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EXTERIOR DIFFERENTIAL SYSTEMS, LIE ALGEBRA COHOMOLOGY, AND THE RIGIDITY OF HOMOGENOUS VARIETIES

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Abstract. These are expository notes from the 2008 Srní Winter School. They have two purposes: (1) to give a quick introduction to exterior differential systems (EDS), which is a collection of techniques for determining local existence to systems of partial differential equations, and (2) to give an exposition of recent work (joint with C. Robles) on the study of the Fubini-Griffiths-Harris rigidity of rational homogeneous varieties, which also involves an advance in the EDS technology.

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1. Introduction

Let G be a complex semi-simple Lie (or algebraic) group, and let $V = V_\lambda$ be an irreducible G-module. The homogeneous variety $G/P = G \cdot [v_\lambda] \subset \mathbb{P}V$ is the orbit of a highest weight line.

For example, let W be a complex vector space, $V = \Lambda^k W$ and let $G = \mathrm{SL}(W)$, then $G/P = G(k, W) \subset \mathbb{P}(\Lambda^k W)$ is the *Grassmannian* of k-planes through the origin in W in its Plücker embedding.

A long term program with my collaborators Laurent Manivel, Colleen Robles and Jerzy Weyman is to study relations between the projective geometry of $G/P \subset \mathbb{P}V$, especially its local differential geometry, and the representation theory of G. More than just the geometry of G/P, we are interested in the geometry of its auxiliary varieties, for example the *tangential variety* $\tau(G/P) \subset \mathbb{P}V$, which is the union of all points on all embedded tangent lines to G/P, and the *r-th secant variety* of G/P, $\sigma_r(G/P) \subset \mathbb{P}V$, which is the Zariski closure of all points on all secant $\mathbb{P}^{r-1}$'s to G/P. The auxilary varieties are all G-varieties, i.e., preserved under the action of G, and thus one can study their ideals, coordinate rings, etc. as G-modules.

1.1. Overview.

These notes are focused on the local projective differential geometry of homogeneously embedded rational homogeneous varieties $G/P \subset \mathbb{P}V$. Specifically, they address the question how much of the local geometry is needed to recover G/P. We begin by describing many examples of rational homogeneous varieties in §1.2. The main question we deal with is rigidity, but before discussing rigidity questions, we give descriptions of related projects in §2 to give context to this work. The rigidity results and questions are described in §3. In §4 and §5 we give a crash course on *exterior differential systems* (EDS). Roughly speaking, EDS is a collection of techniques for determining the space of local solutions to systems of partial differential equations. The techniques usually involve extensive computations that can be simplified by exploiting group actions when such are present, as with the rigidity questions that will be the focus of this paper. In §6 we describe moving frames for submanifolds of projective space and a set of "rigidity" EDS that are natural from the point of view of projective differential geometry. We also describe flexibility results obtained using standard EDS techniques. A different method for resolving certain EDS associated to determining the rigidity of compact Hermitian symmetric spaces (CHSS) was introduced by Hwang and Yamaguchi in [20] that avoided lengthy calculations by reducing the proof to establishing the vanishing of certain Lie algebra cohomology groups. At first, it appeared that their methods would not extend beyond the CHSS cases, but the machinery was finally extended in [24]. *This extension is the central point of these lectures.* Several problems had to be overcome to enable the extension - the problems and their solutions are discussed in detail in in §9. The first problem is that the EDS natural for geometry is not natural for representation theory, once one moves beyond CHSS. This problem is (partially) resolved in §9.1 with the introduction of the systems (I_p, J_p) which are natural for representation theory. The next problem is that even these natural systems do not lead one to Lie algebra cohomology, except in the case of CHSS. However a refined version of the (I_p, J_p) systems, the *filtered systems* (I^f_p, Ω) do. This is explained in §9.2, which then leads to our main theorem, Theorem 9.10. Before discussing these systems, we describe and compare, for $G/P \subset \mathbb{P}V$ the filtration of V induced by the osculating sequence and a filtration induced by the Lie algebra in §7, and briefly review Lie algebra cohomology in §8.

1.2. **Examples of rational homogeneous varieties.**

1.2.1. *Generalized cominuscule varieties.* The simplest rational homogeneous varieties are the *generalized cominuscule varieties*, which are the homogeneously embedded compact Hermitian symmetric spaces. In addition to the Grassmannians mentioned above, the *cominuscule varieties*, which are the irreducible CHSS in their minimal homogeneous embeddings, are

- the *Lagrangian Grassmannians* $G_\omega(n, W) = C_n/P_n \subset \mathbb{P}(\Lambda^n W/\omega \wedge \Lambda^{n-2} W)$, where W is a $2n$ dimensional vector space equipped with a symplectic form $\omega \in \Lambda^2 W^*$, C_n is the group preserving the form, and $G_\omega(n, W) \subset G(n, W)$ are the n-planes on which ω restricts to be zero. (Note that we may use ω to identify W with W^* so $\omega \wedge \Lambda^{n-2} W$ makes sense.)
- the *Spinor varieties* $\mathbb{S}_n = D_n/P_n \simeq D_n/P_{n-1}$ which are also isotropic Grassmannians, only for a symmetric quadratic form, where W again has dimension $2n$. Their minimal homogeneous embedding is in a space smaller than $\mathbb{P}(\Lambda^n W)$.
- the *quadric hypersurfaces* $Q^{n-1} = G_Q(1, W) \subset \mathbb{P}W$, (which are B_m/P_1 and D_m/P_1 depending if $n = 2m + 1$ or $n = 2m$)

– the *Cayley plane* $\mathbb{OP}^2 = E_6/P_6 \simeq E_6/P_1 \subset \mathbb{P}\mathcal{J}_3(\mathbb{O})$ which are the octonionic lines in $\mathbb{O}^2$ embedded as the rank one elements of the exceptional Jordan algebra $\mathcal{J}_3(\mathbb{O})$, see, e.g., [30] for details.
– the *Freudenthal variety* $E_7/P_7 \subset \mathbb{P}^{55}$ which may be thought of as an octonionic Lagrangian Grassmanian $G_w(\mathbb{O}^3, \mathbb{O}^6)$, see [30].

1.2.2. *Products of homogeneous varieties.* An elementary, but important generalized cominuscule variety is the *Segre variety.* Let V, W be vector spaces, the Segre variety $\mathrm{Seg}(\mathbb{P}V \times \mathbb{P}W) \subset \mathbb{P}(V \otimes W)$ as an abstract variety is simply the product of two projective spaces. It is embedded as the set of rank one elements of $V \otimes W$. In general, if $G/P \subset \mathbb{P}V$ and $G'/P' \subset \mathbb{P}V'$, we may form the product $\mathrm{Seg}(G/P \times G'/P') \subset \mathbb{P}(V \otimes V')$, which is of course a subvariety of $\mathrm{Seg}(\mathbb{P}V \times \mathbb{P}V')$.

1.2.3. *Veronese re-embeddings of homogeneous varieties.* Considering S^dV as the space of homogeneous polynomials of degree d on V^*, we can consider the variety of d-th powers inside $\mathbb{P}(S^dV)$, this is isomorphic to $\mathbb{P}V$ via the map $v_d : \mathbb{P}V \to \mathbb{P}S^dV$, $[x] \mapsto [x^d]$, called the *Veronese embedding.* If $X \subset \mathbb{P}V$ is a subvariety we can consider $v_d(X)$. Its linear span $\langle v_d(X)\rangle \subset S^dV$ has the geometric interpretation of the annihilator of $I_d(X) \subset S^dV^*$, the ideal of X in degree d. In particular, if $X = G/P \subset \mathbb{P}V_\lambda$ is homogeneous, then $\langle v_d(G/P)\rangle = V_{d\lambda}$, the d-th *Cartan power* of V_λ.

1.2.4. *Generalized flag varieties.* Given two Grassmannians $G(k, V)$ and $G(\ell, V)$ with say $k < \ell$, we may form the incidence variety $\mathrm{Flag}_{k,\ell}(V) = \{(E, F) \in G(k, V) \times G(\ell, V) \mid E \subset F\}$. Of course $\mathrm{Flag}_{k,\ell}(V) \subset \mathbb{P}(\Lambda^kV \otimes \Lambda^\ell V)$. Write $\Lambda^kV = V_{\omega_k}$. Then in fact $\langle \mathrm{Flag}_{k,\ell}(V)\rangle = V_{\omega_k+\omega_\ell}$ giving a geometric realization of the Cartan product of the two modules V_{ω_k} and V_{ω_ℓ}. This generalizes to arbitrary Cartan products as follows:

The cominuscule varieties are special cases of "generalized Grassmannians", that is varieties G/P where P is a maximal parabolic. Such varieties always admit interpretations as subvarieties of some Grassmannian, usually given in terms of the set of k planes annihilated by some tensor(s). Given two such for the same group, $G/P_i \subset \mathbb{P}V_{\omega_i}$ and $G/P_j \subset \mathbb{P}V_{\omega_j}$, we may form an incidence variety $G/P_{i,j}$ and again we will have $\langle G/P_{i,j}\rangle = V_{\omega_i+\omega_j}$. Thus *Cartan powers and products of modules can be constructed geometrically.*

1.2.5. *Adjoint varieties.* After the generalized cominuscule varieties, the next simplest rational homogeneous varieties are the *adjoint varieties*, where V is taken to be $\mathfrak{g}$, the adjoint representation of G. We write $G/P = X^{\mathrm{ad}}_G \subset \mathbb{P}\mathfrak{g}$ to denote adjoint varieties. The adjoint varieties can also be characterized as the homogeneous compact complex contact manifolds. It is conjectured (see, e.g., [41, 22]) that they are essentially the only compact complex contact manifolds other then projectivized cotangent bundles. Many of these have simple geometric interpretations.
– $X^{\mathrm{ad}}_{\mathrm{SL}(W)} = \mathrm{Flag}_{1,n-1}(W)$ is the variety of flags of lines in hyperplanes in the n-dimensional vector space W.

– $X^{\mathrm{ad}}_{SO(W,Q)} = G_Q(2,W) \subset \mathbb{P}(\Lambda^2 W) = \mathbb{P}\mathfrak{so}(W)$ is the Grassmannian of isotropic 2-planes in W.
– $X^{\mathrm{ad}}_{G_2} = G_{\mathrm{null}}(2, \Im\mathbb{O})$ is the Grassmannian of two planes in the imaginary octonions on which the multiplication is zero. It may also be seen as the projectivization of the set of rank two derivations of $\mathbb{O}$, or as the set of six dimensional subalgebras of $\mathbb{O}$, see [37], Theorem 3.1.
– $X^{\mathrm{ad}}_{\mathrm{Sp}(W,\omega)} = v_2(\mathbb{P}W) \subset \mathbb{P}S^2W = \mathbb{P}\mathfrak{c}_n$ is the variety of quadratic forms of rank one.

Note that other than the pathological groups A_n, C_n, all adjoint representations are fundamental. Also note that the adjoint variety of $\mathfrak{c}_n$ is generalized cominuscule for $\mathfrak{a}_{2n-1}$.

1.3. **Notational conventions.** We work over the complex numbers throughout, all functions are holomorphic functions and manifolds are complex manifolds (although much of the theory carries over to $\mathbb{R}$, with some rigidity results even carrying over to the C^∞ setting). In particular the notion of a *general point* of an analytic manifold makes sense, which is a point off of a finite union of analytic subvarieties. We use the labeling and ordering of roots and weights as in [2]. For subsets $X \subset \mathbb{P}V$, $\hat{X} \subset V$ denotes the corresponding cone. For a manifold X, T_xX denotes its tangent space at x. For a submanifold $X \subset \mathbb{P}V$, $\hat{T}_xX = T_p\hat{X} \subset V$, denotes its affine tangent space, and $p \in \hat{x} =: L_x$. In particular, $T_xX = \hat{x}^* \otimes \hat{T}_xX/\hat{x}$. If $Y \subset \mathbb{P}W$, then $\langle Y \rangle \subset W$ denotes its linear span. We use the summation convention throughout: indices occurring up and down are to be summed over. If G is semi-simple of rank r, we write $P = P_I \subset G$ for the parabolic subgroup obtained by deleting negative root spaces corresponding to roots having a nonzero coefficient on any of the simple roots α_{i_s}, $i_s \in I \subset \{1,...,r\}$.

