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LIE GROUP EXTENSIONS ASSOCIATED TO PROJECTIVE MODULES OF CONTINUOUS INVERSE ALGEBRAS

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Abstract. We call a unital locally convex algebra A a continuous inverse algebra if its unit group $A^\times$ is open and inversion is a continuous map. For any smooth action of a, possibly infinite-dimensional, connected Lie group G on a continuous inverse algebra A by automorphisms and any finitely generated projective right A-module E, we construct a Lie group extension $\widehat{G}$ of G by the group $\mathrm{GL}_A(E)$ of automorphisms of the A-module E. This Lie group extension is a "non-commutative" version of the group $\mathrm{Aut}(\mathbb{V})$ of automorphism of a vector bundle over a compact manifold M, which arises for $G = \mathrm{Diff}(M)$, $A = C^\infty(M,\mathbb{C})$ and $E = \Gamma\mathbb{V}$. We also identify the Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ and explain how it is related to connections of the A-module E.

Introduction

In [1] it is shown that for a finite-dimensional K-principal bundle P over a compact manifold M, the group $\mathrm{Aut}(P)$ of all bundle automorphisms carries a natural Lie group structure whose Lie algebra is the Fréchet-Lie algebra of $\mathcal{V}(P)^K$ of K-invariant smooth vector fields on M. This applies in particular to the group $\mathrm{Aut}(\mathbb{V})$ of automorphisms of a finite-dimensional vector bundle with fiber V because this group can be identified with the automorphisms group of the corresponding frame bundle $P = \mathrm{Fr}\,\mathbb{V}$ which is a $\mathrm{GL}(V)$-principal bundle.

In this paper, we turn to variants of the Lie groups $\mathrm{Aut}(\mathbb{V})$ arising in non-commutative geometry. In view of [19], the group $\mathrm{Aut}(\mathbb{V})$ can be identified with the group of semilinear automorphisms of the $C^\infty(M,\mathbb{R})$-module $\Gamma(\mathbb{V})$ of smooth sections of $\mathbb{V}$, which, according to Swan's Theorem, is a finitely generated projective module. Here the gauge group $\mathrm{Gau}\,(\mathbb{V})$ corresponds to the group of $C^\infty(M,\mathbb{R})$-linear module isomorphisms.

This suggests the following setup: Consider a unital locally convex algebra A and a finitely generated projective right A-module E. When can we turn groups of semilinear automorphisms of E into Lie groups? First of all, we have to restrict our attention to a natural class of algebras whose unit groups $A^\times$ carry natural Lie group structures, which is the case if $A^\times$ is an open subset of A and the inversion map is continuous. Such algebras are called *continuous inverse algebras*, CIAs, for

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short. The Fréchet algebra $C^\infty(M,\mathbb{R})$ is a CIA if and only if M is compact. Then its automorphism group $\mathrm{Aut}(C^\infty(M,\mathbb{R})) \cong \mathrm{Diff}(M)$ carries a natural Lie group structure with Lie algebra $\mathcal{V}(M)$, the Lie algebra of smooth vector fields on M. Another important class of CIAs whose automorphism groups are Lie groups are smooth 2-dimensional quantum tori with generic diophantine properties (cf. [10], [4]). Unfortunately, in general, automorphism groups of CIAs do not always carry a natural Lie group structure, so that it is much more natural to consider triples (A, G, μ_A), where A is a CIA, G a possibly infinite-dimensional Lie group, and $\mu_A \colon G \to \mathrm{Aut}(A)$ a group homomorphism defining a smooth action of G on A.

For any such triple (A, G, μ_A) and any finitely generated projective A-module E, the subgroup G_E of all elements of G for which $\mu_A(g)$ lifts to a semilinear automorphism of E is an open subgroup. One of our main results (Theorem 3.3) is that we thus obtain a Lie group extension

$$\mathbf{1} \to \mathrm{GL}_A(E) = \mathrm{Aut}_A(E) \to \widehat{G}_E \to G_E \to \mathbf{1}\,,$$

where $\widehat{G}_E$ is a Lie group acting smoothly on E by semilinear automorphisms. For the special case where M is a compact manifold, $A = C^\infty(M,\mathbb{R})$, $E = \Gamma(\mathbb{V})$ for a smooth vector bundle $\mathbb{V}$, and $G = \mathrm{Diff}(M)$, the Lie group $\widehat{G}$ is isomorphic to the group $\mathrm{Aut}(\mathbb{V})$ of automorphisms of the vector bundle $\mathbb{V}$, but our construction contains a variety of other interesting settings. From a different perspective, the Lie group structure on $\widehat{G}$ also tells us about possible smooth actions of Lie groups on finitely generated projective A-modules by semilinear maps which are compatible with a smooth action on the algebra A.

