varphi^{-1}(s)\\ &= d_L f \otimes s + f\nabla^\varphi s\end{aligned}$$

for every $f \in C^\infty(M,\mathbb{K})$ and $s \in \Omega^0_L(M,E)$, hence it is an L-connection.

Now we take up the question of reducible connections. Given an L-connection $\nabla \in \mathcal{A}(E,L)$, we shall consider the *isotropy subgroup*

$$\mathrm{Gau}(E)_\nabla = \{\varphi \in \mathrm{Gau}(E);\ \varphi \cdot \nabla = \nabla\} \tag{19}$$

of ∇. Every such subgroup contains the subgroup $\mathbb{K}^* \cdot \mathrm{id}_E$ of $\mathrm{Gau}(E)$.

Definition 3. An L-connection ∇ is called *irreducible* or *simple*, if $\mathrm{Gau}(E)_\nabla = \mathbb{K}^* \cdot \mathrm{id}_E$, otherwise ∇ is called *reducible*. We will denote the set of all irreducible L-connections by $\hat{\mathcal{A}}(E,L)$.

Lemma 3. *Let ∇ be an L-connection on a vector bundle E over a compact manifold M. Then the following are equivalent:*

(1) $\mathrm{Gau}(E)_\nabla = \mathbb{K}^* \cdot \mathrm{id}_E$,

(2) $\ker \nabla^{\mathrm{End}(E)} = \mathbb{K} \cdot \mathrm{id}_E$.

Proof. Consider $\varphi \in \mathrm{Gau}(E)$. Then the relation $\varphi \cdot \nabla = \nabla$ means that $(\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla \circ \varphi^{-1} = \nabla$, and this is equivalent to $(\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla = \nabla \circ \varphi$, or even $\nabla^{\mathrm{End}(E)}\varphi = 0$. Therefore $\varphi \in \mathrm{Gau}(E)_\nabla$ if and only if $\nabla^{\mathrm{End}(E)}\varphi = 0$ and $\varphi \in \mathrm{Gau}(E)$.

Suppose that $\varphi \in \mathrm{Gau}(E)_\nabla$. Then $\nabla^{\mathrm{End}(E)}\varphi = 0$ and, provided that $\ker \nabla^{\mathrm{End}(E)} = \mathbb{K} \cdot \mathrm{id}_E$, we get $\varphi = c \cdot \mathrm{id}_E$ for some $c \in \mathbb{K}^*$. Hence we obtain $\mathrm{Gau}(E)_\nabla \subset \mathbb{K}^* \cdot \mathrm{id}_E$ and because the converse inclusion is trivial, we have proved (2) $\Rightarrow$ (1).

To prove the opposite implication, we use the compactness of the manifold M. Suppose that $\varphi \in \ker \nabla^{\mathrm{End}(E)}$. Because M is compact, there exists $c \in \mathbb{K}$ (with $|c|$ sufficiently large) so that $c \cdot \mathrm{id}_E + \varphi \in \mathrm{Gau}(E)$. Moreover, $\nabla^{\mathrm{End}(E)}(c \cdot \mathrm{id}_E + \varphi) = 0$ and from the previous considerations, it follows that $c \cdot \mathrm{id}_E + \varphi \in \mathrm{Gau}(E)_\nabla$. Besides, if we suppose $\mathrm{Gau}(E)_\nabla = \mathbb{K}^* \cdot \mathrm{id}_E$, we obtain $\ker \nabla^{\mathrm{End}(E)} \subset \mathbb{K} \cdot \mathrm{id}_E$. The converse is trivial so the proof is finished. □

Remark. A trivial observation $\mathrm{Gau}(E)_{\nabla^\varphi} = \varphi \cdot \mathrm{Gau}(E)_\nabla \cdot \varphi^{-1}$ for all $\varphi \in \mathrm{Gau}(E)$ and $\nabla \in \mathcal{A}(E,L)$ implies immediately that $\hat{\mathcal{A}}(E,L)$ is invariant under the action of the gauge group $\mathrm{Gau}(E)$.

With these preliminaries, we can introduce the object of real interest for us. The *moduli space*

$$\mathcal{B}(E,L) = \mathcal{A}(E,L)/\mathrm{Gau}(E) \tag{20}$$

of L-connections on E is the set of all gauge equivalence classes of elements of $\mathcal{A}(E,L)$ modulo the action of $\mathrm{Gau}(E)$. Similarly, the orbit space of $\hat{\mathcal{A}}(E,L)$ under the action $\mathrm{Gau}(E)$ is the *moduli space*

$$\hat{\mathcal{B}}(E,L) = \hat{\mathcal{A}}(E,L)/\mathrm{Gau}(E) \tag{21}$$

of irreducible L-connections on E.