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2. Related projects

2.1. **Representation theory and computational complexity.** These projects with Manivel and Weyman address questions about G-varieties motivated by problems in computer science and algebraic statistics, specifically the complexity of matrix multiplication and the study of phylogenetic invariants. For a survey on this work see [29].

2.2. **Sphericality and tangential varieties.** For work related to Joachim Hilgert's lectures [17], recall that a normal projective G-variety Z is *G-spherical* if for all degrees d, $\mathbb{C}[Z]_d$, the component of the coordinate ring of Z in degree d, is a multiplicity free G-module, see [3]. Note that this property for $Z = \tau(X)$ *a priori* depends both on G and the embedding of X.

Theorem 2.1. [40] *Let $X = G/P \subset \mathbb{P}V$ be a homogeneously embedded rational homogeneous variety. Then $\tau(X)$ is G-spherical iff X admits the structure of a CHSS, and no factor of X is G_2/P_1.*

In [40] we also show that if G/P is cominuscule then $\tau(G/P)$ is normal, with rational singularities, and give explicit and uniform descriptions of the coordinate rings for all cases in the spirit of the project described in §2.3 below.

An interesting class of $\tau(G/P)$'s occurs for the *subexceptional series*, the third row of Freudenthal's magic chart: $\mathrm{Seg}(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$, $G_\omega(3,6)$, $G(3,6)$, $D_6/P_6=\mathbb{S}_6$, $E_7/P_7=G_w(\mathbb{O}^3,\mathbb{O}^6)$ where $\tau(G/P)$ is a quartic hypersurface whose equation is given by a generalized hyperdeterminant. See [30] for details. The equations of these varieties will play an important role in what follows, as the Fubini quartic forms for X_G^{ad} when G is an exceptional group (see §6.2).

2.3. **Vogelia.** This project, joint with Manivel, is inspired conjectural categorical generalizations of Lie algebras proposed by P. Deligne (for the exceptional series) [12, 13] and P. Vogel (for all simple super Lie algebras) [46]. It has relations Pierre Loday's lectures [42] because both conjectures appear to inspired by operads.

Let $\mathfrak{g}$ be a complex simple Lie algebra. Vogel derived a universal decomposition of $S^2\mathfrak{g}$ into (possibly virtual) Casimir eigenspaces, $S^2\mathfrak{g}=\mathbb{C}\oplus Y_2\oplus Y_2'\oplus Y_2''$ which turns out to be a decomposition into irreducible modules. If we let $2t$ denote the Casimir eigenvalue of the adjoint representation (with respect to some invariant quadratic form), these modules respectively have Casimir eigenvalues $4t-2\alpha$, $4t-2\beta$, $4t-2\gamma$, which we may take as the definitions of α, β, γ. Vogel showed that $t=\alpha+\beta+\gamma$. For example, for $\mathfrak{so}(n)$ we may take $(\alpha,\beta,\gamma)=(-2,4,n-4)$ and for the exceptional series $\mathfrak{so}_8$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ we may take $(\alpha,\beta,\gamma)=(-2,m+4,2m+4)$ where $m=0,1,2,4,8$ respectively. Vogel then went on to find Casimir eigenspaces Y_3, Y_3', $Y_3''\subset S^3\mathfrak{g}$ with eigenvalues $6t-6\alpha$, $6t-6\beta$, $6t-6\gamma$ (which again turn out to be irreducible), and computed their dimensions:

$$\dim\ \mathfrak{g}=\frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}\,,$$

$$\dim\ Y_2=-\frac{t(\beta-2t)(\gamma-2t)(\beta+t)(\gamma+t)(3\alpha-2t)}{\alpha^2\beta\gamma(\alpha-\beta)(\alpha-\gamma)}\,,$$

$$\dim\ Y_3=-\frac{t(\alpha-2t)(\beta-2t)(\gamma-2t)(\beta+t)(\gamma+t)(t+\beta-\alpha)(t+\gamma-\alpha)(5\alpha-2t)}{\alpha^3\beta\gamma(\alpha-\beta)(\alpha-\gamma)(2\alpha-\beta)(2\alpha-\gamma)}\,.$$

In [32, 36] we showed that some of the phenomena observed by Vogel and Deligne persist in all degrees. For example, let $\overline{\alpha}$ denote the highest root of $\mathfrak{g}$ (here we have fixed a Cartan subalgebra and a set of positive roots). Let Y_k be the k-th Cartan power $\mathfrak{g}^{(k)}$ of $\mathfrak{g}$ (the module with highest weight $k\overline{\alpha}$).

Theorem 2.2 ([36]). *Use Vogel's parameters α, β, γ as above. The k-th symmetric power of $\mathfrak{g}$ contains three (virtual) modules Y_k, Y_k', Y_k'' with Casimir eigenvalues $2kt-(k^2-k)\alpha$, $2kt-(k^2-k)\beta$, $2kt-(k^2-k)\gamma$. Using binomial coefficients defined by $\binom{y+x}{y}=(1+x)\dots(y+x)/y!$, we have:*

$$\dim\ Y_k=\frac{t-(k-\frac{1}{2})\alpha}{t+\frac{\alpha}{2}}\,\frac{\binom{-\frac{2t}{\alpha}-2+k}{k}\binom{\frac{\beta-2t}{\alpha}-1+k}{k}\binom{\frac{\gamma-2t}{\alpha}-1+k}{k}}{\binom{-\frac{\beta}{\alpha}-1+k}{k}\binom{-\frac{\gamma}{\alpha}-1+k}{k}},$$

and $\dim Y'_k$, $\dim Y''_k$ *are respectively obtained by exchanging the role of* α *with* β (*resp.* γ).

The modules Y'_k, Y''_k are described in [36]. This dimension formula is also the Hilbert function of $X^{\mathrm{ad}}_{G([\alpha,\beta,\gamma])}$.

2.4. **Cartan-Killing classification via projective geometry.** If $X = G(k, W) \subset \mathbb{P}(\Lambda^k W)$ then the variety of tangent directions to lines through a point $E \in X$ is $Y = \mathrm{Seg}\big(\mathbb{P}E^* \times \mathbb{P}(W/E)\big) \subset \mathbb{P}(E^* \otimes W/E) = \mathbb{P}T_E X$. Moreover one can recover X from Y as the image of the rational map $\mathbb{P}(T \oplus \mathbb{C}) \dashrightarrow \mathbb{P}^N$ given by the ideals in degree $r + 1$ of the varieties $\sigma_r(Y)$, multiplied by a suitable power of a linear form coming from the $\mathbb{C}$-factor to give them all the same degree. In [31] we showed that the same is true for any irreducible cominuscule variety. This enabled us to give a new, constructive proof of the classification of CHSS, without having to first classify complex simple Lie algebras. Moreover, a second construction constructs the adjoint varieties and gives a new proof of the Killing-Cartan classification of complex simple Lie algebras without classifying root systems. Here is the construction for adjoint varieties:

Let $Y \subset \mathbb{P}^{n-2} = \mathbb{P}T_1$ be a generalized cominuscule variety. Define Y to be *admissible* if the span of the embedded tangent lines to Y, as a subvariety of the Grassmannian, has codimension one in $\Lambda^2 T_1$. For generalized cominuscule varieties, this condition is equivalent to Y being embedded as a *Legendrian variety.* In particular $\tau(Y)$ is a quartic hypersurface (for the exceptional series, it is the quartic hypersurface described in §2.2). Linearly embedded $T_1 \subset \mathbb{C}^n \subset \mathbb{C}^{n+1}$ respectively as the hyperplanes $\{x_n = 0\}$ and $\{x_0 = 0\}$ and consider the rational map

$$\phi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^N \subset \mathbb{P}(S^4\mathbb{C}^{n+1*})$$

$$\begin{aligned}[x_0, ..., x_n] \mapsto [&x_0^4, x_0^3 T_1^*, x_0^3 x_n, x_0^2 I_2(Y, \mathbb{P}T_1), x_0^2 x_n T_1^* \\ &- x_0 I_3(\tau(Y)_{\mathrm{sing}}, \mathbb{P}T_1), x_0^2 x_n^2 - I_4(\tau(Y), \mathbb{P}T_1)]\end{aligned}$$

In [31] we showed that the image is an adjoint variety and that all adjoint varieties arise in this way. Here are the Legendrian varieties Y and the Lie algebras of the X^{ad}_G that they produce:

Y	$\subset$	$\mathbb{P}^{n-2}$	$\mathfrak{g}$
$v_3(\mathbb{P}^1)$	$\subset$	$\mathbb{P}^3$	$\mathfrak{g}_2$
$\mathbb{P}^1 \times \mathbb{Q}^{m-4}$	$\subset$	$\mathbb{P}^{2m-5}$	$\mathfrak{so}_m$
$G_\omega(3,6)$	$\subset$	$\mathbb{P}^{13}$	$\mathfrak{f}_4$
$G(3,6)$	$\subset$	$\mathbb{P}^{19}$	$\mathfrak{e}_6$
$\mathbb{S}_6$	$\subset$	$\mathbb{P}^{31}$	$\mathfrak{e}_7$
$G_w(\mathbb{O}^3, \mathbb{O}^6)$	$\subset$	$\mathbb{P}^{55}$	$\mathfrak{e}_8$.

The two exceptional (i.e., non-fundamental) cases are

$\mathbb{P}^{k-3} \sqcup \mathbb{P}^{k-3}$	$\subset$	$\mathbb{P}^{2k-3}$	$\mathfrak{sl}_k$
$\emptyset$	$\subset$	$\mathbb{P}^{2m-1}$	$\mathfrak{sp}_{2m}$.

See [31, 34] for details. The varieties $Y \subset \mathbb{P}^{n-1}$ are the asymptotic directions $B(II^{X_G^{\mathrm{ad}}}) \subset \mathbb{P}T_x X_G^{\mathrm{ad}}$ defined in the next section.

3. Projective differential geometry and results

3.1. The Gauss map and the projective second fundamental form. Let $X^n \subset \mathbb{P}V$ be an n-dimensional subvariety or complex manifold. The *Gauss map* is defined by

$$\gamma\colon X \dashrightarrow G(n+1,V)$$
$$x \mapsto \hat{T}_x X$$

Here $\hat{T}_x X \subset V$ is the affine tangent space to X at x, it is related to the intrinsic tangent space $T_x X \subset T_x\mathbb{P}V$ by $T_x X = (\hat{T}_x X/\hat{x}) \otimes \hat{x}^* \subset V/\hat{x} \otimes \hat{x}^*$ where $\hat{x} \subset V$ is the line corresponding to $x \in \mathbb{P}V$. Similarly $N_x X = T_x(\mathbb{P}V)/T_x X = \hat{x}^* \otimes (V/\hat{T}_x X)$. The dashed arrow is used because the Gauss map is not defined at singular points of X, but does define a rational map.

Now let $x \in X_{\mathrm{smooth}}$ and consider

$$d\gamma_x\colon T_x X \to T_{\hat{T}_x X} G(n+1,V) \simeq (\hat{T}_x X)^* \otimes (V/\hat{T}_x X)$$

Since, for all $v \in T_x X$, $\hat{x} \subset \ker d\gamma_x(v)$, where $d\gamma_x(v)\colon \hat{T}_x X \to V/\hat{T}_x X$, we may quotient by $\hat{x}$ to obtain

$$d\underline{\gamma}_x \in T_x^* X \otimes (\hat{T}_x X/\hat{x})^* \otimes V/(\hat{T}_x X) = (T_x^* X)^{\otimes 2} \otimes N_x X\,.$$

In fact, essentially because mixed partial derivatives commute, we have

$$d\underline{\gamma}_x \in S^2 T_x^* X \otimes N_x X$$

and we write $II_x = d\underline{\gamma}_x$, the *projective second fundamental form* of X at $x \cdot II_x$ describes how X is moving away from its embedded tangent space to first order at x.

One piece of geometric information that II_x encodes is the following: Think of $\mathbb{P}T_x X \subset \mathbb{P}T_x(\mathbb{P}V)$ as the set of tangent directions in $T_x\mathbb{P}V$ where there exists a line having contact to X at x to order at least one. Then $B(II_x) := \mathbb{P}\{v \in T_x X \mid II(v,v) = 0\}$, often called the set of *asymptotic directions*, is the set of tangent directions where there exists a line having contact to X at x to order at least two. To study the (macroscopic) geometry of X, we may study the smaller variety $B(II_x)$ and ask: *What does $B(II_x)$ tell us about the geometry of X?* Note that $B(II_x)$ is usually the zero set of $\operatorname{codim} X$ quadratic polynomials and thus we expect it to have codimension equal to $\operatorname{codim}(X, \mathbb{P}V)$ (assuming the codimension of X is sufficiently small, otherwise we expect it to be empty).

