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GL_n-INVARIANT TENSORS AND GRAPHS

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Abstract. We describe a correspondence between GL_n-invariant tensors and graphs. We then show how this correspondence accommodates various types of symmetries and orientations.

Introduction

Let V be a finite dimensional vector space over a field $\mathbf{k}$ of characteristic zero and $\mathrm{GL}(V)$ the group of invertible linear endomorphisms of V. The classical (Co)Invariant Tensor Theorem recalled in Section 1 states that the space of $\mathrm{GL}(V)$-invariant linear maps between tensor products of copies of V is generated by specific 'elementary invariant tensors' and that these elementary tensors are linearly independent if the dimension of V is big enough.

We will observe that elementary invariant tensors are in one-to-one correspondence with contraction schemes for indices which are, in turn, described by graphs. We then show how this translation between invariant tensors and linear combination of graphs accommodates various types of symmetries and orientations.

The above type of description of invariant tensors by graphs was systematically used by M. Kontsevich in his seminal paper [4]. Graphs representing tensors appeared also in the work of several other authors, let us mention at least J. Conant, A. Hamilton, A. Lazarev, J.-L. Loday, S. Mahajan M. Mulase, M. Penkava, K. Vogtmann, A. Schwarz and G. Weingart.

We were, however, not able to find a suitable reference containing all details. The need for such a reference appeared in connection with our paper [5] that provided a vocabulary between natural differential operators and graph complexes. Indeed, this note was originally designed as an appendix to [5], but we believe that it might be of independent interest. It supplies necessary details to [5] and its future applications, and also puts the 'abstract tensor calculus' attributed to R. Penrose onto a solid footing.

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1. Invariant Tensor Theorem: A recollection

Recall that, for finite-dimensional $\mathbf{k}$-vector spaces U and W, one has canonical isomorphisms

(1) $Lin(U,W)^* \cong Lin(W,U)\,,\ Lin(U,V) \cong U^* \otimes V \quad \text{and} \quad (U \otimes W)^* \cong U^* \otimes V^*,$

where $Lin(-,-)$ denotes the space of $\mathbf{k}$-linear maps, $(-)^*$ the linear dual and $\otimes$ the tensor product over $\mathbf{k}$. The first isomorphism in (1) is induced by the non-degenerate pairing

$$Lin(U,W) \otimes Lin(W,U) \to \mathbf{k}$$

that takes $f \otimes g \in Lin(U,W) \otimes Lin(W,U)$ into the trace of the composition $Tr(f \circ g)$, the remaining two isomorphisms are obvious. In this note, by a *canonical isomorphism* we will usually mean a composition of isomorphisms of the above types. Einstein's convention assuming summation over repeated (multi)indices is used. We will also assume that the ground field $\mathbf{k}$ is of characteristic zero.

In what follows, V will be an n-dimensional $\mathbf{k}$-vector space and $\mathrm{GL}(V)$ the group of linear automorphisms of V. We start by considering the vector space $Lin(V^{\otimes k}, V^{\otimes l})$ of $\mathbf{k}$-linear maps $f : V^{\otimes k} \to V^{\otimes l}$, $k,l \geq 0$. Since both $V^{\otimes k}$ and $V^{\otimes l}$ are $\mathrm{GL}(V)$-modules, it makes sense to study the subspace $Lin_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes l}) \subset Lin(V^{\otimes k}, V^{\otimes l})$ of $\mathrm{GL}(V)$-equivariant maps.

As there are no $\mathrm{GL}(V)$-equivariant maps in $Lin(V^{\otimes k}, V^{\otimes l}) = 0$ if $k \neq l$ (see, for instance, [3, §24.3]), the only interesting case is $k = l$. For a permutation $\sigma \in \Sigma_k$, define the *elementary invariant tensor* $t_\sigma \in Lin(V^{\otimes k}, V^{\otimes k})$ as the map given by

$$t_\sigma(v_1 \otimes \cdots \otimes v_k) := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}\,, \quad \text{for} \quad v_1, \ldots, v_k \in V\,. \tag{2}$$

It is simple to verify that t_σ is $\mathrm{GL}(V)$-equivariant. The following theorem is a celebrated result of H. Weyl [7].

Invariant Tensor Theorem. *The space $Lin_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes k})$ is spanned by elementary invariant tensors t_σ, $\sigma \in \Sigma_k$. If $\dim(V) \geq k$, the tensors $\{t_\sigma\}_{\sigma \in \Sigma_k}$ are linearly independent.*

This form of the Invariant Tensor Theorem is a straightforward translation of [1, Theorem 2.1.4] describing invariant tensors in $V^{*\otimes k} \otimes V^{\otimes k}$ and remarks following this theorem, see also [3, Theorem 24.4]. The Invariant Tensor Theorem can be reformulated into saying that the map

$$\mathcal{R}_n : \mathbf{k}[\Sigma_k] \to Lin_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes k}) \tag{3}$$

from the group ring of Σ_k to the subspace of $\mathrm{GL}(V)$-equivariant maps given by $\mathcal{R}_n(\sigma) := t_\sigma$, $\sigma \in \Sigma_k$, is always an epimorphism and is an isomorphism for $n \geq k$ (recall n denoted the dimension of V).

The tensors $\{t_\sigma\}_{\sigma\in\Sigma_k}$ are not linearly independent if $\dim(V) < k$. For a subset $S \subset \{1,\dots,k\}$ such that $\mathrm{card}(S) > \dim(V)$, denote by Σ_S the subgroup of Σ_k consisting of permutations that leave the complement $\{1,\dots,k\} \setminus S$ fixed. It is simple to verify that then

$$\sum_{\sigma\in\Sigma_S} \mathrm{sgn}(\sigma)\cdot t_\sigma = 0 \tag{4}$$

in $\mathit{Lin}_{\mathrm{GL}(V)}(V^{\otimes k}, V^{\otimes k})$. By [1, II.1.3], all relations between the elementary invariant tensors are induced by the relations of the above type. In other words, the kernel of the map $\mathcal{R}_n$ in (3) is generated by the expressions

$$\sum_{\sigma\in\Sigma_S} \mathrm{sgn}(\sigma)\cdot \sigma \in \mathbf{k}[\Sigma_k]\,,$$

where S and Σ_S are as above. Observe that, with the convention used in (2) involving the inverses of σ in the right hand side, $\mathcal{R}_n$ is a ring homomorphism.

1.1. **Definition.** By the *stable range* we mean the situation when $\dim(V) \geq k$, that is, when the map $\mathcal{R}_n$ in (3) is a monomorphism.