Given $\varphi \in \mathrm{Gau}(E)$, the transformed L-connection ∇^φ can be clearly written as

$$\nabla^\varphi = \nabla + (\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla^{\mathrm{End}(E)}\varphi^{-1} = \nabla - \nabla^{\mathrm{End}(E)}\varphi \circ \varphi^{-1}\,, \tag{22}$$

where the last equality follows by differentiating the identity $\varphi \circ \varphi^{-1} = \mathrm{id}_E$. Similarly, for the covariant derivative we get

$$(\nabla^\varphi)_\xi = \nabla_\xi + \varphi \circ \nabla_\xi^{\mathrm{End}(E)}\varphi^{-1} = \nabla_\xi - \nabla_\xi^{\mathrm{End}(E)}\varphi \circ \varphi^{-1}\,, \tag{23}$$

where $\xi \in \mathfrak{X}_L(M)$. More generally, if we fix some L-connection ∇_0 and express another L-connection ∇ as $\nabla = \nabla_0 + \alpha$, then

$$\nabla^\varphi = \nabla_0 + (\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla_0^{\mathrm{End}(E)}\varphi^{-1} + (\mathrm{id}_{L^*} \otimes \varphi) \circ \alpha \circ \varphi^{-1}\,, \tag{24}$$

hence, writing $\nabla^\varphi = \nabla_0 + \alpha^\varphi$, we have

$$\alpha^\varphi = (\mathrm{id}_{L^*} \otimes \varphi) \circ \nabla_0^{\mathrm{End}(E)}\varphi^{-1} + (\mathrm{id}_{L^*} \otimes \varphi) \circ \alpha \circ \varphi^{-1}\,. \tag{25}$$

Remark. If we define the *reduced gauge group* $\mathrm{Gau}(E)^{\mathrm{r}}$ by

$$\mathrm{Gau}(E)^{\mathrm{r}} = \mathrm{Gau}(E)/\mathbb{K}^* \cdot \mathrm{id}_E\,, \tag{26}$$

then the left action of $\mathrm{Gau}(E)$ on $\mathcal{A}(E,L)$ factors trough an action of the reduced gauge group $\mathrm{Gau}(E)^{\mathrm{r}}$, since $\mathbb{K}^* \cdot \mathrm{id}_E$ acts trivially on $\mathcal{A}(E,L)$. The set $\hat{\mathcal{A}}(E,L)$ of all irreducible L-connections is the maximal subset of $\mathcal{A}(E,L)$ with the property that the reduced gauge group $\mathrm{Gau}(E)^{\mathrm{r}}$ acts on it freely. Moreover, we can write

$$\mathcal{B}(E,L) = \mathcal{A}(E,L)/\mathrm{Gau}(E)^{\mathrm{r}} \tag{27}$$

for the moduli space of L-connections and

$$\hat{\mathcal{B}}(E,L) = \hat{\mathcal{A}}(E,L)/\mathrm{Gau}(E)^{\mathrm{r}} \tag{28}$$

for the moduli space of irreducible L-connections.

3. Moduli spaces

The moduli spaces $\mathcal{B}(E,L)$ and $\hat{\mathcal{B}}(E,L)$ introduced in the previous section were only sets of gauge equivalence classes of L-connections. We would like to define the structure of a smooth manifold on these sets.

From now on, we will assume that M is a connected compact manifold. To construct the space of gauge equivalence classes of L-connections it is the most convenient, and standard practice, to work within the framework of Sobolev spaces.

Let E be a real or complex vector bundle over a compact manifold M, L a real or complex Lie algebroid over M, g a Riemannian metric on M and h_E, respectively h_L, an Euclidean, or a Hermitian metric, on E, respectively L. These metrics induce metrics on E^*, $\mathrm{End}(E) \cong E^* \otimes E$ and furthermore on $\Lambda^k L^* \otimes \mathrm{End}(E)$. The metric g on M defines the density $\mathrm{vol}(g)$ of the Riemannian metric and the density $\mathrm{vol}(g)$ even induces a (regular) Borel measure μ_g on M.

Then for each nonnegative integer ℓ, we denote by $L^2_\ell(M,E)$ the vector space of equivalence classes of Borel measurable sections (a section ψ is Borel measurable, if $\psi^{-1}(U)$ is Borel measurable for any open subset $U \subset E$) whose weak derivatives of order $\leq \ell$ are square integrable. Thus, $L^2_\ell(M,E)$ are the Hilbert space completions of $\Gamma(M,E)$ with respect to the scalar product

$$\langle \psi, \varphi\rangle_\ell = \sum_{j=0}^{\ell} \langle \nabla^j \psi, \nabla^j \varphi\rangle \,, \tag{29}$$

where $\langle \nabla^j \psi, \nabla^j \varphi\rangle$ is computed using the scalar product on $T^*M^{\otimes j} \otimes E$. The Hilbert space $L^2_\ell(M, \Lambda^k L^* \otimes \mathrm{End}(E))$ will be denoted by $\Omega^k_L\big(M, \mathrm{End}(E)\big)_\ell$.

The space $\Omega^0_L\big(M, \mathrm{End}(E)\big)$ can be endowed with a scalar product given by

$$(f_1, f_2) = \int_M \mathrm{tr}(f_1 \circ f_2^*)\, \mathrm{d}\mu_g \tag{30}$$

for all $f_1, f_2 \in \Omega^0_L\big(M, \mathrm{End}(E)\big)$, where * denotes the adjoint with respect to h_E. If we define the space $\Omega^0_L\big(M, \mathrm{End}(E)\big)^0$ of traceless endomorphisms by

$$\Omega^0_L\big(M, \mathrm{End}(E)\big)^0 = \Big\{ f \in \Omega^0_L\big(M, \mathrm{End}(E)\big);\ \int_M \mathrm{tr}(f)\, \mathrm{d}\mu_g = 0 \Big\}\,, \tag{31}$$

then obviously

$$\Omega^0_L\big(M, \mathrm{End}(E)\big) = \Omega^0_L\big(M, \mathrm{End}(E)\big)^0 \oplus \mathbb{K} \cdot \mathrm{id}_E \tag{32}$$

and the decomposition is L^2-orthogonal with respect to the scalar product (30). The orthogonal projection p_{r} of $\Omega^0_L\big(M, \mathrm{End}(E)\big)$ onto $\Omega^0_L\big(M, \mathrm{End}(E)\big)^0$ is defined by the following formula

$$p_{\mathrm{r}}(f) = f - \frac{1}{n \cdot \mathrm{vol}(M)} \bigg(\int_M \mathrm{tr}(f)\, \mathrm{d}\mu_g \bigg) \cdot \mathrm{id}_E\,, \tag{33}$$

where $n = \mathrm{rk}\, E$ and $\mathrm{vol}(M)$ is the volume of the manifold M.