Now let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety. In particular we have $II_{X,x} = II_{X,y}$ for all $x, y \in X$ so we will simply write $II^X = II_{X,x}$. To what extent is X characterized by II^X?

Aside. If the ideal of a projective variety $X \subset \mathbb{P}V$ is generated in degrees at most d, then any line having contact with X to order d at a point must be contained in X. By an unpublished theorem of Kostant, the ideals of rational homogeneous varieties are generated in degree two, so $B(II^{G/P})$ corresponds to the tangent directions to lines through a point. Thus, for example, when $X = G(k, V)$, $B(II_E) = \mathrm{Seg}\big(\mathbb{P}E^* \times \mathbb{P}(V/E)\big)$.

3.2. **Second order rigidity.** For the Segre variety, $B(II^{\mathrm{Seg}(\mathbb{P}^2\times\mathbb{P}^2)}) \subset \mathbb{P}^3$ is the union of two disjoint lines ($\mathbb{P}^1$'s). The Segre has codimension four, and normally the common zero set of four quadratic polynomials on $\mathbb{P}^3$ is empty. This prompted Griffiths and Harris to conjecture:

Conjecture 3.1 ([15])**.** Let $X = \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^3)$. Let $Z^4 \subset \mathbb{P}V$ be a variety such that at $z \in Z_{\mathrm{general}}$, $II_{Z,z} = II^X$, then Z is projectively equivalent to the Segre.

Theorem 3.2 ([26, 27])**.** *The conjecture is true, moreover the same result holds when X is any rank two cominuscule variety except for $Q^n \subset \mathbb{P}^{n+1}$ and* $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^m) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^{m+1})$.

One can pose more generally the question: *Given a homogeneous variety $G/P \subset \mathbb{P}V$, an unknown variety $Z \subset \mathbb{P}W$ and a general point $z \in Z$, how many derivatives must we take at z to conclude $Z \simeq G/P$?*

3.3. **History of projective rigidity questions.** The problem of projective rigidity dates back 200 years when Monge showed $v_2(\mathbb{P}^1) \subset \mathbb{P}^2$, the conic curve in the plane, is characterized by a fifth order ODE, i.e., it is rigid at order five. More recently, about 100 years ago [14], Fubini showed that in dimensions greater than one, quadric hypersurfaces are rigid at order three, i.e., characterized by a third order system of PDE.

A vast generalization of Theorem 3.2 was obtained by Hwang and Yamaguchi:

Theorem 3.3 ([20])**.** *Let $X \subset \mathbb{P}V$ be an irreducible homogeneously embedded CHSS, other than a quadric hypersurface or projective space, with osculating sequence of length f. Then X is rigid at order f.*

See §7.1 for the definition of the osculating sequence.

Even more exciting than the theorem of Hwang and Yamaguchi are the methods they used to prove it. More on this in §6.4.

If one changes the hypotheses slightly, one gets a second order result:

Corollary 3.4 ([28])**.** *Let $X \subset \mathbb{P}V$ be a cominuscule variety, other than a quadric hypersurface. Let $Y \subset \mathbb{P}W$ be an unknown variety such that dim $\langle Y\rangle = \dim V$, and such that for $y \in Y_{\mathrm{general}}$, $II_{Y,y} = II^X$. Then Y is projectively equivalent to X.*

The proof of this result uses two facts: that the higher fundamental forms of cominuscule varieties are the (full) prolongations of the second, and that any variety with such fundamental forms must be the homogeneous model (which follows from Theorem 3.3). See [28] for details.

3.4. **Rigidity and flexibility of adjoint varieties.** For the adjoint varieties, it is easy to see that order two rigidity fails (see [33]), even though they have osculating sequence of length two. These lectures will be centered around the proof of the following theorem:

Theorem 3.5 ([24]). *For simple groups G, the adjoint varieties $X_G^{\mathrm{ad}} \subset \mathbb{P}\mathfrak{g}$ (other than $G = A_1$) are rigid at order three.*

In the case $G = A_1$, $X_{A_1}^{\mathrm{ad}} = v_2(\mathbb{P}^1)$, which Monge showed to be rigid at order five but not four.

Robles and I originally wrote a "brute force" proof of this theorem in December 2006, although we had been attempting to use the methods of Hwang and Yamaguchi. Finally, when A. Čap visited us in June 2007, in what can only be described as an incredible synchronicity, we made the breakthrough needed, in parallel with Čap making a breakthrough in his work on BGG operators with maximal kernel. In §8 I describe the methods, which involve a reduction to a Lie algebra cohomology calculation, and which should be useful for other EDS questions. I conclude this section with the description of a result that was obtained using traditional EDS techniques:

The adjoint varieties are the homogeneous models for certain *parabolic geometries*, a much discussed topic at this conference. In particular they are equipped with an intrinsic geometry that includes a holomorphic contact structure. All the intrinsic geometry is visible at order two (including the distinguished hyperplane) *except* for the contact structure. This inspires the modified question:

Assume $Z \subset \mathbb{P}V$ is such that at $z \in Z_{\mathrm{general}}$ we have $II_{Z,z} = II^{X_G^{\mathrm{ad}}}$ and the resulting hyperplane distribution is contact, can we conclude $Z \simeq X_G^{\mathrm{ad}}$? Of course for $G = A_1$, we know the answer is no, thanks to Monge.

Theorem 3.6 ([24]). *If $G \neq A_1, A_2$, then YES! If $G = \mathrm{SL}_3 = A_2$, then NO!; there exist "functions worth" of impostors.*

Remark 3.7. Although the results are formulated in the holomorphic category, the exact same result holds in the real analytic category.

The second conclusion has interesting consequences for geometry. A 3-manifold M equipped with a contact distribution which has two distinguished line sub-bundles is the path space for a *path geometry* in the plane. Such structures have two "curvature" functions, call them J_1, J_2, which are differential invariants that measure the difference between M and the homogeneous model, which is $X_{\mathrm{SL}_3}^{\mathrm{ad}} = \mathrm{Flag}_{1,2}(\mathbb{C}^3)$. This geometry has been well studied by many authors, including E. Cartan [10]. For example, if $J_1 \equiv 0$, then the paths are the geodesics of a projective connection. See [21], Chapter 8 for more.

Theorem 3.8 ([24]). *The general impostor above has J_1, J_2 nonzero, although they do satisfy differential relations.*

This is interesting because it is difficult to come up with natural restrictions on the invariants J_1, J_2 short of imposing that one or the other is zero. A classical analog of this situation, where the condition of being extrinsically realizable gives rise to a natural system of PDE, is the set of surfaces equipped with Riemannian metrics that admit a local isometric immersion into Euclidean 3-space such that the image is a minimal surface. Ricci discovered that for this to happen, the Gauss curvature K of the Riemannian metric of the surface must satisfy the PDE

$$\Delta \log(-K) = 4K$$

where Δ is the Riemannian Laplacian. See [11, 21] for details.

4. From PDE to EDS

Exercise: show that any system of PDE can be expressed as a first order system. (Hint: add variables.) Thus we only discuss first order systems. We want to study them from a geometric perspective, that of submanifold geometry.

Let $\mathbb{R}^n$ have coordinates $(x^1 \dots x^n) = (x^i)$ and $\mathbb{R}^m$ coordinates $(u^1 \dots u^m) = (u^a)$, let

$$F^r(x^i, u^a, p^a_i) = 0 \qquad 1 \leq r \leq R \tag{4.1}$$

be a system of equations in $n + m + nm$ unknowns. We view this as a system of PDE by stating that a map

$$f\colon \mathbb{R}^n \to \mathbb{R}^m$$

$$x \mapsto u = f(x)$$

is a solution of the system determined by (4.1) if (4.1) holds when we set $u^a = f^a(x)$ and $p^a_i = \frac{\partial u^a}{\partial x^i}$.

To rephrase slightly, let $J^1(\mathbb{R}^n, \mathbb{R}^m) := \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm}$ have coordinates (x^i, u^a, p^a_i). Consider the differential forms

$$\theta^a := du^a - p^a_i dx^i \in \Omega^1\big(J^1(\mathbb{R}^n, \mathbb{R}^m)\big) \qquad 1 \leq a \leq m\,.$$

Then we have the following correspondences:

Graphs of maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}^m$	$\leftrightarrow$	immersions $i\colon M^n \to J^1(\mathbb{R}^n, \mathbb{R}^m)$ such that $i^*(\theta^a) = 0$ and $i^*(dx^1 \wedge \cdots \wedge dx^n)$ is nonvanishing.

Graphs of maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}^m$ satisfying the PDE system determined by(4.1)	$\leftrightarrow$	immersions $i\colon M^n \to \Sigma \subset J^1(\mathbb{R}^n, \mathbb{R}^m)$ such that $i^*(\theta^a) = 0$ and $i^*(dx^1 \wedge \cdots \wedge dx^n)$ is nonvanishing, where Σ is the zero set of (4.1).

Now we are ready for EDS:

Definition 4.2. A *Pfaffian EDS with independence condition* on a manifold Σ is a sequence of sub-bundles $I \subset J \subseteq T^*\Sigma$. Write $n = \operatorname{rank}(J/I)$.

An *integral manifold* of (I, J) is an immersed n-dimensional submanifold $i\colon M \to \Sigma$ such that $i^*(I) = 0$ and $i^*(J/I) = T^*M$.

In the motivating example we had $I = \{\theta^a\}$ and $J = \{\theta^a, dx^i\}$.

Thus we have transformed questions about the existence of solutions to a system of PDE to questions about the existence of submanifolds tangent to a distribution.

We next show how to determine existence. But first, here are a few successes of EDS:

- Determination of existence of local isometric embeddings of (analytic) Riemannian manifolds into Euclidean space and other space forms (e.g., Cartan-Janet theorem), see, e.g., [1, 6].
- Proving the existence of Riemannian manifolds with holonomy G_2 and Spin_7 (Bryant [4]).
- Rigidity/flexibility of Schubert varieties in Grassmannians and other symmetric spaces (Bryant [5]).
- Proving existence of special Lagrangian and other calibrated submanifolds (Harvey and Lawson, [16]).

5. The Cartan algorithm to determine local existence of integral manifolds to an EDS

The essence of the Cartan algorithm is to systematically understand the additional conditions imposed by a system of PDE by the fact that mixed partial derivatives commute. In the language of differential forms, this is the statement

$$i^*(\theta) = 0 \;\Rrightarrow\; i^*(d\theta) = 0 \quad \forall \theta \in I\,.$$

For example, in §4, $i^*(d\theta) = 0$ forces $\partial p^a_i/\partial x^j = \partial p^a_j/\partial x^i$. On integral manifolds $p^a_i = \partial u^a/\partial x^i$.

5.1. Linear Pfaffian systems.

Among Pfaffian systems, there are those where the set of integral elements through a point forms an affine space, the *linear* systems.

Definition 5.1. A Pfaffian EDS is linear if the map

$$I \to \Lambda^2(T^*\Sigma/J)$$
$$\theta \mapsto d\theta \bmod J$$

is zero.

To simplify the exposition we will restrict to linear Pfaffian systems. (This is theoretically no loss of generality, see [21], Chapter 5.)

Although some of the theory is valid in the C^∞ category (see, e.g., [48]), we will work in the real or complex analytic category where the theory works best, and in the applications of this paper, we will actually work in the holomorphic category. In particular, it makes sense to talk of a *general point* of an analytic manifold, where general is with respect to the EDS on it (e.g., points where the system does not drop rank, where the derivatives of the forms in the system don't drop rank, etc.)

Fix $x \in \Sigma_{\text{general}}$. To determine the integral manifolds through x we work infinitesimally and reduce to problems in linear algebra (as one does with most problems in mathematics).

Definition 5.2. An n-plane $E \subset T_x\Sigma$ is called an *integral element* if $\theta_x \mid_E = 0$ and $d\theta_x \mid_E = 0$ for all $\theta \in I$.