A starting point of our construction is the observation that the connected components of the set $\mathrm{Idem}(A)$ of idempotents of a CIA coincide with the orbits of the identity component $A_0^\times$ of $A^\times$ under the conjugation action. Using the natural manifold structure on $\mathrm{Idem}(A)$ (cf. [15]), the action of $A^\times$ on $\mathrm{Idem}(A)$ even is a smooth Lie group action.

On the Lie algebra side, the semilinear automorphisms of E correspond to the Lie algebra $\mathrm{DEnd}(E)$ of derivative endomorphisms, i.e., those endomorphisms $\phi \in \mathrm{End}_{\mathbb{K}}(E)$ for which there is a continuous derivation $D \in \mathrm{der}(A)$ with $\phi(s \cdot a) = \phi(s) \cdot a + s \cdot (D \cdot a)$ for $s \in E$ and $a \in A$. The set $\widehat{\mathrm{DEnd}}(E)$ of all pairs $(\phi, D) \in \mathrm{End}_{\mathbb{K}}(E) \times \mathrm{der}(A)$ satisfying this condition is a Lie algebra and we obtain a Lie algebra extension

$$\mathbf{0} \to \mathrm{End}_A(E) = \mathfrak{gl}_A(E) \hookrightarrow \widehat{\mathrm{DEnd}}(E) \twoheadrightarrow \mathrm{der}(A) \to \mathbf{0}\,.$$

Pulling this extension back via the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{der}(A)$ induced by the action of G on A yields the Lie algebra $\widehat{\mathfrak{g}}$ of the group $\widehat{G}$ from above (Proposition 4.7).

In Section 5 we briefly discuss the relation between linear splittings of the Lie algebra extension $\widehat{\mathfrak{g}}$ and covariant derivatives in the context of non-commutative geometry (cf. [5], [21], [8]).

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Preliminaries and notation. Throughout this paper we write $I := [0, 1]$ for the unit interval in $\mathbb{R}$ and $\mathbb{K}$ either denotes $\mathbb{R}$ or $\mathbb{C}$. A locally convex space E is said to be *Mackey complete* if each smooth curve $\gamma\colon I \to E$ has a (weak) integral in E. For a more detailed discussion of Mackey completeness and equivalent conditions, we refer to [20, Th. 2.14].

A *Lie group* G is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element and $\lambda_g(x) = gx$, resp., $\rho_g(x) = xg$ for the left, resp., right multiplication on G. Then each $x \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field x_l with $x_l(g) := d\lambda_g(\mathbf{1}) \cdot x$, $g \in G$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. In this sense we obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[x, y]_l = [x_l, y_l]$ for $x, y \in \mathfrak{g}$. We shall also use the functorial notation $\mathbf{L}(G)$ for the Lie algebra of G and, accordingly, $\mathbf{L}(\phi) = T_{\mathbf{1}}(\phi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ for the Lie algebra morphism associated to a morphism $\phi\colon G_1 \to G_2$ of Lie groups.

A Lie group G is called *regular* if for each $\xi \in C^\infty(I, \mathfrak{g})$, the initial value problem

$$\gamma(0) = \mathbf{1}\,, \quad \gamma'(t) = \gamma(t) \cdot \xi(t) = T(\lambda_{\gamma(t)})\xi(t)$$

has a solution $\gamma_\xi \in C^\infty(I, G)$, and the evolution map

$$\mathrm{evol}_G\colon C^\infty(I, \mathfrak{g}) \to G, \quad \xi \mapsto \gamma_\xi(1)$$

is smooth (cf. [22]). For a locally convex space E, the regularity of the Lie group $(E, +)$ is equivalent to the Mackey completeness of E ([25, Prop. II.5.6]). We also recall that for each regular Lie group G its Lie algebra $\mathfrak{g}$ is Mackey complete and that all Banach–Lie groups are regular (cf. [25, Rem. II.5.3] and [12]).

A smooth map $\exp_G\colon \mathbf{L}(G) \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. The Lie group G is said to be *locally exponential* if it has an exponential function for which there is an open 0-neighborhood U in $\mathbf{L}(G)$ mapped diffeomorphically by $\exp_G$ onto an open subset of G.