For a fixed L-connection ∇_0 in $\mathcal{A}(E,L)$, we define Sobolev completions of the space of L-connections, using (9), as

$$\mathcal{A}(E,L)_\ell = \big\{ \nabla_0 + \alpha;\ \alpha \in \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell \big\} \tag{34}$$

for $\ell \in \mathbb{N}_0$. Thus $\mathcal{A}(E,L)_\ell$ is an affine Hilbert space and therefore a Hilbert manifold whose tangent space at ∇ is

$$T_\nabla \mathcal{A}(E,L)_\ell = \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell\,. \tag{35}$$

Sobolev completions of the gauge group $\mathrm{Gau}(E)$ take a bit more work since it can not be identified with the space of sections of a vector bundle, nevertheless $\mathrm{Gau}(E) \subset \Omega^0_L\big(M, \mathrm{End}(E)\big)$. For $\ell > \frac{1}{2}\dim M$, the Sobolev space $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}$ consists of continuous sections[3] and using the Sobolev multiplication theorem, we get that

[3]Note that this is still true for $\ell + 1 > \frac{1}{2}\dim M$.

the product $\varphi \cdot \psi = \varphi \circ \psi$ in $\Omega^0_L\big(M, \mathrm{End}(E)\big)$ can be extended to a continuous bilinear map

$$\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \times \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \to \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \tag{36}$$

making $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}$ into a Banach algebra with unit id_E and the subset of all invertible elements forms an open subset. Accordingly, we define $\mathrm{Gau}(E)_{\ell+1}$ by

$$\begin{aligned}(37)\quad &\mathrm{Gau}(E)_{\ell+1} = \\ &\big\{\varphi \in \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1};\, \exists\, \psi \in \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1},\, \varphi\cdot\psi = \psi\cdot\varphi = \mathrm{id}_E\big\}\,.\end{aligned}$$

Because $\mathrm{Gau}(E)_{\ell+1}$ is an open subset in the Hilbert space, $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}$, $\mathrm{Gau}(E)_{\ell+1}$ is a Hilbert manifold. In fact, one can show that $\mathrm{Gau}(E)_{\ell+1}$ is a Hilbert–Lie group with Lie algebra

$$\mathfrak{gau}(E)_{\ell+1} = \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}\,. \tag{38}$$

The Lie bracket is given by an extension of the Lie bracket (14) on $\Omega^0_L\big(M, \mathrm{End}(E)\big)$ to a continuous map

$$\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \times \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \to \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \tag{39}$$

(with the help of Sobolev multiplication theorem in the range $\ell > \frac{1}{2}\dim M$). This Lie bracket agrees with the commutator bracket of the Banach algebra $\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}$.

As it follows from the formula (24), the action of $\mathrm{Gau}(E)$ on $\mathcal{A}(E,L)$ extends to an action of $\mathrm{Gau}(E)_{\ell+1}$ on $\mathcal{A}(E,L)_\ell$ via

$$\varphi\cdot\nabla = \varphi\cdot(\nabla_0+\alpha) = \nabla_0 + \varphi\cdot\big(d^{\nabla_0}\varphi^{-1}\big) + \varphi\cdot\alpha\cdot\varphi^{-1}\,, \tag{40}$$

where $\alpha \in \Omega^1_L(M, \mathrm{End}(E))_\ell$, $d^{\nabla_0}\colon \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \to \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$ is a continuous extension of the linear operator d^{∇_0} defined on $\Omega^0_L\big(M, \mathrm{End}(E)\big)$ and the multiplication $\cdot$ is an extension of (13) to a continuous map $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \times \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell \to \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$, respectively $\Omega^0_L\big(M, \mathrm{End}(E)\big)_\ell \times \Omega^1_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \to \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$, in the range $\ell > \frac{1}{2}\dim M$. In this range for ℓ, $\Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$ is a topological $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}$-bimodule.

It is easy to see that this action is a smooth map of Hilbert manifolds because, if we expres the action (40) in local charts, we obtain $(\varphi,\alpha) \mapsto \varphi\cdot(d^{\nabla_0}\varphi^{-1}) + \varphi\cdot\alpha\cdot\varphi^{-1}$ and this is a composition of smooth maps (the multiplication $\cdot$ and the mapping d^{∇_0} are smooth, because they are continuous linear maps). If $\nabla \in \mathcal{A}(E,L)_\ell$ is fixed, the map of $\mathrm{Gau}(E)_{\ell+1}$ to $\mathcal{A}(E,L)_\ell$ given by $\varphi \mapsto \varphi\cdot\nabla$ has a tangent map at id_E equal to

$$-\,d^{\nabla}\colon \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \to \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell\,, \tag{41}$$

where d^∇ is defined as

$$d^\nabla\gamma = d^{\nabla_0}\gamma + [\alpha,\gamma] \tag{42}$$

and $[\cdot,\cdot]\colon \Omega^1_L\big(M,\mathrm{End}(E)\big)_\ell \times \Omega^0_L\big(M,\mathrm{End}(E)\big)_{\ell+1} \to \Omega^1_L\big(M,\mathrm{End}(E)\big)_\ell$ is a continuous extension of (14) by means of Sobolev multiplication theorem in the range $\ell > \frac{1}{2}\dim M$.