Let $\mathcal{V}(I)_x \subset G(n, T_x\Sigma)$ denote the set of all integral elements at x. As remarked above, if (I, J) is linear, then $\mathcal{V}(I)_x$ is an affine space. Set

$$V = (J/I)^*_x$$
$$W = I^*_x.$$

Fix a splitting $T^*_x\Sigma = J_x \oplus J^c_x$ and define a bundle map

$$W^* \to \Lambda^2 V$$
$$\theta_x \mapsto d\theta_x \bmod I, J^c$$

we may consider this map as a tensor $T \in W \otimes \Lambda^2 V^*$, which we call the *apparent torsion* of (I, J) at x. Since the apparent torsion changes if we change the splitting, we instead consider

$$[T] \in W \otimes \Lambda^2 V^* / \sim \tag{5.3}$$

called the *torsion* of the system at x, which is well defined. The equivalence $\sim$ is precisely over the different choices of splittings and is made explicit in (5.5) below.

Since on the one hand we are requiring I to vanish on integral elements but J/I to be of maximal rank, if $[T] \neq 0$, there are no integral elements over x, i.e., $\mathcal{V}(I)_x = \emptyset$. If this is the case, we start over on the submanifold (analytic subvariety) $\Sigma' \subset \Sigma$ defined by $[T] = 0$.

Now consider the bundle map given by exterior differentiation, $\theta \mapsto d\theta \bmod I$, a component of which is $I \to (T^*\Sigma/J) \otimes J$. Pointwise this is a linear map $W^* \to (T^*\Sigma/J)_x \otimes V^*$, which we may consider as a linear map $(T^*\Sigma/J)^*_x \to W \otimes V^*$. Let

$$A \subset W \otimes V^*$$

denote the image of this map at x, which is called the *tableau* of (I, J) at x. For linear Pfaffian systems A corresponds to $\mathcal{V}(I)_x$, where we transform the affine space to a linear space by picking a base integral element $\theta^a_x = \pi^\epsilon_x = 0$, where the θ^a_x give a basis of I_x and π^ϵ_x give a basis for a choice of J^c_x. The quantity $\dim A$ gives us an answer to the infinitesimal version of the question: *How many integral manifolds of (I, J) pass through x ?*

Example 5.4. Say $J = T^*\Sigma$, which is the situation of the Frobenius theorem. Then the tableau A is zero, which corresponds to the uniqueness part of the theorem. There exist integral manifolds if and only if $T = [T] = 0$.

We may think of the tableau A as parametrizing the choices of admissible first order terms in the Taylor series of an integral manifold at x expressed in terms of a graph. From this perspective, the next question is: What are the admissible second order terms in the Taylor series? At the risk of being repetitive, the condition to check is that: *Mixed partials commute!*

5.2. **Prolongations and the Cartan-Kähler theorem.** Let

$$\delta\colon W\otimes V^*\otimes V^*\to W\otimes\Lambda^2V^*$$

denote the skew-symmetrization map. Define

$$A^{(1)}:=\ker\delta\mid_{A\otimes V^*}=(A\otimes V^*)\cap(W\otimes S^2V^*)$$

the *prolongation* of A. We may think of $A^{(1)}$ as parametrizing the admissible second order terms in the Taylor series.

At this point we can make explicit the equivalence in (5.3). It is

$$[T]\in W\otimes\Lambda^2V^*/\delta(A\otimes V^*) \tag{5.5}$$

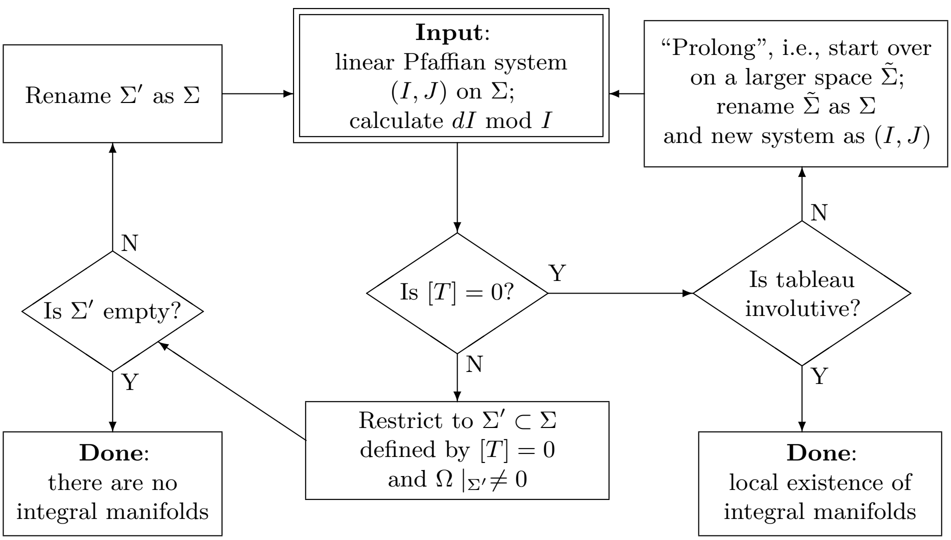
Now we know how to determine the admissible third order Taylor terms etc., but should we keep going on forever? When can we stop working? The answer is given by the following theorem:

Theorem 5.6 (Cartan, Cartan-Kähler (see, e.g., [6, 21])). *Let (I,J) be an analytic linear Pfaffian system on Σ, let $x\in\Sigma_{\text{general}}$. Assume $[T]_x=0$. Choose an A-generic flag $V^*=V^0\supset V^1\supset\cdots\supset V^{n-1}\supset 0$. Let $A_j:=A\cap(W\otimes V^j)$. Then*

$$dim\,A^{(1)}\le dim\,A+dim\,A_1+\cdots+dim\,A_{n-1}\,.$$

If equality holds then we say (I,J) is involutive *at x and then there exist local integral manifolds through x that depend roughly on $dim(A_r/A_{r-1})$ functions of r variables, where r is the unique integer such that $A_{r-1}\neq A_r=A_{r+1}$.*

5.3. **Flowchart and exercises.** Here $\Omega\in\Lambda^n(J/I)$ encodes the independence condition:



Exercises 5.7:
Set up the EDS and perform the Cartan algorithm in the following problems:

1. The Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. (Work on a codimension two submanifold of $J^1(\mathbb{R}^2, \mathbb{R}^2)$.)
2. Find all surfaces $M^2 \subset \mathbb{E}^3$ such that every point is an umbillic point.
3. Determine the local existence of special Lagrangian submanifolds of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$.
4. (For the more ambitious.) Pick your favorite $G \subset SO(p,q)$ and determine local existence of pseudo-Riemannian manifolds with holonomy $\subseteq G$.
5. After you read §6, show that $\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$ is rigid to order two. Then roll up your sleeves to show that $\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^n)$ is flexible at order two.

5.4. **For fans of bases.** Here is a recap in bases: take a local coframing of Σ adapted to the flag $I \subset J \subset T^*\Sigma$, i.e., write $I = \{\theta^a\}$, $1 \le a \le \mathrm{rank}\ I$, $J = \{\theta^a, \omega^i\}$, $1 \le i \le \mathrm{rank}\,(J/I)$, $T^*\Sigma = \{\theta^a, \omega^i, \pi^\epsilon\}$, $1 \le \epsilon \le \mathrm{rank}\,(T^*\Sigma/J)$. Then there exist functions $A, ..., H$ such that

$$\begin{aligned} d\theta^a &= A^a_{\epsilon i}\pi^\epsilon \wedge \omega^i + T^a_{ij}\omega^i \wedge \omega^j + E^a_{\epsilon,\delta}\pi^\epsilon \wedge \pi^\delta \\ &\quad + F^a_{bi}\theta^b \wedge \omega^i + G^a_{b\epsilon}\theta^b \wedge \pi^\epsilon + H^a_{bc}\theta^b \wedge \theta^c\,. \end{aligned}$$

Since we only care about $d\theta^a$ mod I we ignore the second row. The system is linear iff $E^a_{\varepsilon\delta} = 0$. The apparent torsion is $T = T^a_{ij} v^i \wedge v^j \otimes w_a \in \Lambda^2 V^* \otimes W$. The tableau is

$$A = \{A^a_{\epsilon i} v^i \otimes w_a \mid 1 \le \epsilon \le \mathrm{rank}\,(T^*\Sigma/J)\} \subset V^* \otimes W\,.$$

The torsion is

$$\begin{aligned} [T] &= T^a_{ij} w_a \otimes v^i \wedge v^j \ \mathrm{mod}\ \left\{(A^a_{j\epsilon} e^\epsilon_i - A^a_{i\epsilon} e^\epsilon_j) w_a \otimes v^i \wedge v^j \mid e^\epsilon_j \in \mathbb{F}\right\} \\ &\in W \otimes \Lambda^2 V^* / \delta(A \otimes V^*)\,. \end{aligned}$$

Here $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

6. Moving frames for submanifolds of projective space

Let $U = \mathbb{C}^{N+1}$. Let $G \subseteq \mathrm{GL}(U)$ have *Maurer-Cartan form* $\omega_{\mathfrak{g}} \in \Omega^1(G, \mathfrak{g})$. Recall that the Maurer-Cartan form has the following properties:

– left invariance: $L_g^*\omega = \omega$ where $L_g\colon G \to G$ is the map $a \mapsto ga$.
– $\omega_{\mathrm{Id}}\colon T_{\mathrm{Id}}G \to \mathfrak{g}$ is the identity map
– $d\omega = -\omega \wedge \omega$ or equivalently, $d\omega = -\frac{1}{2}[\omega, \omega]$ (Maurer-Cartan equation)

(Here $[\omega, \eta](v, w) := [\omega(v), \eta(w)] - [\omega(w), \eta(v)]$.)

6.1. **Adapted frame bundles.** We want to study the geometry of submanifolds $Y \subset \mathbb{P}U$ from the perspective of Klein, that is we consider $Y \sim Z$ if there exists $g \in \mathrm{GL}(U)$ such that $g.Y = Z$. In order to efficiently incorporate the group action, we will work "upstairs" on $\mathrm{GL}(U)$. Consider the projection map

$$\begin{aligned} \pi\colon\ \mathrm{GL}(U) &\to \mathbb{P}U \\ (e_0 \dots e_N) &\mapsto [e_0] \end{aligned}$$

where we view the e_j as column vectors. Fixing a reference basis, we may identify $\mathrm{GL}(U)$ with the set of all bases of U. We will restrict ourselves to submanifolds of $\mathrm{GL}(U)$ consisting of bases adapted to the local differential geometry of $Y \subset \mathbb{P}U$.

First, consider $\mathcal{F}^0_Y := \pi^{-1}(Y)$, the 0-th order adapted frames (bases). Let $n = \dim Y$. Next consider

$$\mathcal{F}^1_Y := \{f = (e_0 \dots e_N) \in \mathcal{F}^0_Y \mid \hat{T}_{[e_0]}Y = \{e_0 \dots e_n\}\}$$

the frames adapted to the flag $\hat{x} \subset \hat{T}_xY \subset U$ over each point, called the *first order adapted frame bundle.* Write $L = \hat{x}$, $T = \hat{T}_xY/\hat{x}$, $N = U/\hat{T}_xY$. Adopt index ranges $1 \le \alpha,\ \beta \le n = \dim T_xY$, $n+1 \le \mu,\ \nu \le \dim U - 1$. Write $\mathfrak{gl}(U) = (L \oplus T \oplus N)^* \otimes (L \oplus T \oplus N)$ and let, for example, $\omega_{L^* \otimes T}$ denote the component of ω taking values in $L^* \otimes T \subset U^* \otimes U = \mathfrak{gl}(U)$. Write

$$\omega_{\mathfrak{gl}(U)} = \begin{pmatrix} \omega^0_0 & \omega^0_\beta & \omega^0_\nu \\ \omega^\alpha_0 & \omega^\alpha_\beta & \omega^\alpha_\nu \\ \omega^\mu_0 & \omega^\mu_\beta & \omega^\mu_\nu \end{pmatrix} = \begin{pmatrix} \omega_{L^* \otimes L} & \omega_{T^* \otimes L} & \omega_{N^* \otimes L} \\ \omega_{L^* \otimes T} & \omega_{T^* \otimes T} & \omega_{N^* \otimes T} \\ \omega_{L^* \otimes N} & \omega_{T^* \otimes N} & \omega_{N^* \otimes N} \end{pmatrix}. \tag{6.1}$$

Write $i\colon \mathcal{F}^1_Y \to \mathrm{GL}(U)$ as the inclusion. We have $i^*(\omega^\mu_0) = i^*(\omega_{L^* \otimes N}) = 0$. (Note that at each $f \in \mathcal{F}^1_Y$ we actually have a splitting $U = L \oplus T \oplus N$.)