If A is an associative algebra with unit, we write $\mathbf{1}$ for the identity element, $A^\times$ for its group of units, $\mathrm{Idem}(A) = \{p \in A\colon p^2 = p\}$ for its set of idempotents and $\eta_A(a) = a^{-1}$ for the inversion map $A^\times \to A$. A homomorphism $\rho\colon A \to B$ is unital algebras is called *isospectral* if $\rho^{-1}(B^\times) = A^\times$. We write $\mathrm{GL}_n(A) := M_n(A)^\times$ for the unit group of the unital algebra $M_n(A)$ of $n \times n$-matrices with entries in A.

Throughout, G denotes a (possibly infinite-dimensional) Lie group, A a Mackey complete unital continuous inverse algebra (CIA for short) and $G \times A \to A, (g, a) \mapsto g.a = \mu_A(g)(a)$ is a smooth action of G on A.

1. Idempotents and finitely generated projective modules

The set $\mathrm{Idem}(A)$ of idempotents of a CIA A plays a central role in (topological) K-theory. In [15, Satz 2.13], Gramsch shows that this set always carries a natural manifold structure, which implies in particular that its connected components are open subsets. Since we shall need it in the following, we briefly recall some basic facts on $\mathrm{Idem}(A)$ (cf. [15]; see also [3, Sect. 4]).

Proposition 1.1. *For each $p \in \mathrm{Idem}(A)$, the set*

$$U_p := \big\{q \in \mathrm{Idem}(A) : pq + (\mathbf{1}-p)(\mathbf{1}-q) \in A^\times\big\}$$

is an open neighborhood of p in $\mathrm{Idem}(A)$ and, for each $q \in U_p$, the element

$$s_q := pq + (\mathbf{1}-p)(\mathbf{1}-q) \in A^\times \quad \textit{satisfies} \quad s_q q s_q^{-1} = p\,.$$

The connected component of p in $\mathrm{Idem}(A)$ coincides with the orbit of the identity component $A_0^\times$ of $A^\times$ under the conjugation action $A^\times \times \mathrm{Idem}(A) \to \mathrm{Idem}(A)$, $(g,p) \mapsto c_g(p) := g \cdot p := gpg^{-1}$.

Proof. Since the map $q \mapsto pq + (\mathbf{1}-p)(\mathbf{1}-q)$ is continuous, it maps p to $\mathbf{1}$ and since $A^\times$ is open, U_p is an open neighborhood of p. Hence, for each $q \in U_p$, the element s_q is invertible, and a trivial calculation shows that $s_q q = p s_q$.

If q is sufficiently close to p, then $s_q \in A_0^\times$ because $s_p = \mathbf{1}$ and $A_0^\times$ is an open neighborhood of $\mathbf{1}$ in A (recall that A is locally convex and $A^\times$ is open). This implies that $q = s_q^{-1} p s_q$ lies in the orbit $\{c_g(p) \colon g \in A_0^\times\}$ of p under $A_0^\times$. Conversely, since the orbit map $A^\times \to \mathrm{Idem}(A), g \mapsto c_g(p)$ is continuous, it maps the identity component $A_0^\times$ into the connected component of p in $\mathrm{Idem}(A)$. □

Lemma 1.2. *Let A be a CIA, $n \in \mathbb{N}$, and $p = p^2 \in M_n(A)$ an idempotent. Then the following assertions hold:*

1. *The subalgebra $pM_n(A)p$ is a CIA with identity element p.*
2. *The unit group $(pM_n(A)p)^\times$ is a Lie group.*

Proof. Since $M_n(A)$ also is a CIA ([30]; [11]) which is Mackey complete if A is so, it suffices to prove the assertion for $n = 1$.

(1) From the decomposition of the identity $\mathbf{1}$ as a sum $\mathbf{1} = p + (\mathbf{1}-p)$ of two orthogonal idempotents, we obtain the direct sum decomposition

$$A = pAp \oplus pA(\mathbf{1}-p) \oplus (\mathbf{1}-p)Ap \oplus (\mathbf{1}-p)A(\mathbf{1}-p)\,.$$

We claim that an element $a \in pAp$ is invertible in this algebra if and only if the element $a + (\mathbf{1}-p)$ is invertible in A. In fact, if $b \in pAp$ satisfies $ab = ba = p$, then

$$\big(a + (\mathbf{1}-p)\big)\big(b + (\mathbf{1}-p)\big) = ab + (\mathbf{1}-p) = \mathbf{1} = \big(b + (\mathbf{1}-p)\big)\big(a + (\mathbf{1}-p)\big)\,.$$