Analogously to the smooth case, we define the notion of irreducibility of L-connection. A stabilizer $\mathrm{Gau}(E)^{\nabla}_{\ell+1}$ of any Sobolev L-connection ∇ contains the subgroup $\mathbb{K}^*\cdot \mathrm{id}_E$ of $\mathrm{Gau}(E)_{\ell+1}$. When $\mathrm{Gau}(E)^{\nabla}_{\ell+1} = \mathbb{K}^*\cdot \mathrm{id}_E$, we will say that the connection ∇ is *irreducible*; otherwise, ∇ is *reducible*. We can prove the following characterization of irreducibility.

Lemma 4. *Let $\nabla \in \mathcal{A}(E,L)_\ell$ be a Sobolev L-connection. Then the following are equivalent:*

(1) $\mathrm{Gau}(E)^{\nabla}_{\ell+1} = \mathbb{K}^*\cdot \mathrm{id}_E$,
(2) $\ker d^{\nabla} = \mathbb{K}\cdot \mathrm{id}_E$.

Proof. The proof goes along the same line as in Lemma 3, so we shall skip it. □

We will denote by $\hat{\mathcal{A}}(E,L)_\ell$ the subset of $\mathcal{A}(E,L)_\ell$ consisting of irreducible L-connections. It follows from $\mathrm{Gau}(E)^{\nabla^\varphi}_{\ell+1} = \varphi\cdot \mathrm{Gau}(E)^{\nabla}_{\ell+1}\cdot\varphi^{-1}$ that notion of irreducibility of a connection is invariant under gauge transformations.

Following an analogy with (20) and (21), we define the *moduli space*

$$\mathcal{B}(E,L)_\ell = \mathcal{A}(E,L)_\ell / \mathrm{Gau}(E)_{\ell+1} \tag{43}$$

of L-connections on E, and similarly the *moduli space*

$$\hat{\mathcal{B}}(E,L)_\ell = \hat{\mathcal{A}}(E,L)_\ell / \mathrm{Gau}(E)_{\ell+1} \tag{44}$$

of irreducible L-connections on E. Each of these is assumed to have the quotient topology and in the following, we shall show that if L is a transitive Lie algebroid on M, then $\hat{\mathcal{B}}(E,L)_\ell$ is open in $\mathcal{B}(E,L)_\ell$.

We use notation

$$p\colon \mathcal{A}(E,L)_\ell \to \mathcal{B}(E,L)_\ell \quad \text{resp.} \quad \hat{p}\colon \hat{\mathcal{A}}(E,L)_\ell \to \hat{\mathcal{B}}(E,L)_\ell \tag{45}$$

for the canonical projections.

Remark. From now on we will assume that L is a transitive Lie algebroid, i.e. $a\colon L\to TM$, resp. $a\colon L \to TM_{\mathbb{C}}$ is surjective.

For $\alpha \in \Omega^1_L\big(M,\mathrm{End}(E)\big)$, the zero order operator $\mathrm{ad}(\alpha)^*\colon \Omega^1_L\big(M,\mathrm{End}(E)\big) \to \Omega^0_L\big(M,\mathrm{End}(E)\big)$, defined as a formal adjoint of $\mathrm{ad}(\alpha)\colon \Omega^0_L\big(M,\mathrm{End}(E)\big) \to \Omega^1_L(M,\mathrm{End}(E))$, $\mathrm{ad}(\alpha)(\varphi) = [\alpha,\varphi]$, with respect to the Hermitian metric on $\mathrm{End}(E)$ given by $(f_1,f_2)\mapsto \mathrm{tr}(f_1\circ f_2^*)$, yields a map $m\colon \Omega^1_L\big(M,\mathrm{End}(E)\big)\times \Omega^1_L\big(M,\mathrm{End}(E)\big) \to \Omega^0_L\big(M,\mathrm{End}(E)\big)$, $(\alpha,\psi)\mapsto \mathrm{ad}(\alpha)^*(\psi)$, which is $C^\infty(M,\mathbb{K})$-sesquilinear. This mapping can be extended by the Sobolev multiplication theorem to a continuous sesquilinear-linear map

$$\Omega^1_L\big(M,\mathrm{End}(E)\big)_\ell \times \Omega^1\big(M,\mathrm{End}(E)\big)_\ell \to \Omega^0_L\big(M,\mathrm{End}(E)\big)_\ell$$

hence the map $\mathrm{ad}(\alpha)^*\colon \Omega^1_L\big(M,\mathrm{End}(E)\big)_\ell \to \Omega^0_L\big(M,\mathrm{End}(E)\big)_\ell$ for every $\alpha \in \Omega^1_L\big(M,\mathrm{End}(E)\big)_\ell$ is continuous.