6.2. **Fubini Forms.** Now anytime you ever see a quantity equal to zero, *Differentiate it!* We have

$$i^*(\omega^\mu_0) = 0 \Rrightarrow i^*(d\omega^\mu_0) = 0$$

which, using the Maurer-Cartan equation tells us that

$$i^*(\omega^\mu_\alpha \wedge \omega^\alpha_0) = 0$$

(note use of summation convention). We are assuming that the forms $i^*(\omega^\alpha_0)$ are linearly independent (as they span the pullback of T^*Y by our choice of adaptation) so we must have

$$i^*(\omega^\mu_\alpha) = q^\mu_{\alpha\beta} i^*(\omega^\beta_0)$$

for some functions $q^\mu_{\alpha\beta}\colon \mathcal{F}^1_Y \to \mathbb{C}$. Moreover (exercise) $q^\mu_{\alpha\beta} = q^\mu_{\beta\alpha}$ for all α, β (this is often called the *Cartan Lemma*). The functions $q^\mu_{\alpha\beta}$ vary on the fiber, but they do contain geometric information. If we form the tensor field

$$F_2 := q^\mu_{\alpha\beta}\omega^\alpha_0 \circ \omega^\beta_0 \otimes e^0 \otimes (e_\mu \bmod \hat{T}_xY) \in \Gamma\big(\mathcal{F}^1_Y, \pi^*(S^2T^*Y \otimes NY)\big)$$

a short calculation shows that F_2 is constant on the fibers, i.e. $F_2 = \pi^*(II)$ for some tensor $II \in \Gamma(Y, S^2T^*Y \otimes NY)$. II is indeed the projective second fundamental form defined as the derivative of the Gauss map in §3.

Unlike with the case of the Gauss map, where it was not clear how to continue, here it is - we have a quantity equal to zero: $\omega^\mu_\alpha - q^\mu_{\alpha\beta}\omega^\beta_0$ so we differentiate it! (From now on we drop the i^* when describing pullbacks of differential forms to simplify notation.) The result is that there exist functions

$$r^\mu_{\alpha\beta\gamma}\colon \mathcal{F}^1_Y \to \mathbb{C}$$

such that

$$-dq^\mu_{\alpha\beta} - q^\mu_{\alpha\beta}\omega^0_0 - q^\nu_{\alpha\beta}\omega^\mu_\nu + q^\mu_{\alpha\epsilon}\omega^\epsilon_\beta + q^\mu_{\beta\epsilon}\omega^\epsilon_\alpha = r^\mu_{\alpha\beta\gamma}\omega^\gamma_0$$

which gives rise to a tensor field

$$F_3 \in \Gamma\big(\mathcal{F}^1_Y, \pi^*(S^3T^*Y \otimes NY)\big).$$

This tensor, called the *Fubini cubic form* does not descend to be well defined on Y, but it does contain important geometric information.

6.3. **Second order Fubini systems.** Fix vector spaces L, T, N of dimensions 1, n, a and fix an element $F_2 \in S^2T^* \otimes N \otimes L$. Let $U = L \oplus T \oplus N$, and let $\omega \in \Omega^1\big(\mathrm{GL}(U), \mathfrak{gl}(U)\big)$ denote the Maurer-Cartan form.

Writing the Maurer-Cartan equation component-wise yields, for example,

$$d\omega_{L^*\otimes T} = -\omega_{L^*\otimes T} \wedge \omega_{L^*\otimes L} - \omega_{T^*\otimes T} \wedge \omega_{L^*\otimes T} - \omega_{N^*\otimes T} \wedge \omega_{L^*\otimes N}\,.$$

Given $F_2 \in L \otimes S^2T^* \otimes N$, the *second order Fubini system* for F_2 is

$$I_{\mathrm{Fub}_2} = \big\{\omega_{L^*\otimes N}, \omega_{T^*\otimes N} - F_2(\omega_{L^*\otimes T})\big\}\,, \quad J_{\mathrm{Fub}_2} = \{I_{\mathrm{Fub}_2}, \omega_{L^*\otimes T}\}\,.$$

Its integral manifolds are submanifolds $\mathcal{F}^2 \subset \mathrm{GL}(U)$ that are adapted frame bundles of submanifolds $X \subset \mathbb{P}U$ having the property that at each point $x \in X$, the projective second fundamental form $F_{2,X,x}$ is equivalent to F_2. (The tautological system for frame bundles of arbitrary n dimensional submanifolds is given by $I = \{\omega_{L^*\otimes N}\}$, $J = \{I, \omega_{L^*\otimes T}\}$.)

Let $R \subset \mathrm{GL}(L) \times \mathrm{GL}(T) \times \mathrm{GL}(N)$ denote the subgroup stabilizing F_2 and let

$$\mathfrak{r} \subset (L^* \otimes L) \oplus (T^* \otimes T) \oplus (N^* \otimes N) =: \mathfrak{gl}(U)_{0,*}$$

denote its subalgebra. These are the elements of $\mathfrak{gl}(U)_{0,*}$ annihilating F_2. (The motivation for the notation $\mathfrak{gl}(U)_{0,*}$ is explained in §7.1.) Assume $\mathfrak{r}$ is reductive so that we may decompose $\mathfrak{gl}(U)_{0,*} = \mathfrak{r} \oplus \mathfrak{r}^\perp$ as an $\mathfrak{r}$-module.

In the case of homogeneous varieties G/P, $F_2 \in S^2T^* \otimes N \otimes L$ will correspond to a trivial representation of the Levi factor of P, which we denote G_0. For example, let $G/P = G(2, M) \subset \mathbb{P}\Lambda^2 M$ be the Grassmannian of 2-planes. Then $R = G_0 = \mathrm{GL}(E) \times \mathrm{GL}(F)$, $T = E^* \otimes F$, $N = \Lambda^2 E^* \otimes \Lambda^2 F$, and we have the decomposition $S^2T^* = (\Lambda^2 E \otimes \Lambda^2 F^*) \oplus (S^2 E \otimes S^2 F)$, and $F_2 \in S^2T^* \otimes N$ corresponds to the trivial representation in $(\Lambda^2 E \otimes \Lambda^2 F^*) \otimes (\Lambda^2 E \otimes \Lambda^2 F^*)^*$.

In the notation of §5,

$$V \simeq L^* \otimes T\,, \quad W \simeq (L^* \otimes N) \oplus (T^* \otimes N)\,, \quad A \simeq \mathfrak{r}^\perp\,,$$

with $L^* \otimes N \subset W$ in the first derived system. That $\mathfrak{r}^\perp \subset V^* \otimes W$ may be seen as follows

$$\begin{aligned}
d\big(\omega_{T^*\otimes N} - F_2(\omega_{L^*\otimes T})\big) &= -\,\omega_{T^*\otimes L} \wedge \omega_{L^*\otimes N} - \omega_{T^*\otimes T} \wedge \omega_{T^*\otimes N} \\
&\quad - \omega_{T^*\otimes N} \wedge \omega_{N^*\otimes N} \;+\; F_2(\omega_{L^*\otimes L} \wedge \omega_{L^*\otimes T}) \\
&\quad + F_2(\omega_{L^*\otimes T} \wedge \omega_{T^*\otimes T} \;+ \omega_{L^*\otimes N} \wedge \omega_{N^*\otimes T}) \\
&\equiv -\,\omega_{T^*\otimes T} \wedge F_2(\omega_{L^*\otimes T}) \;- F_2(\omega_{L^*\otimes T}) \wedge \omega_{N^*\otimes N} \\
&\quad - F_2(-\omega_{L^*\otimes L} \wedge \omega_{L^*\otimes T} - \omega_{L^*\otimes T} \wedge \omega_{T^*\otimes T}) \mod I \\
&\equiv (\omega_{0,*} \cdot F_2) \wedge \omega_{L^*\otimes T} \mod I \\
&\equiv (\omega_{\mathfrak{r}^\perp} \cdot F_2) \wedge \omega_{L^*\otimes T} \mod I\,.
\end{aligned} \tag{6.2}$$

To understand the last two lines, $\omega_{0,*}\cdot F_2$ denotes the action of the $\mathfrak{gl}(U)_{0,*}$-valued component $\omega_{0,*}$ of the Maurer-Cartan form on $F_2\in S^2T^*\otimes N$. Recall that $\mathfrak{r}$ is the annihilator of this action. By definition $\omega_{0,*}\cdot F_2=(\omega_{\mathfrak{r}}+\omega_{\mathfrak{r}^\perp})\cdot F_2=\omega_{\mathfrak{r}^\perp}\cdot F_2$.

For the Cartan algorithm we need to calculate $A^{(1)}=\ker\delta$ where

$$\delta\colon \mathfrak{r}^\perp\otimes V^*\to W\otimes\Lambda^2V^*\,.$$

One can check directly that A is *never* involutive for any F_2 system. (One has not yet uncovered all commutation relations among mixed partials. This is essentially because we have yet to look at the entire Maurer-Cartan form).

Thus we need to prolong, introducing elements of $A^{(1)}$ as new variables and differential forms to force variables representing the elements of $A^{(1)}$ to behave properly, just as the θ^a's forced the p^a_i's to be derivatives in §4.

Before doing so, we simplify our calculations by exploiting the group action to normalize $A^{(1)}\sim F_3$ as much as possible. Write $\mathfrak{gl}(U)_{1,*}:=T^*\otimes L+N^*\otimes T$. Consider the linear map

$$\delta\colon \mathfrak{gl}(U)_{1,*}\to(L^*\otimes T)\otimes\mathfrak{r}^\perp=V^*\otimes A$$

defined as the transpose of the Lie bracket

$$[\ ,\ ]\colon \mathfrak{gl}(U)_{1,*}\times L\otimes T^*\to\mathfrak{r}^\perp\subset\mathfrak{gl}(U)_{0,*}\,.$$

Now $L\otimes T^*\subset\mathfrak{gl}(U)_{-1\cdot *}:=\mathfrak{gl}(U)^*_{1\cdot *}$. Then we define

$$A^{(1)}_{\mathrm{red}}:=\frac{\ker\delta\colon A\otimes V^*\to W\otimes\Lambda^2V^*}{\mathrm{Image}\,\delta\colon \mathfrak{gl}(U)_{1,*}\to A\otimes V^*}\,.$$

One can calculate directly, that if X is a rank 2 CHSS in its minimal homogeneous embedding (other than a quadric or $\mathbb{P}^1\times\mathbb{P}^m$) and $\overline{F}_2=II^X$, then $A^{(1)}_{\mathrm{red}}=0$. In these cases, we begin again with a new system

$$\tilde{I}:=\{I,\omega_{\mathfrak{r}^\perp}\}$$

on $\mathrm{GL}(U)$. Again, one can check that $\tilde{A}$ is never involutive, but that $\tilde{A}_{\mathrm{red}}{}^{(1)}=0$. Finally, one defines

$$\tilde{\tilde{I}}=\{\tilde{I},\omega_{\mathfrak{r}^\perp}\}$$

which turns out to be Frobenius in the case of rank 2 CHSS, i.e. $\tilde{\tilde{A}}{}^{(1)}=0$, which implies rigidity.

6.4. **An easier path to rigidity?** A better way to obtain the same conclusion is to observe that $A^{(1)}_{\mathrm{red}}$ looks like the graded Lie algebra cohomology group $H^1_1(\mathfrak{g}_-,\mathfrak{g}^\perp)$ defined in §8. In the CHSS case, it indeed is this cohomology group, but in all other cases it is not. In the next few sections we will see that the correspondence is exact in the CHSS case, and how it fails in all other cases - it fails in two ways, but none the less, with the introduction of certain *filtered* EDS, the use of Lie algebra cohomology can be recovered.

7. Osculating gradings and root gradings

As mentioned above, for homogeneously embedded CHSS, the osculating filtration and a filtration induced by the Lie algebra coincide, but that these two differ for all other homogeneous varieties. In this section we explain the two filtrations.