2. Graphs appear: An example

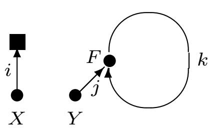
In this section we analyze an example that illustrates how the Invariant Tensor Theorem leads to graphs. We are going to describe invariant tensors in $\mathit{Lin}\big(V^{\otimes 2}\otimes \mathit{Lin}(V^{\otimes 2}, V), V\big)$. The canonical identifications (1) determine a $\mathrm{GL}(V)$--equivariant isomorphism

$$\Phi\colon \mathit{Lin}\big(V^{\otimes 2}\otimes \mathit{Lin}(V^{\otimes 2}, V), V\big) \cong \mathit{Lin}(V^{\otimes 3}, V^{\otimes 3})\,.$$

Applying the Invariant Tensor Theorem to $\mathit{Lin}(V^{\otimes 3}, V^{\otimes 3})$, one concludes that the subspace $\mathit{Lin}_{\mathrm{GL}(V)}(V^{\otimes 2}\otimes \mathit{Lin}(V^{\otimes 2}, V), V)$ is spanned by $\Phi^{-1}(t_\sigma)$, $\sigma\in\Sigma_3$, and that these generators are linearly independent if $\dim(V)\geq 3$. It is a simple exercise to calculate the tensors $\Phi^{-1}(t_\sigma)$ explicitly. The results are shown in the second column of the table in Figure 1 in which $X\otimes Y\otimes F$ is an element of $V^{\otimes 2}\otimes \mathit{Lin}(V^{\otimes 2}, V)$ and $\mathit{Tr}(-)$ the trace of a linear map $V\to V$.

Let us fix a basis $\{e_1,\dots,e_n\}$ of V and write $X = X^a e_a$, $Y = Y^a e_a$ and $F(e_a, e_b) = F^c_{ab} e_c$, for some scalars $X^a, Y^a, F^c_{ab}\in\mathbf{k}$, $1\leq a,b,c\leq n$. The corresponding coordinate forms of the elementary tensors are shown in the third column of the table. Observe that the expressions in this column are all possible *contractions of indices* of the tensors X, Y and F.

The contraction schemes for indices are encoded by the rightmost column as follows. Given a graph G from this column, decorate its edges by symbols i, j, k. For example, for the graph in the bottom right corner of the table, choose the decoration



.

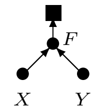
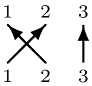
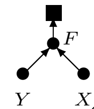
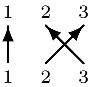
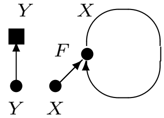
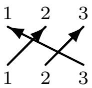
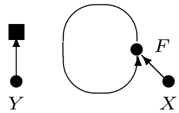
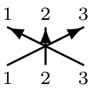
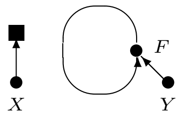
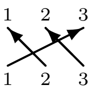
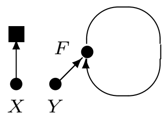
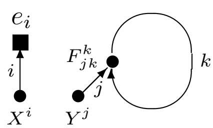
	$\Phi^{-1}(t_\sigma)$:	coordinate form:	graph:
$\sigma = identity$	$X \otimes Y \otimes F \mapsto F(X,Y)$	$X^j Y^k F^i_{jk} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto F(Y,X)$	$X^j Y^k F^i_{kj} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto Y \otimes Tr(F(X,-))$	$X^j Y^i F^k_{jk} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto Y \otimes Tr(F(-,X))$	$X^j Y^i F^k_{kj} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto X \otimes Tr(F(-,Y))$	$X^i Y^j F^k_{kj} e_i$	
$\sigma =$ 	$X \otimes Y \otimes F \mapsto X \otimes Tr(F(Y,-))$	$X^i Y^j F^k_{jk} e_i$	

FIG. 1: Invariant tensors in $Lin(V^{\otimes 2} \otimes Lin(V^{\otimes 2}, V), V)$. The meaning of vertical braces on the right is explained in Example 4.1.

To each vertex of this edge-decorated graph we assign the coordinates of the corresponding tensors with the names of indices determined by decorations of edges adjacent to this vertex. For example, to the F-vertex we assign F^k_{jk}, because its left ingoing edge is decorated by j and its right ingoing edge which happens to be the same as its outgoing edge, is decorated by k. The vertex , called *the anchor*, plays a special role. We assign to it the basis of V indexed by the decoration of its ingoing edge. We get



As the final step we take the product of the factors assigned to vertices and perform the summation over repeated indices. The result is

$$\sum_{1 \le i,j,k \le n} X^i Y^j F^k_{jk} e_i \, .$$

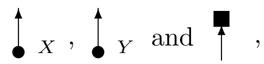
In this formula we made an exception from Einstein's convention and wrote the summation explicitly to emphasize the idea of the construction. A formal general definition of this process of interpreting graphs as contraction schemes is given below.

Let $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}}$ be the vector space spanned by the six graphs in the last column of the table; the hat indicates that the graphs are not oriented. The subscript "ex" is an abbreviation of "example," and distinguishes this space from other spaces with similar names used throughout the note. The procedure described above gives an epimorphism

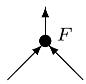
$$\widehat{\mathrm{R}}_n \colon \widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}} \to \mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Lin}(V^{\otimes 2}, V), V\right) \tag{5}$$

which is an isomorphism if $n \geq 3$. The map $\widehat{R}_n$ defined in this way obviously does not depend on the choice of the basis $\{e_1, \dots, e_n\}$ of V.

The space $\widehat{\mathfrak{G}\mathrm{r}}_{\mathrm{ex}}$ can also be defined as the span of all directed graphs with three unary vertices

(6) 

and one "planar" binary vertex

(7) 

whose planarity means that its inputs are linearly ordered. In pictures, this order is determined by reading the inputs from left to right.