If, conversely, $c \in A$ is an inverse of $a + (\mathbf{1}-p)$ in A, then $ca + c(\mathbf{1}-p) = \mathbf{1} = ac + (\mathbf{1}-p)c$ leads after multiplication with p to $ca = p = ac$, which implies $(pcp)a = p = a(pcp)$, so that pcp is an inverse of a in pAp. The preceding argument implies in particular that $(pAp)^\times = pAp \cap (A^\times - (\mathbf{1}-p))$ is an open subset of pAp, and that the inversion map

$$\eta_{pAp} \colon (pAp)^\times \to pAp, \quad a \mapsto a^{-1} = \eta_A(a + \mathbf{1} - p) - (\mathbf{1}-p)$$

is continuous.

(2) is an immediate consequence of (1) (cf. [11], [25], Ex. II.1.4, Th. IV.1.11). $\square$

Let E be a finitely generated projective right A-module. Then there is some $n \in \mathbb{N}$ and an idempotent $p = p^2 \in M_n(A)$ with $E \cong pA^n$, where A acts by multiplication on the right. Conversely, for each idempotent $p \in \mathrm{Idem}(M_n(A))$, the right submodule pA^n of A^n is finitely generated (as a quotient of A^n) and projective because it is a direct summand of the free module $A^n \cong pA^n \oplus (\mathbf{1}-p)A^n$. The following lemma provides some information on A-linear maps between such modules.

Lemma 1.3. *Let $p, q \in \mathrm{Idem}\big(M_n(A)\big)$ be two idempotents. Then the following assertions hold:*

(1) *The map $x \mapsto \lambda_x|_{pA^n}$ (left multiplication) yields a bijection*

$$\alpha_{p,q}\colon qM_n(A)p = \{x \in M_n(A) : qx = x, xp = x\} \to \mathrm{Hom}_A(pA^n, qA^n)\,.$$

(2) *$pA^n \cong qA^n$ if and only if there are $x, y \in M_n(A)$ with $xy = q$ and $yx = p$. If, in particular, $q = gpg^{-1}$ for some $g \in \mathrm{GL}_n(A)$, then $x := gp$ and $y := g^{-1}$ satisfy $xy = q$ and $yx = p$.*

(3) *$pA^n \cong qA^n$ if and only if there are $x \in qM_n(A)p$ and $y \in pM_n(A)q$ with $xy = q$ and $yx = p$.*

(4) *If $pA^n \cong qA^n$, then there exists an element $g \in \mathrm{GL}_{2n}(A)$ with $g\widetilde{p}g^{-1} = \widetilde{q}$, where*

$$\widetilde{p} = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{q} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. (1) Each element of $\mathrm{End}_A(A^n)$ is given by left multiplication with a matrix, so that $M_n(A) \cong \mathrm{End}_A(A^n)$. Since pA^n and qA^n are direct summands of A^n, each element of $\mathrm{Hom}(pA^n, qA^n)$ can be realized by left multiplication with a matrix, and we have the direct sum decomposition

$$\begin{aligned}\mathrm{End}_A(A^n) \cong \mathrm{Hom}(pA^n, qA^n) &\oplus \mathrm{Hom}\big(pA^n, (\mathbf{1}-q)A^n\big) \\ &\oplus \mathrm{Hom}\big((\mathbf{1}-p)A^n, qA^n\big) \oplus \mathrm{Hom}\big((\mathbf{1}-p)A^n, (\mathbf{1}-q)A^n\big)\,,\end{aligned}$$

which corresponds to the direct sum decomposition

$$M_n(A) \cong qM_n(A)p \oplus (\mathbf{1}-q)M_n(A)p \oplus qM_n(A)(\mathbf{1}-p) \oplus (\mathbf{1}-q)M_n(A)(\mathbf{1}-p)\,.$$

Now the assertion follows from $qM_n(A)p = \big\{x \in M_n(A)\colon qx = x, xp = x\big\}$.