7.1. **The osculating filtration.** Given a submanifold $X \subset \mathbb{P}U$, and $x \in X$, the *osculating filtration at* x

$$U_0 \subset U_1 \subset \cdots \subset U_r = U$$

is defined by

$$\begin{aligned} U_0 &= \hat{x}\,, \\ U_1 &= \hat{T}_x X\,, \\ U_2 &= U_1 + F_2(L^* \otimes S^2 T_x X) \\ &\vdots \\ U_r &= U_{r-1} + F_r(L^{*\,\otimes\,(r-1)} \otimes S^r T_x X)\,. \end{aligned}$$

We may reduce the frame bundle $\mathcal{F}^1_X$ to framings adapted to the osculating sequence by restricting to $e = (e_0, e_\alpha, e_{\mu_2}, \ldots, e_{\mu_f}) \in \mathcal{F}^1_X$ such that $[e_0] \in X$, $\hat{T}_{[e_0]}X = \operatorname{span}\{e_0, e_\alpha\}$ and $U_k = \operatorname{span}\{e_0, e_\alpha, e_{\mu_2}, \ldots, e_{\mu_k}\}$. (The indices α and μ_j respectively range over $1, \ldots, n$ and $\dim U_{j-1} + 1, \ldots, \dim U_j$.) From now on we work on this reduced frame-bundle, denoted $\mathcal{F}^r_X \subset \mathcal{F}^1_X$.

At each point of $\mathcal{F}^r_X$ we obtain a splitting of U. This induces a splitting

$$\mathfrak{gl}(U) = \oplus \mathfrak{gl}(U)_{k,*}\,.$$

(The asterisk above is a place holder for a second splitting given by the representation theory when $X = G/P$ that we define in §7.2.)

The osculating filtration of U determines a refinement of the Fubini forms. Let $L = U_0$. Let $N_k = U_k/U_{k-1}$ and define $F_{k,s}\colon N_k^* \to L^{\otimes\,(s-1)} \otimes S^s T^*_x X$ by restricting $F_s \in N_k * \otimes L^{*\,\otimes\,(s-1)} \otimes S^s T_x X$ to N_k^*. Although the Fubini forms do not descend to well-defined tensors on X, the *fundamental forms* $F_{k,k}$ do. By construction, $F_{k,k}\colon L^{*\,\otimes\,(k-1)} \otimes S^k T_x X \to N_{k,x}X$ is surjective.

7.2. **The root grading.** Let $\tilde{\mathfrak{g}}$ be a complex semi-simple Lie algebra with a fixed set of simple roots $\{\alpha_1 \ldots \alpha_r\}$, and corresponding fundamental weights $\{\omega_1, \ldots, \omega_r\}$. Let $I \subset \{1, \ldots, r\}$, and consider the irreducible representation $\mu\colon \tilde{\mathfrak{g}} \to \mathfrak{gl}(U)$ of highest weight $\lambda = \sum_{i\in I} \lambda^i \omega_i$. Set $\mathfrak{g} = \mu(\tilde{\mathfrak{g}})$, and let $\mu(G) \subset \mathrm{GL}(U)$ be the associated Lie group so that $G/P \subset \mathbb{P}U$ is the orbit of a highest weight line. Write $P = P_I \subset G$ for the parabolic subgroup obtained by deleting negative root spaces corresponding to roots having a nonzero coefficient on any of the simple roots α_i, $i \in I$.

Since $\tilde{\mathfrak{g}}$ is reductive, we have a splitting $\mathfrak{gl}(U) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the $\tilde{\mathfrak{g}}$-submodule of $\mathfrak{gl}(U)$ complementary to $\mathfrak{g}$. Let $\omega \in \Omega^1\big(\mathrm{GL}(U), \mathfrak{gl}(U)\big)$ denote the

Maurer-Cartan form of $\mathrm{GL}(U)$, and let $\omega_{\mathfrak{g}}$ and $\omega_{\mathfrak{g}^\perp}$ denote the components of ω taking values in $\mathfrak{g}$ and $\mathfrak{g}^\perp$, respectively.

The bundle $\mathcal{F}^1_{G/P}$ admits a reduction to a bundle $\mathcal{F}^G_{G/P} = \mu(G)$. On this bundle the Maurer-Cartan form pulls-back to take values in $\mathfrak{g}$, that is, $\omega_{\mathfrak{g}^\perp} = 0$. Conversely, all $\dim(G)$ dimensional integral manifolds of the system $I = \{\omega_{\mathfrak{g}^\perp}\}$ are left translates of $\mu(G)$.

Let $Z = Z_I \subset \mathfrak{t}$ be the *grading element* corresponding to $\sum_{i_s \in I} \alpha_{i_s}$. The grading element Z_i for a simple root α_i has the property that $Z_i(\alpha_j) = \delta^i_j$. In general $Z = \sum_{i_s \in I} Z_{i_s}$. Thus, if (c^{-1}) denotes the *inverse of the Cartan matrix*, then given a weight $\nu = \sum \nu^j \omega_j$,

$$Z(\nu) = \sum_{\substack{1 \le j \le r \\ i_s \in I}} \nu^j (c^{-1})_{j, i_s} . \tag{7.1}$$

The grading element induces a $\mathbb{Z}$-grading of $\mathfrak{g} = \oplus^k_{-k} \mathfrak{g}_k$. To determine k in the case $P = P_{\alpha_i}$ is a maximal parabolic, let $\tilde{\alpha} = \sum m_j \alpha_j$ denote the highest root, then $k = m_i$.

The module U inherits a $\mathbb{Z}$-grading

$$U = U_{Z(\lambda)} \oplus U_{Z(\lambda)-1} \oplus \cdots \oplus U_{Z(\lambda)-f} .$$

The U_j are eigen-spaces for Z. This grading is compatible with the action of $\tilde{\mathfrak{g}}$: $\mu(\tilde{\mathfrak{g}}_i) \cdot U_j \subset U_{i+j}$. We adopt the notational convention of shifting the grading on U to begin at zero. The component U_0 (formally named $U_{Z(\lambda)}$) is one dimensional, and corresponds to the highest weight line of U, and $G \cdot \mathbb{P}U_0 = G/P \subset \mathbb{P}U$. (The labeling of the grading on $\mathfrak{gl}(U) = U^* \otimes U$ is independent of our shift convention.)

Note, in particular, that the vector space $\hat{T}_{[\mathrm{Id}]}(G/P)/\hat{\mathrm{Id}} \simeq \mathfrak{g}/\mathfrak{p}$ is graded from -1 to $-k$. The osculating grading on U induces gradings of $\mathfrak{gl}(U)$, $\mathfrak{g}$ and $\mathfrak{g}^\perp$. In Examples 7.3 and 7.4 the summands in $T_x G/P$ appearing are in order from -1 to $-k$

We write

$$\mathfrak{gl}(U) = \bigoplus_{s,j} \mathfrak{gl}(U)_{s,j}$$

where the first index refers to the osculating grading (§7.1) induced by $G/P \subset \mathbb{P}U$ and the second the root grading. We adopt the notational convention

$$\mathfrak{gl}(U)_j = \bigoplus_{s} \mathfrak{gl}(U)_{s,j} ;$$

so if there is only one index, it refers to the root grading. Note that the grading of $\mathfrak{gl}(U)$ is indexed by integers $-f, ..., f$.

7.3. **Examples of tangent spaces and osculating filtrations of homogeneous varieties.**

Example 7.2. Consider $G(k, V) \subset \mathbb{P}\Lambda^k V = \mathbb{P}U$. Fix $E \in G(k, V)$. Then the osculating sequence is

$$U_0 = \Lambda^k E \subset (\Lambda^{k-1} E \wedge V) \subset (\Lambda^{k-2} E \wedge \Lambda^2 V) \subset \cdots \subset (\Lambda^1 E \wedge \Lambda^{k-1} V) \subset \Lambda^k V = U.$$

Remark. The only nonzero Fubini forms of a homogeneously embedded CHSS are the fundamental forms. For the adjoint varieties, the only nonzero Fubini forms are $F_{2,2}, F_{2,3}, F_{2,4}$.

One definition of $G/P \subset \mathbb{P}V$ being cominuscule is that $T_{Id}(G/P)$ is an irreducible P-module. Here are some examples describing tangent spaces and osculating sequences of non-cominuscule varieties.

Example 7.3. For orthogonal Grassmannians $G_Q(k,V) \subset \mathbb{P}\Lambda^k V$ (assume $k < \frac{1}{2}\dim V$),

$$\begin{aligned}\big(T_E(G_Q(k,V))\big)_{-1} &= E^* \otimes (E^\perp/E)\\ \big(T_E(G_Q(k,V))\big)_{-2} &= \Lambda^2 E^*\,,\end{aligned}$$

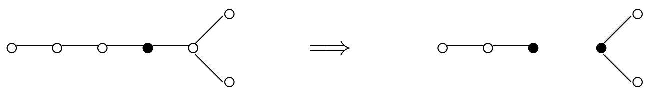
where the $\perp$ refers to the Q-orthogonal complement, gr to the associated graded vector space of the filtered vector space $T_E G_Q(k,V)$. Note that $E \subset E^\perp$ because E is isotropic.

Example 7.4. For the 89 dimensional variety $(E_8/P_3) \subset \mathbb{P}V_{\omega_3} = \mathbb{P}^{6696999}$

$$\begin{aligned}T_{-1} &= U \otimes \Lambda^2 W\\ T_{-2} &= \Lambda^4 W\\ T_{-3} &= U \otimes \Lambda^6 W\\ T_{-4} &= W\end{aligned}$$

where $U = \mathbb{C}^2$ is the standard representation of A_1 and $W = \mathbb{C}^7$, the standard representation of A_6.

Remark. For those familiar with Dynkin diagrams, it is possible to obtain T_{-1} and T_{-f} pictorially, where T_{-f} is the last summand. For simplicity assume P is maximal, take the Dynkin diagram for $\mathfrak{g}$, delete the node for P, and mark the adjacent nodes with the multiplicity of the bond assuming an arrow points towards the marked note, otherwise just mark with multiplicity one.



$X = G_Q(4,12)$ $\qquad\qquad$ $T_{-1} = \mathbb{C}^4 \otimes \Lambda^2\mathbb{C}^4$

The last filtrand, T_{-f} is obtained by marking the node(s) associated to the adjoint representation of $\mathfrak{g}$ and taking the dual module in the new diagram. These coincide iff G/P is CHSS.

7.3.1. *Symplectic Grassmannians.* Here is the full osculating sequence and some details for the symplectic Grassmannians taken from [33] (where many other cases with P maximal may be found as well):

Let $G_\omega(k,2n) = C_n/P_k$ denote the Grassmannian of k-planes isotropic for a symplectic form. Its minimal embedding is to $V_{\omega_k} = \Lambda^{\langle k\rangle}V = \Lambda^k V/(\Omega \wedge \Lambda^{k-2}V)$, the

k-th reduced exterior power of $V = \mathbb{C}^{2n}$, where $\Omega \in \Lambda^2 V$ denotes the symplectic form on V^* induced from $\omega \in \Lambda^2 V^*$.

Let $E \in G_\omega(k, V)$ and write $U = E^\perp/E$. A straightforward computation shows that V_{ω_k} has the following decomposition as an $H = \mathrm{SL}(E) \times \mathrm{Sp}(U)$-module:

$$V_{\omega_k} = \Lambda^{\langle k \rangle} V = \bigoplus_{a,b} \Lambda^b E^* \otimes \Lambda^{a+b} E^* \otimes \Lambda^{\langle a \rangle} U\,.$$

(Here $\dim E = k$ and $\dim U = 2n - 2k$.) Note that U is endowed with a symplectic form induced by the symplectic form on $V = \mathbb{C}^{2n}$.

Proposition 7.5 ([33])**.** *Let $E \in G_\omega(k, 2n)$, let $E^\perp \supset E$ denote the Ω-orthogonal complement to E and let $U = E^\perp/E$. Then the tangent space and normal spaces of $G_\omega(k, 2n)$ are, as G_0-modules,*

$$\begin{aligned}
T_{-1} &= E^* \otimes U\,,\\
T_{-2} &= S^2 E^*\,,\\
N_{2,-2} &= \Lambda^2 E^* \otimes \Lambda^{\langle 2 \rangle} U\,,\\
N_{2,-3} &= S_{21} E^* \otimes U\,,\\
N_{2,-4} &= S_{22} E^*\,,\\
N_{p,*} &= \bigoplus_{a+b+c=p} \Lambda^{\langle a \rangle} U \otimes S_{2^{b-c}1^{a+2c}} E^*\\
&= \bigoplus_{d+e=p} \Lambda^d U \otimes S_{2^e 1^d} E^*\,.
\end{aligned}$$

$S_\pi E$ is the irreducible $\mathrm{GL}(E)$ module associated to the partition π. (Here $S_{2^a 1^b} E$ corresponds to the partition with a 2's and b 1's.) In particular, the length of the osculating sequence is equal to $k+1$, the last non zero term being $N_k \simeq \Lambda^k(\mathbb{C} \oplus U)$.