Then we may write $d^\nabla = d^{\nabla_0} + \mathrm{ad}(\alpha) \circ i$, where $i \colon \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \to \Omega^0\big(M, \mathrm{End}(E)\big)_\ell$ is a compact embedding. Furthermore, we define

$$(d^\nabla)^* \colon \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell \to \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell-1}$$

as $(d^\nabla)^* = (d^{\nabla_0})^* + i \circ \mathrm{ad}(\alpha)^*$, where $i \colon \Omega^0_L\big(M, \mathrm{End}(E)\big)_\ell \to \Omega^0\big(M, \mathrm{End}(E)\big)_{\ell-1}$ is a compact embedding and $(d^{\nabla_0})^*$ is a continuous extension of formal adjoint of d^{∇_0} with respect to the Hermitian metric on $\mathrm{End}(E)$.

For any connection $\nabla = \nabla_0 + \alpha$, we will denote by d_α the covariant derivative d^∇ and d^*_α the operator $(d^\nabla)^*$.

Lemma 5. *For any $\nabla \in \mathcal{A}(E, L)_\ell$, the operator*

$$\big(d^\nabla\big)^* \circ d^\nabla \colon \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} \to \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell-1} \tag{46}$$

is a Fredholm operator for all $\ell > \frac{1}{2} \dim M$.

Proof. For $\nabla = \nabla_0 + \alpha$, we may write $\Delta_\alpha = d^*_\alpha \circ d_\alpha = (d^*_0 + i \circ \mathrm{ad}(\alpha)^*) \circ (d_0 + \mathrm{ad}(\alpha) \circ i)$. Because $\mathrm{ad}(\alpha) \circ i$ and $i \circ \mathrm{ad}(\alpha)^*$ are compact operators, $i \circ \mathrm{ad}(\alpha)^* \circ d_0 + d^*_0 \circ \mathrm{ad}(\alpha) \circ i + i \circ \mathrm{ad}(\alpha)^* \circ \mathrm{ad}(\alpha) \circ i$ is compact operator, as well. So we need only to show that $d^*_0 \circ d_0$ is a Fredholm operator. It is enough to show that $d^*_0 \circ d_0 \colon \Omega^0_L(M, \mathrm{End}(E)) \to \Omega^0_L\big(M, \mathrm{End}(E)\big)$ is an elliptic operator, i.e. that the principal symbol $\sigma_2(d^*_0 \circ d_0)(\xi_x) \colon \mathrm{End}(E)_x \to \mathrm{End}(E)_x$ is an isomorphism for all $x \in M$ and $\xi_x \in T^*_x M \setminus \{0\}$. Obviously, $\sigma_2(d^*_0 \circ d_0)(\xi_x) = \sigma_1(d^*_0)(\xi_x) \circ \sigma_1(d_0)(\xi_x)$ and this is an isomorphism if and only if $\sigma_1(d_0)(\xi_x)$ is an isomorphism. But $\sigma_1(d_0)(\xi_x) = a^*(\xi_x) \otimes$, i.e. the symbol is the tensor multiplication by $a^*(\xi_x)$, hence it is an isomorphism, if $a^*(\xi_x) \neq 0$. Thus $\sigma_2(d^*_0 \circ d_0)$ is an isomorphism for all $x \in M$ and $\xi_x \in T^*_x M \setminus \{0\}$ if a^* is injective, or equivalently if a is surjective. This is true because L is a transitive Lie algebroid. $\square$

Lemma 6. *For any $\nabla \in \mathcal{A}(E, L)_\ell$, we have an L^2-orthogonal decomposition*

$$\Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell = \mathrm{im}\, d^\nabla \oplus \ker \big(d^\nabla\big)^* \tag{47}$$

for all $\ell > \frac{1}{2} \dim M$.

Proof. Since Δ_α is a Fredholm operator, $\dim \ker \Delta_\alpha < +\infty$ and $\mathrm{im}\, \Delta_\alpha$ is a closed subspace in $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell-1}$. Therefore $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1} = \ker \Delta_\alpha \oplus (\ker \Delta_\alpha)^\perp$ is an L^2-orthogonal (not $L^2_{\ell+1}$) decomposition into closed subspaces in $\Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}$. Moreover, $\mathrm{im}\, \Delta_\alpha$ is a closed subspace, thus $\Delta_{\alpha|(\ker \Delta_\alpha)^\perp} \colon (\ker \Delta_\alpha)^\perp \to \mathrm{im}\, \Delta_\alpha$ is a bijective continuous linear operator between Banach spaces and, using the Banach's Open Mapping Theorem, it follows that $G_\alpha = (\Delta_{\alpha|(\ker \Delta_\alpha)^\perp})^{-1}$ is a continuous linear operator. Further, if $X \subset \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$ denotes the closed subspace given by $X = (d^*_\alpha)^{-1}(\mathrm{im}\, \Delta_\alpha)$, then $\mathrm{id}_X - d_\alpha G_\alpha d^*_{\alpha|X}$ is a continuous linear operator and moreover $\mathrm{im}\, d_\alpha = \ker(\mathrm{id}_X - d_\alpha G_\alpha d^*_{\alpha|X})$. Hence $\mathrm{im}\, d_\alpha$ is a closed subspace in $\Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$.