3. The general case

Let us generalize calculations in Section 2 and describe $\mathrm{GL}(V)$-invariant elements in

$$\mathit{Lin}\left(\mathit{Lin}(V^{\otimes h_1}, V^{\otimes p_1}) \otimes \cdots \otimes \mathit{Lin}(V^{\otimes h_r}, V^{\otimes p_r}), \mathit{Lin}(V^{\otimes c}, V^{\otimes d})\right) , \tag{8}$$

where $r, p_1, \dots, p_r, h_1, \dots, h_r, c$ and d are non-negative integers. The above space is canonically isomorphic to

$$V^{*\otimes p_1} \otimes V^{\otimes h_1} \otimes \cdots \otimes V^{*\otimes p_r} \otimes V^{\otimes h_r} \otimes V^{*\otimes c} \otimes V^{\otimes d},$$

which is in turn isomorphic to

$$V^{*\otimes(p_1+\cdots+p_r+c)} \otimes V^{\otimes(h_1+\cdots+h_r+d)}, \tag{9}$$

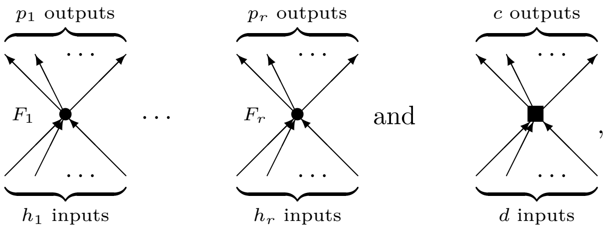
via the isomorphism that moves all V^*-factors to the left, without changing their relative order. By the last and first isomorphisms in (1), the space in (9) is isomorphic to

$$\mathit{Lin}(V^{\otimes(p_1+\cdots+p_r+c)}, V^{\otimes(h_1+\cdots+h_r+d)}) .$$

We will denote the composite isomorphism between (8) and the space in the above display by Φ. Since all isomorphisms above are $\mathrm{GL}(V)$-equivariant, Φ is equivariant, too, thus the space (8) may contain nontrivial $\mathrm{GL}(V)$-equivariant maps only if

$$p_1 + \cdots + p_r + c = h_1 + \cdots + h_r + d\,. \tag{10}$$

Denote by $\widehat{\mathfrak{G}\mathrm{r}}$ the space spanned by all directed graphs with $r+1$ planar vertices



where planarity means that linear orders of the sets of input and output edges are specified. Observe that the number of edges of each graph spanning $\widehat{\mathfrak{G}\mathrm{r}}$ equals the common value of the sums in (10). For each graph $G \in \widehat{\mathfrak{G}\mathrm{r}}$ we define a $\mathrm{GL}(V)$-equivariant map $\widehat{\mathrm{R}}_n(G)$ in the space (8) as follows.

As in Section 2, choose a basis $(e_1, \ldots, e_n)$ of V and let $(e^1, \ldots, e^n)$ be the corresponding dual basis of V^*. For $F_i \in \mathit{Lin}(V^{\otimes h_i}, V^{\otimes p_i})$, $1 \leq i \leq r$, write

$$F_i = F_i{}^{a_1^i, \ldots, a_{p_i}^i}_{b_1^i, \ldots, b_{h_i}^i} \; e_{a_1} \otimes \cdots \otimes e_{a_{p_i}} \otimes e^{b_1} \otimes \cdots \otimes e^{b_{h_i}}$$

with some scalars $F_i{}^{a_1^i, \ldots, a_{p_i}^i}_{b_1^i, \ldots, b_{h_i}^i} \in \mathbf{k}$ or, more concisely, $F_i = F_i{}^{A^i}_{B^i} \; e_{A^i} \otimes e^{B^i}$, where A^i abbreviates the multiindex $(a_1^i, \ldots, a_{p_i}^i)$, B^i the multiindex $(b_1^i, \ldots, b_{h_i}^i)$, $e_{A^i} := e_{a_1} \otimes \cdots \otimes e_{a_{p_i}}$, $e^{B^i} := e^{b_1} \otimes \cdots \otimes e^{b_{h_i}}$ and, as everywhere in this paper, summations over repeated (multi)indices are assumed.

A *labelling* of a graph $G \in \widehat{\mathfrak{G}\mathrm{r}}$ is a function $\ell : \mathit{Edg}(G) \to \{1, \ldots, n\}$, where $\mathit{Edg}(G)$ denotes the set of edges of G. Let $\mathit{Lab}(G)$ be the set of all labellings of G. For $\ell \in \mathit{Lab}(G)$ and $1 \leq i \leq r$, define $A^i(\ell)$ to be the multiindex $(a_1^i, \ldots, a_{p_i}^i)$ such that a_s^i equals $\ell(e)$, where e is the edge that starts at the s-th output of the vertex F_i, $1 \leq s \leq p_i$. Likewise, put $I(\ell) := (i_1, \ldots, i_c)$ with $i_t := \ell(e)$, where now e is the edge that starts at the t-th output of the $\blacksquare$-vertex, $1 \leq t \leq c$. Let $B^i(\ell)$ and $J(\ell)$ have similar obvious meanings, with 'inputs' taken instead of 'outputs.' For $F_1 \otimes \cdots \otimes F_r \in \mathit{Lin}(V^{\otimes h_1}, V^{\otimes p_1}) \otimes \cdots \otimes \mathit{Lin}(V^{\otimes h_r}, V^{\otimes p_r})$ define finally an element $\widehat{\mathrm{R}}_n(G)(F_1 \otimes \cdots \otimes F_r) \in \mathit{Lin}(V^{\otimes c}, V^{\otimes d})$ by

$$\widehat{\mathrm{R}}_n(G)(F_1 \otimes \cdots \otimes F_r) := \sum_{\ell \in \mathit{Lab}(G)} F_1{}^{A^1(\ell)}_{B^1(\ell)} \otimes \cdots \otimes F_r{}^{A^r(\ell)}_{B^r(\ell)} \; e_{J(\ell)} \otimes e^{I(\ell)} . \tag{11}$$

It is easy to check that $\widehat{\mathrm{R}}_n(G)$ is a $\mathrm{GL}(V)$-fixed element of the space (8). The nature of the summation in (11) is close to the *state sum model* for link invariants, see [2, Section I.8], with states being the values of labels of the edges of the graph.

3.1. **Proposition.** *Let $r, p_1, \ldots, p_r, h_1, \ldots, h_r, c$ and d be non-negative integers. Then the map*

$$\widehat{\mathrm{R}}_n \colon \widehat{\mathfrak{G}\mathrm{r}} \to \mathit{Lin}_{\mathrm{GL}(V)}\big(\mathit{Lin}(V^{\otimes h_1}, V^{\otimes p_1}) \otimes \cdots \otimes \mathit{Lin}(V^{\otimes h_r}, V^{\otimes p_r}), \mathit{Lin}(V^{\otimes c}, V^{\otimes d})\big)$$

defined by (11) *is an epimorphism. If $n \geq e$, where e is the number of edges of graphs spanning $\widehat{\mathfrak{G}\mathrm{r}}$ and $n = \dim(V)$, $\widehat{\mathrm{R}}_n$ is also an isomorphism.*

Observe that we do not need to assume (10) in Proposition 3.1. If (10) is not satisfied, then there are no $\mathrm{GL}(V)$-invariant elements in (8) and also the space $\widehat{\mathfrak{Gr}}$ is trivial, thus $\widehat{\mathrm{R}}_n$ is an isomorphism of trivial spaces.