Corollary 7.6 ([33])**.**

$$B\big(II_{G_\omega(k,2n),E}\big) = \mathbb{P}\overline{\{e \otimes u \oplus e^2 \mid e \in E^*\backslash\{0\},\ u \in U\backslash\{0\}\}}\,.$$

This set of asymptotic directions contains an open dense P-orbit, the boundary of which is the union of the two (disjoint) closed H-orbits

$$Y_1 \simeq \mathbb{P}^{k-1} \times \mathbb{P}^{2n-2k-1} \subset \mathbb{P}(T_{-1}) \quad \textit{and} \quad Y_2 \simeq v_2(\mathbb{P}^{k-1}) \subset \mathbb{P}(T_{-2})\,.$$

7.3.2. *The bigrading for adjoint varieties.* For adjoint varieties, $T_x X_G^{\mathrm{ad}}$ has a two step filtration, with the hyperplane being the first filtrand, and the osculating sequence is simply $\hat{x} \subset \hat{T} \subset U$. The induced bi-grading on $\mathfrak{gl}(U)_{\mathrm{osc,alg}}$ is indicated in the table below.

	$\hat{x}^*$	$T_{-1}{}^*$	$T_{-2}{}^*$	$N_{-2}{}^*$	$N_{-3}{}^*$	$N_{-4}{}^*$
$\hat{x}$	(0,0)	(-1,1)	(-1,2)	(-2,2)	(-2,3)	(-2,4)
T_{-1}	(1,-1)	(0,0)	(0,1)	(-1,1)	(-1,2)	(-1,3)
T_{-2}	(1,-2)	(0,-1)	(0,0)	(-1,0)	(-1,1)	(-1,2)
N_{-2}	(2,-2)	(1,-1)	(1,0)	(0,0)	(0,1)	(0,2)
N_{-3}	(2,-3)	(1,-2)	(1,-1)	(0,-1)	(0,0)	(0,1)
N_{-4}	(2,-4)	(1,-3)	(1,-2)	(0,-2)	(0,-1)	(0,0)

In all cases T_{-1}, T_{-2} may be determined by the remark above (T_{-2} is the trivial module as there is no node left to mark), and $N_{-2} = I_2(Y)^*$, $N_{-3} = I_3\big((\tau(Y))_{\mathrm{sing}}\big)$ and N_{-4} is trivial (corresponding to the quartic generating $I_4(\tau(Y))$).

8. Lie algebra cohomology and Kostant's theory

Let $\mathfrak{l}$ be a Lie algebra and let Γ be an $\mathfrak{l}$-module. Define maps

$$\partial^j : \Lambda^j \mathfrak{l}^* \otimes \Gamma \to \Lambda^{j+1} \mathfrak{l}^* \otimes \Gamma$$

in the only natural way possible respecting the Leibniz rule. This gives rise to a complex and we define $H^k(\mathfrak{l}, \Gamma) := \ker \partial^k / \mathrm{Image}\, \partial^{k-1}$. We will only have need of ∂^0 and ∂^1 which are defined explicitly as follows: if $X \in \Gamma$ and $v, w \in \mathfrak{l}$, then

$$\partial^0(X)(v) = v \cdot X\,,$$

and if $\alpha \otimes X \in \Lambda^1 \mathfrak{l}^* \otimes \Gamma$, then

$$\partial^1(\alpha \otimes X)(v \wedge w) = \alpha([v,w])X + \alpha(v) w \cdot X - \alpha(w) v \cdot X \tag{8.1}$$

Now let $\mathfrak{l}$ be a graded Lie algebra and Γ a graded $\mathfrak{l}$-module. The chain complex and Lie algebra cohomology groups inherit gradings as well. Explicitly,

$$\partial^1_d : \ \oplus_i \, (\mathfrak{l}_{-i})^* \otimes \Gamma_{d-i} \to \ \oplus_{j \le m} \, (\mathfrak{l}_{-j})^* \wedge (\mathfrak{l}_{-m})^* \otimes \Gamma_{d-j-m}\,.$$

Kostant [23] shows that under the following circumstances one can compute $H^k(\mathfrak{l}, \Gamma)$ combinatorially:

(1) $\mathfrak{l} = \mathfrak{n} \subset \mathfrak{p} \subset \mathfrak{g}$ is the nilpotent subalgebra of a parabolic subalgebra of a semi-simple Lie algebra $\mathfrak{g}$.

(2) Γ is a $\mathfrak{g}$-module.

Under these conditions, letting $\mathfrak{g}_0 \subset \mathfrak{p}$ be the (reductive) Levi factor of $\mathfrak{p}$, $H^j(\mathfrak{n}, \Gamma)$ is naturally a $\mathfrak{g}_0$-module. Kostant shows that for any irreducible module Γ it is essentially trivial to compute $H^1(\mathfrak{n}, \Gamma)$, one just examines certain simple reflections in the Weyl group. However, in our situation, where we need to compute $H^1(\mathfrak{g}, \mathfrak{g}^\perp)$, there may be numerous components to $\mathfrak{g}^\perp$, and moreover we would like to avoid a case by case decomposition. Here the beauty of the grading element comes in, because it is easy to prove that in many situations $H^1_d(\mathfrak{g}, \Gamma)$ is zero in positive degree. This is well documented in [47, 20, 24] among other places.

9. From the Fubini EDS to Filtered EDS

I now explain how we were led to work with filtered EDS in an effort to use Lie algebra cohomology to determine rigidity of homogeneous varieties.

9.1. **Problem 1: Osculating v.s. root gradings.** As mentioned above, for CHSS, the osculating grading coincides with the root grading, but for all other homogeneously embedded homogeneous varieties this fails. Thus to have any hope to exploit Lie algebra cohomology, we need to work with an EDS that respects the root grading.

From now on we will work on $\mathrm{SL}(U) \subset \mathrm{GL}(U)$ which will not change anything regarding our study of rigidity of subvarieties of $\mathbb{P}U$. Define the (I_p, J_p) system on $\mathrm{SL}(U)$ by $I_p = \{\omega_{\mathfrak{g}^\perp_{\leq p}}\}$, $J_p = \{I_p, \omega_{\mathfrak{g}_-}\}$.

In specific examples, after a short calculation, the k-th order Fubini system can be shown to be strictly stronger than some (I_p, J_p) system (where of course p depends on k). At the moment we have no general method of determining this, but we do so uniformly for adjoint varieties in [24]. In summary, this problem is easy to resolve in specific cases or even classes of cases, but work remains to resolve the general case.

Here is the proof in the adjoint case:

Proposition 9.1. *Every integral manifold of the third-order Fubini system* $(I_{\mathrm{Fub}_3}, J_{\mathrm{Fub}_3})$ *for a given adjoint variety is an integral manifold of the* (I_{-1}, J_{-1}) *system for the same adjoint variety.*

Proof. Suppose that $\mathcal{F} \subset \mathrm{SL}(U)$ is an integral manifold of third-order Fubini system. We wish to show that the $\mathfrak{g}^\perp_{*,<0}$–valued component of the Maurer-Cartan form vanishes when pulled-back to $\mathcal{F}$. That the $\mathfrak{g}^\perp_{>0,*}$–valued component vanishes is an immediate consequence of the injectivity of the second fundamental form F_2 on each homogeneous component.

Referring to the table above, we see that there remain four blocks of the component of the Maurer-Cartan form in $\mathfrak{g}^\perp_{*,<0}$ to consider: the three $(0,-1)$ blocks $\omega_{T_{-2}\otimes T^*_{-1}}$, $\omega_{N_{-3}\otimes N^*_{-2}}$ and $\omega_{N_{-4}\otimes N^*_{-3}}$; and the singleton $(0,-2)$ block $\omega_{N_{-4}\otimes N^*_{-2}}$. The third Fubini form is defined by (3.5) of [21, §3.5]. The vanishing of the $\mathfrak{g}^\perp$–component of the first two blocks is a consequence of the $S^3T^*_{-1}\otimes N_{-3}$ component of F_3. (This is the only nonzero component of F_3.) The vanishing of the $\mathfrak{g}^\perp$-component of the third and fourth blocks is given by the $S^3T^*\otimes N_{-4}$-component of F_3. □

9.2. **Problem 2: Even the systems defined by the root grading do not lead to Lie algebra cohomology.** For simplicity we take $p = -1$ and $k = 2$. Notice that $\mathfrak{g}^\perp_s = \mathfrak{sl}(U)_s$ for all $s \leq -3$. Abbreviate

$$\omega_{\mathfrak{sl}(U)_s} =: \omega_s ,$$

so that $\omega_{\mathfrak{g}^\perp_s} = \omega_s$, for all $s \leq -3$. Thus

$$I_{-1} = \left\{\omega_{\mathfrak{g}^\perp_{-1}}, \omega_{\mathfrak{g}^\perp_{-2}}, \omega_{-3}, \dots, \omega_{-f}\right\}.$$

The calculations that follow utilize the Maurer-Cartan equation (see §6), and that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$ and $[\mathfrak{g},\mathfrak{g}^\perp] \subset \mathfrak{g}^\perp$. It is easy to see that $\mathrm{d}\omega_s \equiv 0$ modulo I_{-1} when $s \leq -3$. Next, computing modulo I_{-1},

$$-\mathrm{d}\,\omega_{\mathfrak{g}^\perp_{-2}} \equiv \left[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}^\perp_0}\right]$$

and

$$-\mathrm{d}\,\omega_{\mathfrak{g}^\perp_{-1}} \equiv \left[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}^\perp_1}\right] + \left[\omega_{\mathfrak{g}_{-1}}, \omega_{\mathfrak{g}^\perp_0}\right].$$

In order for these two equations to be satisfied, on an integral element we must have

$$\omega_{\mathfrak{g}^\perp_0} = \lambda_{0,1}(\omega_{\mathfrak{g}_{-1}}) + \lambda_{0,2}(\omega_{\mathfrak{g}_{-2}}) \tag{9.2}$$

$$\omega_{\mathfrak{g}^\perp_1} = \lambda_{1,1}(\omega_{\mathfrak{g}_{-1}}) + \lambda_{1,2}(\omega_{\mathfrak{g}_{-2}}) \tag{9.3}$$

for some $\lambda_{i,j} \in \mathfrak{g}^\perp_i \otimes \mathfrak{g}^*_{-j}$.

Consider the degree two homogeneous component δ_2 of the Spencer differential $\delta\colon A \otimes V^* \to W \otimes \Lambda^2 V^*$, where $A = \mathfrak{g}^\perp_1 \oplus \mathfrak{g}^\perp_0$, $W = \mathfrak{g}^\perp_{\leq -1}$ (but we may and will ignore the first derived system $\mathfrak{g}^\perp_{\leq -3}$) $V = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$:

$$\begin{aligned}\delta_2\colon (\mathfrak{g}^\perp_1 \otimes \mathfrak{g}^*_{-1}) \oplus (\mathfrak{g}^\perp_0 \otimes \mathfrak{g}^*_{-2}) &\to (\mathfrak{g}^\perp_{-1} \otimes \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-2}) \oplus (\mathfrak{g}^\perp_{-2} \otimes \mathfrak{g}^*_{-2} \wedge \mathfrak{g}^*_{-2})\\ \lambda_{1,1} \oplus \lambda_{0,2} &\mapsto \left\{(u_{-1} \wedge v_{-2}) \mapsto \left[\lambda_{1,1}(u_{-1}), v_{-2}\right] + \left[u_{-1}, \lambda_{0,2}(v_{-2})\right]\right.\\ &\qquad \left.\oplus\, (x_{-2}, y_{-2}) \mapsto \left[\lambda_{0,2}(x_{-2}), y_{-2}\right] + \left[x_{-2}, \lambda_{0,2}(y_{-2})\right]\right\}\end{aligned}$$

Here $x_{-2} \in \mathfrak{g}_{-2}$ etc. This is exactly the Lie algebra cohomology differential ∂^1_2! Now consider the degree one component δ_1

$$\begin{aligned}\delta_1\colon (\mathfrak{g}^\perp_0 \otimes \mathfrak{g}^*_{-1}) &\to (\mathfrak{g}^\perp_{-1} \otimes \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1}) \oplus (\mathfrak{g}^\perp_{-2} \otimes \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-2})\\ \lambda_{0,1} &\mapsto \left\{(u_{-1} \wedge v_{-1}) \oplus (x_{-1}, y_{-2})\right.\\ &\qquad \left.\mapsto \left[\lambda_{0,1}(u_{-1}), v_{-1}\right] + \left[u_{-1}, \lambda_{0,1}(v_{-1})\right] \oplus \left[\lambda_{0,1}(x_{-1}), y_{-2}\right]\right\}\end{aligned}$$

This fails to be the Lie algebra cohomology differential because we are "missing" a term $\lambda_{-1,2}([u_{-1}, v_{-2}])$ on the right hand side. One can try to "fix" this by adding in such a term. At first this appears unnatural, but if one takes into account that there is a natural filtration on our manifold, it is not unreasonable to weaken the condition $\omega_{\mathfrak{g}^\perp_{-1}} = 0$ to the condition $\omega_{\mathfrak{g}^\perp_{-1}}|_{T_{-1}} = 0$, i.e., $\omega_{\mathfrak{g}^\perp_{-1}} = \lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}})$ where $\lambda_{-1,2} \in \mathfrak{g}^\perp_{-1} \otimes \mathfrak{g}^*_{-2}$ at each point of our manifold.