For that reason we can write $\Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell = \mathrm{im}\, d_\alpha \oplus (\mathrm{im}\, d_\alpha)^\perp$ and this decomposition is L^2-orthogonal. On the other hand, for $\varphi \in \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}$

and $\psi \in \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$, we have $(d_\alpha\varphi, \psi) = (\varphi, d^*_\alpha\psi)$, and it follows that $(\mathrm{im}\, d_\alpha)^\perp = \ker d^*_\alpha$. □

Lemma 7. *$\hat{\mathcal{A}}(E, L)_\ell$ is an open subset in $\mathcal{A}(E, L)_\ell$ for all $\ell > \frac{1}{2}\dim M$.*

Proof. It follows from Lemma 5 that Δ_α is a Fredholm operator. Moreover, the map

$$\mathcal{A}(E, L)_\ell \to \mathcal{F}\big(\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}, \Omega^0_L(M, \mathrm{End}(E))_{\ell-1}\big)\,,$$

$$\nabla_0 + \alpha \mapsto \Delta_\alpha\,,$$

where $\mathcal{F}\big(\Omega^0_L(M, \mathrm{End}(E))_{\ell+1}, \Omega^0_L(M, \mathrm{End}(E))_{\ell-1}\big)$ denotes the set of all Fredholm operators between corresponding Hilbert spaces, is a continuous family of Fredholm operators. Hence the map

$$\nabla_0 + \alpha \mapsto \dim\ker \Delta_\alpha$$

is an upper semicontinuous from $\mathcal{A}(E, L)_\ell$ to $\mathbb{R}$, see [1]. Because $\ker d_\alpha = \ker \Delta_\alpha$ and $\dim\ker d_\alpha \geq 1$, the upper semicontinuity implies that $\hat{\mathcal{A}}(E, L)_\ell$ is an open subset. □

Remark. We have just proved that $\hat{\mathcal{A}}(E, L)_\ell$ is an open subset in $\mathcal{A}(E, L)_\ell$. Because $\mathcal{B}(E, L)_\ell$ is assumed to have the quotient topology and $p^{-1}(\hat{\mathcal{B}}(E, L)_\ell) = \hat{\mathcal{A}}(E, L)_\ell$, we get that $\hat{\mathcal{B}}(E, L)_\ell$ is open.

Now, for each $\nabla \in \mathcal{A}(E, L)_\ell$ and for each $\varepsilon > 0$, let us consider the Hilbert submanifold

$$\mathcal{O}_{\nabla,\varepsilon} = \big\{\nabla + \alpha;\ \alpha \in \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell,\ (d^\nabla)^*\alpha = 0,\ \|\alpha\|_\ell < \varepsilon\big\} \tag{48}$$

of the Hilbert manifold $\mathcal{A}(E, L)_\ell$. It clearly satisfies

$$T_\nabla(\mathcal{O}_{\nabla,\varepsilon}) = \ker(d^\nabla)^*. \tag{49}$$

First note that if $\nabla \in \hat{\mathcal{A}}(E, L)_\ell$, we may take ε small enough to ensure $\mathcal{O}_{\nabla,\varepsilon} \subset \hat{\mathcal{A}}(E, L)_\ell$, because $\hat{\mathcal{A}}(E, L)_\ell$ is open in $\mathcal{A}(E, L)_\ell$. Next, we define the *reduced gauge group* $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$ by

$$\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} = {\mathrm{Gau}(E)_{\ell+1}}/{\mathbb{K}^* \cdot \mathrm{id}_E}. \tag{50}$$

Because $\mathbb{K}^* \cdot \mathrm{id}_E$ is a normal Hilbert–Lie subgroup of $\mathrm{Gau}(E)_{\ell+1}$, Theorem 1 below implies that the reduced gauge group is a Hilbert–Lie group with Lie algebra

$$\mathfrak{gau}(E)^{\mathrm{r}}_{\ell+1} = \Omega^0_L\big(M, \mathrm{End}(E)\big)^0_{\ell+1}\,, \tag{51}$$

where the Lie bracket descends from the one on $\mathfrak{gau}(E)_{\ell+1}$. Moreover, if

$$q\colon\ \mathrm{Gau}(E)_{\ell+1} \to \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} = {\mathrm{Gau}(E)_{\ell+1}}/{\mathbb{K}^* \cdot \mathrm{id}_E} \tag{52}$$

denotes the canonical projection, then q is a smooth $\mathrm{Gau}(E)_{\ell+1}$-equivariant map and any map $f\colon\ \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} \to X$ is smooth if and only if $f \circ q\colon\ \mathrm{Gau}(E)_{\ell+1} \to X$ is smooth, where X is a smooth Banach manifold.

Theorem 1. *Let G be a Banach–Lie group over $\mathbb{K}$ with Lie algebra $\mathfrak{g}$ and suppose that N is a normal Banach–Lie subgroup over $\mathbb{K}$ of G with Lie algebra $\mathfrak{n}$. Then G/N is a Banach–Lie group with Lie algebra $\mathfrak{g}/\mathfrak{n}$ in a unique way such that the quotient map $q\colon G \to G/N$ is a smooth map. Moreover, for any Banach manifold X a map $f\colon G/N \to X$ is smooth if and only if $f \circ q$ is smooth.*

Proof. See [5], [4] and [3]. □

Theorem 2. *$\hat{\mathcal{B}}(E,L)_\ell$ is a locally Hausdorff Hilbert manifold and $\hat{p}\colon \hat{\mathcal{A}}(E,L)_\ell \to \hat{\mathcal{B}}(E,L)_\ell$ is a principal $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$-bundle for $\ell > \frac{1}{2}\dim M$.*