Proof of Proposition 3.1. By the above observation, we may assume (10). Consider the diagram

$$
\begin{tikzcd}
\mathbf{k}[\Sigma_k] \arrow[r, "\mathcal{R}_n"] & Lin_{\mathrm{GL}(V)}\big(V^{\otimes(p_1+\cdots+p_r+c)}, V^{\otimes(h_1+\cdots+h_r+d)}\big) \\
\widehat{\mathfrak{Gr}} \arrow[u, "\Psi \cong"] \arrow[r, "\widehat{\mathrm{R}}_n"] & Lin_{\mathrm{GL}(V)}\big(Lin(V^{\otimes h_1},V^{\otimes p_1})\otimes\cdots\otimes Lin(V^{\otimes h_r},V^{\otimes p_r}), Lin(V^{\otimes c},V^{\otimes d})\big) \arrow[u, "\Phi \cong"']
\end{tikzcd}
\tag{12}
$$

in which $\mathcal{R}_n$ is the map (3), $\widehat{\mathrm{R}}_n$ is defined in (11) and Φ is the composition of canonical isomorphisms and reshufflings of factors described on page 453 above. The map Ψ is defined as follows.

Let us denote, for the purposes of this proof only, by $\mathrm{Ou}(F_i)$ the linearly ordered set of outputs of the F_i-vertex, $1 \le i \le r$, and by $\mathrm{Ou}(\blacksquare)$ the linearly ordered set of outputs of $\blacksquare$. The set $\mathrm{Ou} := \mathrm{Ou}(F_1) \cup \cdots \cup \mathrm{Ou}(F_r) \cup \mathrm{Ou}(\blacksquare)$ is linearly ordered by requiring that

$$\mathrm{Ou}(F_1) < \cdots < \mathrm{Ou}(F_r) < \mathrm{Ou}(\blacksquare)$$

(we believe that the meaning of this shorthand is obvious). Let In be the linearly ordered set of inputs defined in the similar way. The orders define unique isomorphisms

$$\mathrm{Ou} \cong (1, \dots, k) \ \text{ and } \mathrm{In} \cong (1, \dots, k) \tag{13}$$

of ordered sets.

Since graphs spanning $\widehat{\mathfrak{Gr}}$ are determined by specifying how the outputs of vertices are connected to its inputs, there exists a one-to-one correspondence $G \leftrightarrow \varphi_G$ between graphs $G \in \widehat{\mathfrak{Gr}}$ and isomorphisms $\varphi_G \colon \mathrm{Ou} \xrightarrow{\cong} \mathrm{In}$. Given (13), such φ_G can be interpreted as an element of the symmetric group Σ_k. The map Ψ is then defined by $\Psi(G) := \varphi_G$.

It is simple to verify that the diagram (12) commutes, so the proposition follows from the Invariant Tensor Theorem. $\square$

4. Symmetries occur

In the light of diagram (12), Proposition 3.1 may look just as a clumsy reformulation of the Invariant Tensor Theorem. Graphs become relevant when symmetries occur.

4.1. **Example.** Let $Sym(V^{\otimes 2}, V) \subset Lin(V^{\otimes 2}, V)$ be the subspace of symmetric bilinear maps, i.e. maps satisfying $f(v', v'') = f(v'', v')$ for $v', v'' \in V$. Let us explain how to use calculations of Section 2 to describe $\mathrm{GL}(V)$-equivariant maps in $Lin\left(V^{\otimes 2} \otimes Sym(V^{\otimes 2}, V), V\right)$.

The right Σ_2-action on $Lin(V^{\otimes 2}, V)$ given by permuting the inputs of bilinear maps is such that the space $Sym(V^{\otimes 2}, V)$ equals the subspace $Lin(V^{\otimes 2}, V)^{\Sigma_2}$ of Σ_2-fixed elements. This right Σ_2-action induces a left Σ_2-action on the space $Lin\left(V^{\otimes 2}\otimes Lin(V^{\otimes 2}, V), V\right)$ which commutes with the $\mathrm{GL}(V)$-action, therefore it restricts to a left Σ_2-action on the subspace $Lin_{\mathrm{GL}(V)}\left(V^{\otimes 2}\otimes Lin(V^{\otimes 2}, V), V\right)$ of $\mathrm{GL}(V)$-equivariant maps.

There is also a left Σ_2-action on the linear space $\widehat{\mathfrak{Gr}}_{\mathrm{ex}}$ interchanging the inputs of the F-vertices of generating graphs. It is simple to check that the map (5) of Section 2 is equivariant with respect to these two Σ_2-actions, hence it induces the map

$$\Sigma_2\backslash\widehat{\mathrm{R}}_n : \Sigma_2\backslash\widehat{\mathfrak{Gr}}_{\mathrm{ex}} \to \Sigma_2\backslash Lin_{\mathrm{GL}(V)}\left(V^{\otimes 2}\otimes Lin(V^{\otimes 2}, V), V\right) \tag{14}$$

of left cosets. Observe that, by a standard duality argument,

$$\Sigma_2\backslash Lin_{\mathrm{GL}(V)}\big(V^{\otimes 2}\otimes Lin(V^{\otimes 2}, V),V\big) \cong Lin_{\mathrm{GL}(V)}\big(V^{\otimes 2}\otimes Sym(V^{\otimes 2}, V),V\big)\,. \tag{15}$$

Let us denote $\widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet} := \Sigma_2\backslash\widehat{\mathfrak{Gr}}_{\mathrm{ex}}$. The bullet $\bullet$ in the subscript signalizes the presence of vertices with fully symmetric inputs. By definition, graphs $G', G'' \in \widehat{\mathfrak{Gr}}_{\mathrm{ex}}$ are identified in the quotient $\widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet}$ if they differ only by the order of inputs of the F-vertex. In Figure 1, this identification is indicated by vertical braces. We see that $\widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet}$ is again a space *spanned by graphs,* this time with no linear order on the inputs of the F-vertex. So we may *define* $\widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet}$ as the space spanned by directed graphs with vertices (6) and one binary (ordinary, non-planar) vertex (7). We conclude by interpreting (14) as the map

$$\widehat{\mathrm{R}}_n : \widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet} \to Lin_{\mathrm{GL}(V)}\left(V^{\otimes 2}\otimes Sym(V^{\otimes 2}, V), V\right)\,. \tag{16}$$

It follows from the properties of the map (5) and the characteristic zero assumption that $\widehat{\mathrm{R}}_n$ is always an epimorphism and is an isomorphism if $n \geq 3$.