We make this "fix" precise and natural with the introduction of filtered EDS:

9.3. **The Fix for problem 2: Filtered EDS.**

Definition 9.4. Let Σ be a manifold equipped with a filtration of its tangent bundle $T^{-1} \subset T^{-2} \subset \cdots \subset T^{-f} = T\Sigma$. Define an *r-filtered* Pfaffian EDS on Σ to be a filtered ideal $I \subset T^*\Sigma$ whose integral manifolds are the immersed submanifolds $i\colon M \to \Sigma$ such that $i^*(I_u)|_{i^*(T^{u-r})} = 0$ for all u, with the convention that $T^{-s} = T\Sigma$ when $-s \leq -f$.

Another way to view filtered EDS is to consider the ordinary EDS on the sum of the bundles $I_u \otimes (T\Sigma/T^{u+r})$. In our case these bundles will be trivial with fixed vector spaces as models.

Define $(I_p^{\mathfrak{f}}, \Omega)$ to be the $(p+1)$-filtered EDS on $\mathrm{GL}(U)$ with filtered ideal $I_p^{\mathfrak{f}} := \omega_{\mathfrak{g}^{\perp}_{\leq p}}$ and independence condition Ω given by the wedge product of the forms in $\omega_{\mathfrak{g}_-}$. We may view this as an ordinary EDS on

$$\mathrm{GL}(U)\times \big([\mathfrak{g}_p^{\perp} \otimes (\mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_{-k})^*] \oplus [\mathfrak{g}_{p-1}^{\perp} \otimes (\mathfrak{g}_{-3} \oplus \cdots \oplus \mathfrak{g}_{-k})^*] \oplus \cdots \oplus [\mathfrak{g}^{\perp}_{p-k+2} \otimes \mathfrak{g}^*_{-k}]\big)$$

where, giving $\mathfrak{g}_i^{\perp} \otimes \mathfrak{g}^*_{-j}$ linear coordinates $\lambda_{i,j}$, we have

$$\begin{aligned} I_p^{\mathfrak{f}} = \big\{ &\omega_{\mathfrak{g}_s^{\perp}}, s \leq p-k+1;\ \omega_{\mathfrak{g}^{\perp}_{p-k+2}} - \lambda_{p-k+2,k}(\omega_{\mathfrak{g}_{-k}}), \\ &\omega_{\mathfrak{g}^{\perp}_{p-k+3}} - \lambda_{p-k+3,k}(\omega_{\mathfrak{g}_{-k}}) - \lambda_{p-k+3,k-1}(\omega_{\mathfrak{g}_{-k+1}}), \dots \\ &\omega_{\mathfrak{g}^{\perp}_p} - \lambda_{p,k}(\omega_{\mathfrak{g}_{-k}}) - \cdots - \lambda_{p,2}(\omega_{\mathfrak{g}_{-2}}) \big\} \end{aligned} \tag{9.5}$$

However, as is explained below, it is more natural to work in the category of filtered EDS.

Returning to the $p=-1$, $k=2$ system, the first derived system is $\omega_{\leq -4}$. Computing similarly to above, only now modulo $I^{\mathfrak{f}}_{-1}$, we obtain

$$-\mathrm{d}\,\omega_{-3} \equiv \big[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}^{\perp}_{-1}}\big] \equiv \big[\omega_{\mathfrak{g}_{-2}}, \lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}})\big], \tag{9.6}$$

$$\begin{aligned} -\mathrm{d}\,\omega_{\mathfrak{g}^{\perp}_{-2}} &\equiv \big[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}^{\perp}_0}\big] + \big[\omega_{\mathfrak{g}_{-1}}, \omega_{\mathfrak{g}^{\perp}_{-1}}\big] + \big[\omega_{\mathfrak{g}^{\perp}_{-1}}, \omega_{\mathfrak{g}^{\perp}_{-1}}\big]_{\mathfrak{g}^{\perp}} \\ &\equiv \big[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}^{\perp}_0}\big] + \big[\omega_{\mathfrak{g}_{-1}}, \lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}})\big] \\ &\quad + \big[\lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}}), \lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}})\big]_{\mathfrak{g}^{\perp}}, \end{aligned} \tag{9.7}$$

$$\begin{aligned} -\mathrm{d}\big(\omega_{\mathfrak{g}^{\perp}_{-1}} - \lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}})\big) &\equiv \big[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}^{\perp}_1}\big] + \big[\omega_{\mathfrak{g}_{-1}}, \omega_{\mathfrak{g}^{\perp}_0}\big] \\ &\quad + \big[\omega_{\mathfrak{g}^{\perp}_{-1}}, \omega_{\mathfrak{g}_0}\big] + \big[\omega_{\mathfrak{g}^{\perp}_{-1}}, \omega_{\mathfrak{g}^{\perp}_0}\big]_{\mathfrak{g}^{\perp}} + \mathrm{d}\lambda_{-1,2}(\wedge\omega_{\mathfrak{g}_{-2}}) \\ &\quad - \lambda_{-1,2}\big(\big[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}_0}\big] + \big[\omega_{\mathfrak{g}_1}, \omega_{\mathfrak{g}_{-1}}\big] + \big[\omega_{\mathfrak{g}^{\perp}_{-1}}, \omega_{\mathfrak{g}^{\perp}_{-1}}\big]_{\mathfrak{g}}\big) \\ &\equiv \big[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}^{\perp}_1}\big] + \big[\omega_{\mathfrak{g}_{-1}}, \omega_{\mathfrak{g}^{\perp}_0}\big] + \big[\lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}}), \omega_{\mathfrak{g}_0}\big] \\ &\quad + \big[\lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}}), \omega_{\mathfrak{g}^{\perp}_0}\big]_{\mathfrak{g}^{\perp}} + \mathrm{d}\lambda_{-1,2}(\wedge\omega_{\mathfrak{g}_{-2}}) \\ &\quad - \lambda_{-1,2}\big(\big[\omega_{\mathfrak{g}_{-2}}, \omega_{\mathfrak{g}_0}\big] + \big[\omega_{\mathfrak{g}_{-1}}, \omega_{\mathfrak{g}_{-1}}\big] \\ &\quad + \big[\lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}}), \lambda_{-1,2}(\omega_{\mathfrak{g}_{-2}})\big]_{\mathfrak{g}}\big) \end{aligned} \tag{9.8}$$

Here $[\cdot,\cdot]_{\mathfrak{g}}$ (resp. $[\cdot,\cdot]_{\mathfrak{g}^{\perp}}$) denotes the component of the bracket taking values in $\mathfrak{g}$ (resp. $\mathfrak{g}^{\perp}$).

Note that if we were to view the filtered EDS as an ordinary EDS on $\mathrm{SL}(U) \times \mathfrak{g}^{\perp}_{-1} \otimes \mathfrak{g}^*_{-2}$, the term $\lambda_{-1,2}$ is part of the torsion whereas here it is simply part of the tableau of the filtered Spencer differential.

The degree one homogeneous component of (9.6,9.7,9.8), is as follows: $\lambda_1 := \oplus_{s=-1}^{0}\lambda_{s,1-s}$ must be in the kernel of the map

$$\delta_1 \colon \oplus_{s=-1}^{0} (\mathfrak{g}_s^{\perp} \otimes \mathfrak{g}_{-s-1}^{*}) \to (\mathfrak{g}_{-1}^{\perp} \otimes \mathfrak{g}_{-1}^{*} \wedge \mathfrak{g}_{-1}^{*}) \oplus (\mathfrak{g}_{-2}^{\perp} \otimes \mathfrak{g}_{-1}^{*} \wedge \mathfrak{g}_{-2}^{*})$$

defined as follows. Given $u_{-1}, v_{-1} \in \mathfrak{g}_{-1}$,

$$\delta_1(\lambda_1)(u_{-1} \wedge v_{-1}) = [\lambda_{0,1}(u_{-1}), v_{-1}] + [u_{-1}, \lambda_{0,1}(v_{-1})] - \lambda_{-1,2}([u_{-1}, v_{-1}]). \tag{9.9}$$

For $u_{-1} \in \mathfrak{g}_{-1}, v_{-2} \in \mathfrak{g}_{-2}$

$$\delta_1(\lambda_1)(u_{-1} \wedge v_{-2}) = [\lambda_{0,1}(u_{-1}), v_{-2}] + [u_{-1}, \lambda_{-1,2}(v_{-2})] .$$

That is, $\delta_1 = \partial_1^1$, where ∂_1^1 is the Lie algebra cohomology differential described in §8.

Moreover, $\mathfrak{g}_1^{\perp} = \mathfrak{n} \cap \mathfrak{gl}(U)_1$, and the Lie algebra cohomology denominator $\partial_1^0(\mathfrak{g}_1^{\perp})$ is the space of admissible normalizations of the prolongation coefficients λ_1. Thus, the vanishing of $H_1^1(\mathfrak{g}_-, \mathfrak{g}^{\perp})$ implies that normalized integral manifolds of the (I_{-1}^f, Ω) system are in one to one correspondence with integral manifolds of the (I_0^f, Ω) system.

Punch line: by working with filtered EDS and by homogeneous degree we do obtain Lie algebra cohomology. The vanishing of the Lie algebra cohomology reduces the system to the (I_0^f, Ω) system, and vanishing of the Lie algebra cohomology group $H_2^1(\mathfrak{g}_-, \mathfrak{g}^{\perp})$ moves one to the (I_1^f, Ω) system etc... Moreover, there was nothing special about beginning with $p = -1$. The final result is:

Theorem 9.10. *Let U be a complex vector space, and $\mathfrak{g} \subset \mathfrak{gl}(U)$ a represented complex semi-simple Lie algebra. Let $Z = G/P \subset \mathbb{P}U$ be the corresponding homogeneous variety (the orbit of a highest weight line). Denote the induced $\mathbb{Z}$-gradings by $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ and $U = U_0 \oplus \cdots \oplus U_{-f}$. Fix an integer $p \geq -1$, and let (I_p^f, Ω) denote the linear Pfaffian system given by* (9.5). *If $H_d^1(\mathfrak{g}_-, \mathfrak{g}^{\perp}) = 0$, for all $d \geq p+2$, then the homogenous variety G/P is rigid for the (I_p^f, Ω) system.*

10. Open questions and problems

- Does $H_d^1(\mathfrak{g}_-, \mathfrak{g}^{\perp})$ nonzero imply flexibility? If so, can one prove this directly and in general without going through the (sometimes quite long) Cartan algorithm?
- Give a uniform description of the F_k for all G/P's to obtain uniform determinations of Fubini rigidity.
- Determine the class of extrinsically realizable non flat parabolic geometries modeled on $\mathrm{Flag}_{1,2}(\mathbb{C}^3)$ as some natural class of parabolic geometries.
- Apply Čap's machinery to study parabolic geometries having families of differential operators whose kernel is large but not maximal.

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