Proof. Consider the smooth map of Hilbert manifolds

$$\Psi_\nabla\colon\ \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} \times \mathcal{O}_{\nabla,\varepsilon} \to \hat{\mathcal{A}}(E,L)_\ell\,, \tag{53}$$

$$\Psi_\nabla(\varphi, \nabla+\alpha) = \varphi\cdot(\nabla+\alpha)\,, \tag{54}$$

then the tangent map at (id_E,∇) equals to

$$T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla\colon \Omega^0_L\big(M,\mathrm{End}(E)\big)^0_{\ell+1} \oplus \ker(d^\nabla)^* \to \Omega^1_L\big(M,\mathrm{End}(E)\big)_\ell\,, \tag{55}$$

$$\big(T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla\big)(\gamma,\beta) = -d^\nabla\gamma + \beta\,. \tag{56}$$

It follows from the Lemma 6 that $T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla$ is surjective. Moreover, because ∇ is assumed to be irreducible, $T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla$ is injective. Hence by the Banach's Open Mapping Theorem $T_{(\mathrm{id}_E,\nabla)}\Psi_\nabla$ is an isomorphism. Therefore the inverse function theorem for Banach manifolds implies that Ψ_∇ is a local diffeomorphism near (id_E,∇). Consequently, there is an open neighborhood $\mathcal{U}_\nabla$ of ∇ in $\hat{\mathcal{A}}(E,L)_\ell$ and an open neighborhood $\mathcal{N}_{\mathrm{id}_E}$ of id_E in $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$ such that

$$\Psi_\nabla\colon \mathcal{N}_{\mathrm{id}_E} \times \mathcal{O}_{\nabla,\varepsilon} \to \mathcal{U}_\nabla \tag{57}$$

is a diffeomorphism for sufficiently small $\varepsilon > 0$.

Next we show that, for ε small enough, the map $p_{\nabla,\varepsilon} = p_{|\mathcal{O}_{\nabla,\varepsilon}}\colon \mathcal{O}_{\nabla,\varepsilon} \to \hat{\mathcal{B}}(E,L)_\ell$ is injective. We have to show that if for two elements $\nabla+\alpha_1, \nabla+\alpha_2 \in \mathcal{O}_{\nabla,\varepsilon}$ there exists a gauge transformation $\varphi \in \mathrm{Gau}(E)_{\ell+1}$ such that

$$\varphi\cdot(\nabla+\alpha_1) = \nabla+\alpha_2\,, \tag{58}$$

then $\alpha_1 = \alpha_2$. First observe that (58) is equivalent to

$$d^\nabla\varphi = \varphi\cdot\alpha_1 - \alpha_2\cdot\varphi\,. \tag{59}$$

Next, because $\Omega^0_L\big(M,\mathrm{End}(E)\big)_{\ell+1} = \ker d^\nabla \oplus (\ker d^\nabla)^\perp$ is an L^2-orthogonal decomposition, we can write $\varphi = c\cdot\mathrm{id}_E + \varphi_0$, where $c \in \mathbb{K}$ and $\varphi_0 \in (\ker d^\nabla)^\perp$. Furthermore, $\operatorname{im} d^\nabla$ is a closed subspace in $\Omega^1_L\big(M,\mathrm{End}(E)\big)_\ell$, hence we get by the Banach's Open Mapping Theorem that

$$d^\nabla\colon (\ker d^\nabla)^\perp \to \operatorname{im} d^\nabla \tag{60}$$

is an isomorphism of Hilbert spaces. It means that it is a lower bounded operator, i.e. there exists a positive constant c_1 such that

$$\|d^\nabla\psi\|_\ell \geq c_1\|\psi\|_{\ell+1} \tag{61}$$

for all $\psi \in (\ker d^\nabla)^\perp$. Thus we may write

$$c_1 \|\varphi_0\|_{\ell+1} \leq \|d^\nabla \varphi_0\|_\ell = \\ \|d^\nabla \varphi\|_\ell = \|\varphi \cdot \alpha_1 - \alpha_2 \cdot \varphi\|_\ell \leq 2\,\tilde{c} \cdot \varepsilon \cdot (|c| \cdot \|\mathrm{id}_E\|_{\ell+1} + \|\varphi_0\|_{\ell+1})\,, \tag{62}$$

where we used the fact $\|\psi \cdot \alpha\|_{\ell+1} \leq \tilde{c} \cdot \|\psi\|_{\ell+1} \|\alpha\|_\ell$, respectively $\||\alpha \cdot \psi\|_{\ell+1} \leq \tilde{c} \cdot \|\alpha\|_\ell \|\psi\|_{\ell+1}$, for all $\psi \in \Omega^0_L\big(M, \mathrm{End}(E)\big)_{\ell+1}$ and $\alpha \in \Omega^1_L\big(M, \mathrm{End}(E)\big)_\ell$. It implies that

$$\|\varphi_0\|_{\ell+1} \leq \frac{2\,\tilde{c} \cdot |c| \cdot \varepsilon}{c_1 - 2\,\tilde{c} \cdot \varepsilon} \|\mathrm{id}_E\|_{\ell+1} \tag{63}$$

for $\varepsilon < \frac{c_1}{2\tilde{c}}$. If $c = 0$, then we obtain immediately $\|\varphi_0\|_{\ell+1} = 0$, thus $\varphi = 0$ and this is a contradiction. Because $c \neq 0$, we get

$$\|c^{-1} \cdot \varphi - \mathrm{id}_E\|_{\ell+1} = \frac{1}{|c|} \|\varphi_0\|_{\ell+1} \leq \frac{2\,\tilde{c} \cdot \varepsilon}{c_1 - 2\,\tilde{c} \cdot \varepsilon} \|\mathrm{id}_E\|_{\ell+1}\,, \tag{64}$$

hence for ε small enough is φ near id_E in $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$, i.e. $\varphi \in \mathcal{N}_{\mathrm{id}_E}$. And if we use that Ψ_∇ is injective, we obtain $\alpha_1 = \alpha_2$.