At this point we want to incorporate, by generalizing the pattern used in Example 4.1, symmetries into Proposition 3.1. Unfortunately, it turns out that treating the space (8) in full generality leads to a notational disaster. To keep the length of formulas within a reasonable limit, we decided to *assume from now on* that $p_1 = \cdots = p_r = 1$, $c = 0$ and $d = 1$. This means that we will restrict our attention to maps in

$$Lin\left(Lin(V^{\otimes h_1}, V)\otimes\cdots\otimes Lin(V^{\otimes h_r}, V), V\right)\,. \tag{17}$$

For graphs this assumption implies that the vertices $F_1,\ldots,F_r$ have precisely one output, and that the anchor $\blacksquare$ has one input and no outputs. The number of inputs of F_i will be called the *arity* of F_i, $1 \leq i \leq r$. Condition (10) reduces to

$$r = h_1 + \cdots + h_r + 1$$

and one also sees that r equals the number of edges of the generating graphs.

The above generality is sufficient for all applications we have in mind. A modification to the general case is straightforward but notationally challenging.

The space $Lin(V^{\otimes h}, V)$ admits, for each $h \geq 0$, a natural right Σ_h-action given by permuting inputs of multilinear maps. A *symmetry* of maps in $Lin(V^{\otimes h}, V)$ will be specified by a subset $\mathfrak{I} \subset \mathbf{k}[\Sigma_h]$. We then denote

$$Lin_{\mathfrak{I}}(V^{\otimes h}, V) := \left\{ f \in Lin(V^{\otimes h}, V);\ f\mathfrak{s} = 0 \text{ for each } \mathfrak{s} \in \mathfrak{I} \right\} .$$

For $\mathfrak{I}$ as above and a left Σ_h-module U, we will abbreviate by $\mathfrak{I}\backslash U$ the left coset $\mathfrak{I}U\backslash U$.

4.2. **Example.** Let $\mathfrak{I} := I_h \subset \mathbf{k}[\Sigma_h]$ be the augmentation ideal. Then the space $Lin_{I_h}(V^{\otimes h}, V)$ is the space of symmetric maps,

$$Lin_{I_h}(V^{\otimes h}, V) = Sym(V^{\otimes h}, V)\,,$$

therefore the augmentation ideal describes the symmetry of the local coordinates of vector fields and their derivatives, see [5, Example 3.2]. We leave as an exercise to describe in this language the spaces of *anti*symmetric maps.

4.3. **Example.** Let $h := v + 2$, $v \geq 0$, and let $\nabla \subset \mathbf{k}[\Sigma_h]$ be the image of the augmentation ideal I_v of $\mathbf{k}[\Sigma_v]$ in $\mathbf{k}[\Sigma_h]$ under the map of group rings induced by the inclusion $\Sigma_v \hookrightarrow \Sigma_v \times \Sigma_2 \hookrightarrow \Sigma_h$ that interprets permutations of $(1, \ldots, v)$ as permutations of $(1, \ldots, v, v+1, v+2)$ keeping the last two elements fixed. Then $Lin_{\nabla}(V^{\otimes h}, V)$ consists of multilinear maps $V^{\otimes(v+2)} \to V$ that are symmetric in the first v inputs, i.e. multilinear maps possessing the symmetry of the Christoffel symbols of linear connections and their derivatives, see again [5, Example 3.2].

4.4. **Remark.** It is clear how to generalize the above notion of symmetry to maps in the left Σ_p- right Σ_h-module $Lin(V^{\otimes h}, V^{\otimes p})$ for general $p, h \geq 0$. A symmetry of these maps will be specified by subsets $\mathfrak{I} \in \mathbf{k}[\Sigma_h]$ and $\mathfrak{O} \in \mathbf{k}[\Sigma_p]$, the corresponding subspaces will then be

$$Lin_{\mathfrak{I}}^{\mathfrak{O}}(V^{\otimes h}, V^{\otimes p}) := \left\{ f \in Lin(V^{\otimes h}, V^{\otimes p}); f\mathfrak{s} = 0 = \mathfrak{t}f \text{ for each } \mathfrak{s} \in \mathfrak{I}, \mathfrak{t} \in \mathfrak{O} \right\} .$$

Suppose we are given subsets $\mathfrak{I}_i \subset \mathbf{k}[\Sigma_{h_i}]$, $1 \leq i \leq r$. Our aim is to describe $\mathrm{GL}(V)$-invariant elements in the space

$$Lin\left(Lin_{\mathfrak{I}_1}(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin_{\mathfrak{I}_r}(V^{\otimes h_r}, V), V\right) . \tag{18}$$

Let

$$\mathfrak{I} := \mathfrak{I}_1 \cup \cdots \cup \mathfrak{I}_r \subset \mathbf{k}[\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}]\,,$$

where $\mathfrak{I}_i$ is, for $1 \leq i \leq r$, identified with its image in $\mathbf{k}[\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}]$ under the map induced by the group inclusion $\Sigma_{h_i} \hookrightarrow \Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$.

As in Example 4.1, we use the fact that, for $1 \leq i \leq r$, each $Lin(V^{\otimes h_i}, V)$ is a right Σ_{h_i}-space, hence the tensor product $Lin(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin(V^{\otimes h_r}, V)$ has a natural right $\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$-action which induces a left $\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$-action on the space (17). This action restricts to the subspace of $\mathrm{GL}(V)$-equivariant maps.

There is also a left $\Sigma_{h_1} \times \cdots \times \Sigma_{h_r}$-action on the space $\widehat{\mathfrak{Gr}}$ given by permuting, in the obvious manner, the inputs of the vertices $F_1, \ldots, F_r$ of generating graphs. The map $\widehat{\mathrm{R}}_n$ of Proposition 3.1 is equivariant with respect to the above two actions and induces the map

$$\mathfrak{I}\backslash\widehat{\mathrm{R}}_n \colon \mathfrak{I}\backslash\widehat{\mathfrak{Gr}} \to \mathfrak{I}\backslash Lin_{\mathrm{GL}(V)}\left(Lin(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin(V^{\otimes h_r}, V), V\right)$$

of left quotients. Denoting $\widehat{\mathfrak{Gr}}_{\mathfrak{J}} := \mathfrak{J}\backslash\widehat{\mathfrak{Gr}}$ and realizing that, by duality, the codomain of $\mathfrak{J}\backslash\widehat{\mathrm{R}}_n$ is isomorphic to the subspace of $\mathrm{GL}(V)$-fixed elements in (18), we obtain the map (denoted again $\widehat{\mathrm{R}}_n$)

$$\widehat{\mathrm{R}}_n \colon \widehat{\mathfrak{Gr}}_{\mathfrak{J}} \to \mathit{Lin}_{\mathrm{GL}(V)}\left(\mathit{Lin}_{\mathfrak{J}_1}(V^{\otimes h_1}, V) \otimes \cdots \otimes \mathit{Lin}_{\mathfrak{J}_r}(V^{\otimes h_r}, V), V\right) \tag{19}$$

which is, by Proposition 3.1, an epimorphism and is an isomorphism if $\dim(V) \geq r$.

4.5. **Remark.** As in Example 4.1, it turns out that the quotient $\widehat{\mathfrak{Gr}}_{\mathfrak{J}} = \mathfrak{J}\backslash\widehat{\mathfrak{Gr}}$ is a *space of graphs* though, for general symmetries, "space of graphs" means a free wheeled operad on a certain Σ-module [6]. In the cases relevant for our paper, we however remain in the realm of 'classical' graphs, as shown in the following example, see also the proof of Corollary 5.1.

4.6. **Example.** Suppose that, for some $1 \leq i \leq r$, $\mathfrak{J}_i$ equals the augmentation ideal I_{h_i} of $\mathbf{k}[\Sigma_{h_i}]$ as in Example 4.2. Then, in the quotient $\mathfrak{J}\backslash\widehat{\mathfrak{Gr}}$, one identifies graphs that differ by the order of inputs of the vertex F_i. In other words, modding out by $\mathfrak{J}_i \subset \mathfrak{J}$ erases the order of inputs of F_i, turning F_i into an ordinary (non-planar) vertex. If $\mathfrak{J}_i = \nabla$ as in Example 4.3, one gets a vertex of arity $v+2$, $v \geq 0$, whose first v inputs are symmetric.

For applications, we still need one more level of generalization that will reflect the antisymmetry of the Chevalley-Eilenberg complex [5, Section 2] in the Lie algebra variables. As a motivation for our construction, we offer the following continuation of the calculations in Section 2 and Example 4.1.

4.7. **Example.** We will consider the tensor product $V \otimes V$ as a left Σ_2-module, with the action $\tau(v' \otimes v'') := -(v'' \otimes v')$, for $v', v'' \in V$ and the generator $\tau \in \Sigma_2$. The subspace $(V \otimes V)^{\Sigma_2}$ of Σ_2-fixed elements is then precisely the second exterior power $\wedge^2 V$. This left action induces a $\mathrm{GL}(V)$-equivariant right Σ_2-action on the space $\mathit{Lin}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)$ such that

$$\mathit{Lin}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)/\Sigma_2 \cong \mathit{Lin}\left(\wedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right).$$

The above isomorphism restricts to an isomorphism

$$\mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)/\Sigma_2 \cong \mathit{Lin}_{\mathrm{GL}(V)}\left(\wedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right). \tag{20}$$

of the subspaces of $\mathrm{GL}(V)$-equivariant maps.

Likewise, $\widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet}$ carries a right Σ_2-action that interchanges the labels X and Y of the $\overset{\uparrow}{\bullet}$-vertices of graphs in the last column of Figure 1 and multiplies the sign of the corresponding generator by -1. The map (16) is Σ_2-equivariant, therefore it induces the map

$$\widehat{\mathrm{R}}_n/\Sigma_2 : \widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet}/\Sigma_2 \to \mathit{Lin}_{\mathrm{GL}(V)}\left(V^{\otimes 2} \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)/\Sigma_2 .$$

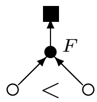
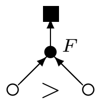
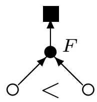
Let us denote $\mathfrak{Gr}^2_{\mathrm{ex},\bullet} := \widehat{\mathfrak{Gr}}_{\mathrm{ex},\bullet}/\Sigma_2$ and $\mathrm{R}^2_n := \widehat{\mathrm{R}}_n/\Sigma_2$. Using (20), one rewrites the above map as an epimorphism

$$\mathrm{R}^2_n \colon \mathfrak{Gr}^2_{\mathrm{ex},\bullet} \twoheadrightarrow \mathit{Lin}_{\mathrm{GL}(V)}\left(\wedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V), V\right)$$

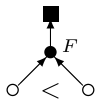
which is an isomorphism if $n \geq 3$.

The space $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$ is isomorphic to the span of the set of directed, oriented graphs with one (non-planar) binary vertex F, an anchor , and two 'white' vertices . By an *orientation* we mean a linear order of white vertices. A graph with the opposite orientation is identified with the original one taken with the opposite sign. It is clear that, with $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$ defined in this way, the map $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet} \to \widehat{\mathcal{G}\mathrm{r}}_{\mathrm{ex},\bullet}/\Sigma_2$ that replaces the first (in the linear order given by the orientation) white vertex by the black vertex labelled by X, and the second white vertex by the black vertex labelled by Y, is an isomorphism.

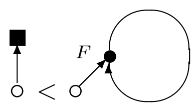
The symmetry of the inputs of the vertex F implies the following identities in $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$:

 $= -$  $= -$  ,

from which one concludes that

 $= 0.$

Therefore $\mathcal{G}\mathrm{r}^2_{\mathrm{ex},\bullet}$ is in this case one-dimensional, spanned by the equivalence class of the oriented directed graph

 .

In the notation of Figure 1, the above graph represents the map that sends $(X \wedge Y) \otimes F \in \wedge^2 V \otimes \mathit{Sym}(V^{\otimes 2}, V)$ into

$$X \otimes \mathit{Tr}(F(Y,-)) - Y \otimes \mathit{Tr}(F(X,-)) \in V\,.$$

Let us turn to our final task. We want to describe $\mathrm{GL}(V)$-invariant elements in the space

$$\mathit{Lin}\Big(\bigwedge_{i=1}^{m} \mathit{Sym}(V^{\otimes h_i}, V) \otimes \bigotimes_{i=m+1}^{r} \mathit{Lin}_{\mathfrak{I}_i}(V^{\otimes h_i}, V), V\Big) \tag{21}$$

where, as before, $r, h_1, \ldots, h_r$ are positive integers, $\mathfrak{I}_i \subset \mathbf{k}[\Sigma_{h_i}]$ for $m+1 \leq i \leq r$, and m is an integer such that $1 \leq m \leq r$. Having in mind the description of the space of symmetric multilinear maps given in Example 4.2, we extend the definition of $\mathfrak{I}_i$ also to $1 \leq i \leq m$, by putting $\mathfrak{I}_i := I_{h_i}$. The first step is to identify the exterior power $\bigwedge_{1 \leq i \leq m} \mathit{Sym}(V^{\otimes h_i}, V)$ with the fixed point set of an action of a suitable finite group. This can be done as follows.