Let $\mathcal{U}_{\nabla,\varepsilon} = p_{\nabla,\varepsilon}(\mathcal{O}_{\nabla,\varepsilon})$, then $\mathcal{U}_{\nabla,\varepsilon}$ is open in $\hat{\mathfrak{B}}(E,L)_\ell$. It is easy to see that $p^{-1}(\mathcal{U}_{\nabla,\varepsilon}) = \lambda(\mathrm{Gau}(E)_{\ell+1} \times \mathcal{O}_{\nabla,\varepsilon})$, where $\lambda\colon\ \mathrm{Gau}(E)_{\ell+1} \times \hat{\mathcal{A}}(E,L)_\ell \to \hat{\mathcal{A}}(E,L)_\ell$ is the left action, is an open subset in $\hat{\mathcal{A}}(E,L)_\ell$. The map

$$\Psi_\nabla\colon\ \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} \times \mathcal{O}_{\nabla,\varepsilon} \to p^{-1}_{\nabla,\varepsilon}(\mathcal{U}_{\nabla,\varepsilon})\,, \tag{65}$$

$$\Psi_\nabla(\varphi, \nabla + \alpha) = \varphi \cdot (\nabla + \alpha) \tag{66}$$

is surjective because $p^{-1}(\mathcal{U}_{\nabla,\varepsilon}) = \lambda(\mathrm{Gau}(E)_{\ell+1} \times \mathcal{O}_{\nabla,\varepsilon})$, the injectivity follows from the previous considerations and from the fact that the action of $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$ on $\hat{\mathcal{A}}(E,L)_\ell$ is free. We will show that it is in fact a diffeomorphism of Hilbert manifolds.

For any $\varphi \in \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$, we find an open neighborhood $\mathcal{W}$ of φ such that $\Psi_{\nabla|L_{\varphi^{-1}}(\mathcal{W}) \times \mathcal{O}_{\nabla,\varepsilon}}$ is a diffeomorphism, where $L_{\varphi^{-1}}$ is the left translation by φ^{-1} in $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$. In particular, we can take $\mathcal{W} = L_\varphi(\mathcal{N}_{\mathrm{id}_E})$. Further, if $\tilde{\lambda}_\varphi$ denotes the left multiplication by φ in $\hat{\mathcal{A}}(E,L)_\ell$, then we have

$$\Psi_{\nabla|\mathcal{W} \times \mathcal{O}_{\nabla,\varepsilon}} = \tilde{\lambda}_\varphi \circ \Psi_{\nabla|\mathcal{N}_{\mathrm{id}_E} \times \mathcal{O}_{\nabla,\varepsilon}} \circ (L_{\varphi^{-1}} \times \mathrm{id}_{\hat{\mathcal{A}}(E,L)_\ell})_{|\mathcal{W} \times \mathcal{O}_{\nabla,\varepsilon}}\,, \tag{67}$$

which is a diffeomorphism.

Now to show that $\hat{p}\colon \hat{\mathcal{A}}(E,L)_\ell \to \hat{\mathfrak{B}}(E,L)_\ell$ is a principal $\mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$-bundle over a Hilbert manifold, we only need to glue together the local charts $\sigma_\nabla : \mathcal{U}_{\nabla,\varepsilon} \to \mathcal{O}_{\nabla,\varepsilon}$, $\sigma_\nabla = p^{-1}_{\nabla,\varepsilon}$. Consider the smooth map

$$g_\nabla = \mathrm{pr} \circ \Psi_\nabla^{-1} \colon p^{-1}_{\nabla,\varepsilon}(\mathcal{U}_{\nabla,\varepsilon}) \to \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}\,, \tag{68}$$

where $\mathrm{pr}\colon\ \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1} \times \mathcal{O}_{\nabla,\varepsilon} \to \mathrm{Gau}(E)^{\mathrm{r}}_{\ell+1}$ is the projection. Then for any $\nabla' \in \hat{\mathcal{A}}(E,L)_\ell$ with $p(\nabla') \in \mathcal{U}_{\nabla,\varepsilon}$ we have

$$\sigma_\nabla\big(p(\nabla')\big) = \big(g_\nabla(\nabla')\big)^{-1} \cdot \nabla'\,. \tag{69}$$

Hence it is easy to see that we obtain

$$\sigma_\nabla \circ \sigma_{\nabla'}^{-1}(\nabla' + \alpha) = \sigma_\nabla\big(p(\nabla' + \alpha)\big) = \big(g_\nabla(\nabla' + \alpha)\big)^{-1} \cdot (\nabla' + \alpha) \tag{70}$$

over $\sigma_{\nabla'}(\mathcal{U}_{\nabla',\varepsilon'} \cap \mathcal{U}_{\nabla,\varepsilon})$, and this is clearly smooth in α. □

In this paper we have considered the spaces of irreducible Lie algebroid connections for a transitive Lie algebroid over a connected compact manifold. The next step would be to investigate the moduli spaces of flat L-connections in the subsequent work. These moduli spaces are more interesting from the viewpoint of applications in physics, especially in mirror symmetry, as was already mentioned in the introduction.

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