For $1 \leq w \leq m$, let $A(w) \subset \{1, \ldots, m\}$ be the subset $A(w) := \{1 \leq i \leq m;\ h_i = h_w\}$. Then

$$\{1, \ldots, m\} = \textstyle\bigcup_{1 \leq w \leq m} A(w)$$

is a decomposition of $\{1, \ldots, m\}$ into not necessarily distinct subsets. Let $\widehat{\Sigma} \subset \Sigma_m$ be the subgroup of permutations of $\{1, \ldots, m\}$ preserving this decomposition.

The group $\widehat{\Sigma}$ acts on $\bigotimes_{1\leq i\leq m} Sym(V^{\otimes h_i}, V)$ by permuting the corresponding factors. If we consider this tensor product as a left $\widehat{\Sigma}$-module with this permutation action twisted by the signum representation, then

$$\bigwedge_{i=1}^{m} Sym(V^{\otimes h_i}, V) \cong \Big(\bigotimes_{i=1}^{m} Sym(V^{\otimes h_i}, V)\Big)^{\widehat{\Sigma}}.$$

The above left $\widehat{\Sigma}$-action on $\bigotimes_{1\leq i\leq m} Sym(V^{\otimes h_i}, V)$ induces a dual right $\mathrm{GL}(V)$--equivariant $\widehat{\Sigma}$-action on the space (21).

There is a right $\widehat{\Sigma}$-action on the quotient $\widehat{\mathfrak{Gr}}_{\mathfrak{I}} = \mathfrak{I}\backslash\widehat{\mathfrak{Gr}}$ defined as follows. For a graph $G \in \widehat{\mathfrak{Gr}}$ representing an element $[G] \in \widehat{\mathfrak{Gr}}_{\mathfrak{I}}$ and for $\sigma \in \widehat{\Sigma}$, let G^σ be the graph obtained from G by permuting the vertices $F_1, \ldots, F_m$ according to σ. We then put $[G]\sigma := \mathrm{sgn}(\sigma)[G^\sigma]$. Since, by the definition of $\widehat{\Sigma}$, σ may interchange only vertices with the same number of inputs and the same symmetry, our definition of G^σ makes sense.

It is simple to see that the map $\widehat{\mathrm{R}}_n$ in (19) is $\widehat{\Sigma}$-equivariant, giving rise to the map

$$\widehat{\mathrm{R}}_n/\widehat{\Sigma}\colon \widehat{\mathfrak{Gr}}_{\mathfrak{I}}/\widehat{\Sigma} \to Lin_{\mathrm{GL}(V)}(Lin_{\mathfrak{I}_1}(V^{\otimes h_1}, V) \otimes \cdots \otimes Lin_{\mathfrak{I}_r}(V^{\otimes h_r}, V), V)/\widehat{\Sigma}$$

of right cosets. The codomain of $\widehat{\mathrm{R}}_n/\widehat{\Sigma}$ is easily seen to be isomorphic to the subspace of $\mathrm{GL}(V)$-equivariant elements in (21). The above calculations are summarized in the following proposition in which $\mathfrak{Gr}^m_{\mathfrak{I}} := \widehat{\mathfrak{Gr}}_{\mathfrak{I}}/\widehat{\Sigma}$ and $\mathrm{R}^m_n := \widehat{\mathrm{R}}_n/\widehat{\Sigma}$.

4.8. **Proposition.** *Let $r, h_1, \ldots, h_r$ be non-negative integers, $1 \leq m \leq r$, and $\mathfrak{I}_i \subset \mathbf{k}[\Sigma_{h_i}]$ for $m+1 \leq i \leq r$. Then the map*

$$\mathrm{R}^m_n : \mathfrak{Gr}^m_{\mathfrak{I}} \to Lin_{\mathrm{GL}(V)}\Big(\bigwedge_{i=1}^{m} Sym(V^{\otimes h_i}, V) \otimes \bigotimes_{i=m+1}^{r} Lin_{\mathfrak{I}_i}(V^{\otimes h_i}, V), V\Big) \tag{22}$$

constructed above is an epimorphism. If, moreover, the dimension n of $V \geq$ the number of edges of graphs spanning $\mathfrak{Gr}^m_{\mathfrak{I}}$, R^m_n is also an isomorphism.

The following result says that the presence of vertices with symmetric inputs miraculously *extends* the stability range (Definition 1.1). In applications, these vertices will represent the Lie algebra generators in the Chevalley-Eilenberg complex.

4.9. **Proposition.** *Suppose that $h_1, \ldots, h_m \geq 2$. If $n \geq e - m$, where n is the dimension of V and e the number of edges of graphs spanning $\mathfrak{Gr}^m_{\mathfrak{I}}$, then the map R^m_n in Proposition 4.8 is an isomorphism.*

Proof. Let G be a graph spanning $\mathfrak{Gr}^m_{\mathfrak{I}}$ and $S \subset Edg(G)$ a subset of edges of G such that $\mathrm{card}(S) > n$. For each permutation σ of elements of S, denote by G_σ the graph obtained by cutting the edges belonging to S in the middle and regluing them following the automorphism σ. The linear combination

$$\sum_{\sigma\in\Sigma_S} \mathrm{sgn}(\sigma)\cdot G_\sigma \in \mathfrak{Gr}^m_{\mathfrak{I}} \tag{23}$$

is then a graph-ical representation of the expression in (4), thus the kernel of R^m_n is generated by expressions of this type. Since, by assumption, $\mathrm{card}(S) \leq n+m$ and $h_1, \ldots, h_m \geq 2$, the set S must necessarily contain two input edges of the *same* symmetric vertex of G. This implies that the sum (23) vanishes, because with each graph G_σ it contains the same graph with the opposite sign. This shows that the kernel of R^m_n is trivial. $\square$

4.10. **Remark.** By an absolutely straightforward generalization of the above constructions, one can obtain versions of Proposition 4.8 and Proposition 4.9 describing the space

$$Lin_{\mathrm{GL}(V)}\Big(\bigwedge_{i=1}^{m} Sym(V^{\otimes h_i}, V) \otimes \bigotimes_{i=m+1}^{r} Lin^{\mathfrak{D}_i}_{\mathfrak{I}_i}(V^{\otimes h_i}, V^{\otimes p_i}), Lin^{\mathfrak{D}}_{\mathfrak{I}}(V^{\otimes c}, V^{\otimes d})\Big) \tag{24}$$

in terms of a space spanned by graphs. Since the notational aspects of such a generalization are horrendous, we must leave the details as an exercise to the reader.

5. A particular case

We finish this note by a corollary tailored for the needs of [5]. For non-negative integers m, b and c, denote by $\mathcal{G}\mathrm{r}^m_{\bullet(b)\nabla(c)}$ the space spanned by directed, oriented graphs with

(i) m unlabeled 'white' vertices with fully symmetric inputs and arities ≥ 2,
(ii) b 'black' labelled vertices with fully symmetric inputs and arities ≥ 0,
(iii) c labelled ∇-vertices, and
(iv) the anchor $\overset{\blacksquare}{\uparrow}$.

In item (iii), a ∇-vertex means a vertex with the symmetry described in Example 4.3, see also Example 4.6. As in Example 4.7, an *orientation* is given by a linear order on the set of white vertices. If G' and G'' are graphs in $\mathcal{G}\mathrm{r}^m_{\bullet(b)\nabla(c)}$ whose orientations differ by an odd number of transpositions, then we identify $G' = -G''$ in $\mathcal{G}\mathrm{r}^m_{\bullet(b)\nabla(c)}$.

5.1. **Corollary.** *For each non-negative integers m, b and c there exists a natural epimorphism*

$$\mathrm{R}^m_{\bullet(b)\nabla(c),n} : \mathcal{G}\mathrm{r}^m_{\bullet(b)\nabla(c)} \twoheadrightarrow$$
$$\bigoplus_{\vec{h}\in\mathfrak{H}} Lin_{\mathrm{GL}(V)}\Big(\bigwedge_{i=1}^{m} Sym(V^{\otimes h_i}, V) \otimes \bigotimes_{i=m+1}^{m+b} Sym(V^{\otimes h_i}, V) \bigotimes_{i=m+b+1}^{m+b+c} Lin_\nabla(V^{\otimes h_i}, V), V\Big),$$

with the direct sum taken over the set $\mathfrak{H}$ of all multiindices $\vec{h} = (h_1, \ldots, h_{m+b+c})$ such that

$$h_1, \ldots, h_m \geq 2\,, \quad h_{m+1}, \ldots, h_{m+b} \geq 0 \quad \text{and} \quad h_{m+b+1}, \ldots, h_{m+b+c} \geq 2\,.$$

The map $\mathrm{R}^m_{\bullet(b)\nabla(c),n}$ is an isomorphism if $n = \dim(V) \geq b + c$.

Proof. The map $\mathrm{R}^m_{\bullet(b)\nabla(c),n}$ is constructed by assembling the maps R^m_n from Proposition 4.8 as follows. For a multiindex $\vec{h} = (h_1, \ldots, h_{m+b+c}) \in \mathfrak{H}$ as in the corollary take, in Proposition 4.8, $r := m + b + c$ and

$$\mathfrak{I}_i = \mathfrak{I}_i(\vec{h}) := \begin{cases} I_{h_i}, & \text{for } m+1 \leq i \leq m+b \quad \text{and} \\ \nabla, & \text{for } m+b+1 \leq i \leq r, \end{cases}$$

see Examples 4.2 and 4.3 for the notation. Let $\mathrm{R}^m_n(\vec{h})$ be the map (22) corresponding to the above choices and $\mathrm{R}^m_{\bullet(b)\nabla(c),n} := \bigoplus_{\vec{h}\in\mathfrak{H}} \mathrm{R}^m_n(\vec{h})$. We only need to show that the graph space $\mathfrak{Gr}^m_{\bullet(b),\nabla(c)}$ is isomorphic to the direct sum of the double quotients $\mathfrak{Gr}^m_{\mathfrak{I}(\vec{h})} = \mathfrak{I}(\vec{h})\backslash\widehat{\mathfrak{Gr}}/\widehat{\Sigma}$.

As we argued in Example 4.6, the left quotient $\widehat{\mathfrak{Gr}}_{\mathfrak{I}(\vec{h})} = \mathfrak{I}(\vec{h})\backslash\widehat{\mathfrak{Gr}}$ is spanned by directed graphs with r labelled vertices $F_1, \ldots, F_r$ such that the 1st type vertices $F_1, \ldots, F_m$ ('white' vertices) have fully symmetric inputs and arities $h_1, \ldots, h_m$, and the remaining vertices $F_{m+1}, \ldots, F_r$ are as in items (ii)–(iv) of the definition of $\mathfrak{Gr}^m_{\bullet(b)\nabla(c)}$ but with fixed arities $h_{m+1}, \ldots, h_r$.

Modding out $\widehat{\mathfrak{Gr}}_{\mathfrak{I}(\vec{h})}$ by $\widehat{\Sigma}$ identifies graphs that differ by a relabelling of white vertices of the same arity and the sign given by to the signum of this relabelling. This clearly means that the map

$$\mathfrak{Gr}^m_{\bullet(b),\nabla(c)} \to \bigoplus_{\vec{h}\in\mathfrak{H}} \mathfrak{Gr}^m_{\mathfrak{I}(\vec{h})} = \bigoplus_{\vec{h}\in\mathfrak{H}} \widehat{\mathfrak{Gr}}_{\mathfrak{I}(\vec{h})}/\widehat{\Sigma}$$

that assigns to the first (in the linear order given by the orientation) white vertex of graphs generating $\mathfrak{Gr}^m_{\bullet(b),\nabla(c)}$ label F_1, to the second white vertex label F_2, etc., is an isomorphism. By simple combinatorics, graphs spanning $\mathfrak{Gr}^m_{\bullet(b),\nabla(c)}$ have precisely $m + b + c$ edges which completes the proof of the corollary. □

5.2. **Remark.** Proposition 4.8 and its Corollary 5.1 was obtained by applying the double-coset reduction $\mathfrak{I}\backslash -/\widehat{\Sigma}$ and standard duality to the map $\widehat{\mathrm{R}}_n$ of Proposition 3.1. Backtracking all the constructions involved, one can see that, in Corollary 5.1, the invariant linear map $\mathrm{R}^m_{\bullet(b)\nabla(c),n}(G)$ corresponding to a graph $G \in \mathfrak{Gr}^m_{\bullet(b)\nabla(c)}$ is given by the 'state sum' (11) *antisymmetrized* in the white vertices.

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