(2), (3) If pA^n and qA^n are isomorphic, there is some $x \in qM_n(A)p \cong \mathrm{Hom}(pA^n,\, qA^n)$ for which $\lambda_x\colon pA^n \to qA^n, s \mapsto xs$ is an isomorphism. Writing λ_x^{-1} as λ_y for some $y \in \mathrm{Hom}(qA^n, pA^n) \cong pM_n(A)q$, we get

$$p = \lambda_y \circ \lambda_x(p) = yxp = yx \quad \text{and} \quad q = \lambda_x \circ \lambda_y(q) = xyq = xy\,.$$

If, conversely, $p = yx$ and $q = xy$ hold for some $x, y \in M_n(A)$, then $p^2 = p$ implies $p = yx = yxyx = yqx$ and likewise $q = xy = xpy$, which leads to

$$(pyq)(qxp) = pyqxp = p^3 = p \quad \text{and} \quad (qxp)(pyq) = qxpyq = q^3 = q\,.$$

Therefore, we also have $p = y'x'$ and $q = x'y'$ with $x' := qxp \in qM_n(A)p$ and $y' := pyq \in pM_n(A)q$. Then $\lambda_{x'}\colon pA^n \to qA^n$ and $\lambda_{y'}\colon qA^n \to pA^n$ are module homomorphisms with $\lambda_{x'} \circ \lambda_{y'} = \lambda_{x'y'} = \lambda_q = \mathrm{id}_{qA^n}$ and $\lambda_{y'} \circ \lambda_{x'} = \lambda_{y'x'} = \lambda_p = \mathrm{id}_{pA^n}$.

(4) (cf. [3, Prop. 4.3.1]) Pick x, y as in (3). Let

$$\alpha := \begin{pmatrix} \mathbf{1} - q & x \\ y & \mathbf{1} - p \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} \mathbf{1} - p & p \\ p & \mathbf{1} - p \end{pmatrix} \in M_{2n}(A)\,.$$

Then a direct calculation yields $\alpha^2 = \mathbf{1} = \beta^2$. Therefore $z := \beta\alpha \in \mathrm{GL}_{2n}(A)$. Moreover, we have

$$\alpha\widetilde{q}\alpha^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \quad \text{and} \quad \beta \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \beta^{-1} = \widetilde{p}\,,$$

so that $z\widetilde{q}z^{-1} = \widetilde{p}$. □

Proposition 1.4. *For each finitely generated projective right A-module E, we pick some idempotent $p \in M_n(A)$ with $E \cong pA^n$. Then we topologize* $\mathrm{End}_A(E)$ *by declaring*

$$\alpha_{p,p}\colon pM_n(A)p \to \mathrm{End}_A(E), \quad x \mapsto \lambda_x|_{pA^n}$$

to be a topological isomorphism. Then the algebra $\mathrm{End}_A(E)$ *is a CIA and* $\mathrm{GL}_A(E)$ *is a Lie group. This topology does not depend on the choice of p and if A is Mackey complete, then* $\mathrm{GL}_A(E)$ *is locally exponential.*

Proof. We simply combine Lemma 1.2 with Lemma 1.3(1) to see that we obtain a CIA structure on $\mathrm{End}_A(E)$, so that $\mathrm{GL}_A(E)$ is a Lie group which is locally exponential if A is Mackey complete ([11]).

To verify the independence of the topology on $\mathrm{End}_A(E)$ from p, we first note that for any matrix

$$\widetilde{p} = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in M_N(A)\,, \quad N > n\,,$$

we have a natural isomorphism $\widetilde{p}M_N(A)\widetilde{p} \cong pM_n(A)p$, because all non-zero entries of matrices of the form $\widetilde{p}X\widetilde{p}$, $X \in M_N(A)$, lie in the upper left $(p \times p)$-submatrix and depend only on the corresponding entries of X.

If $q \in \mathrm{Idem}\big(M_\ell(A)\big)$ is another idempotent with $qA^\ell \cong E$, then the preceding argument shows that, after passing to $\max(n, \ell)$, we may w.l.o.g. assume that $\ell = n$. Then Lemma 1.3 yields a $g \in \mathrm{GL}_{2n}(A)$ with $g\overline{p}g^{-1} = \overline{q}$, and then conjugation with g induces a topological isomorphism

$$pM_n(A)p \cong \overline{p}M_{2n}(A)\overline{p} \xrightarrow{c_g} \overline{q}M_{2n}(A)\overline{q} \cong qM_n(A)q\,.$$ □

Example 1.5. (a) Let M be a smooth paracompact finite-dimensional manifold. We endow $A := C^\infty(M, \mathbb{K})$ with the smooth compact open topology, i.e., the topology inherited by the natural embedding

$$C^\infty(M, \mathbb{K}) \hookrightarrow \prod_{k=0}^{\infty} C(T^kM, T^k\mathbb{K})\,, \quad f \mapsto \big(T^k(f)\big)_{k \in \mathbb{N}_0}\,,$$

where all spaces $C(T^k M, T^k \mathbb{K})$ carry the compact open topology which coincides with the topology of uniform convergence on compact subsets.

If E is the space of smooth sections of a smooth vector bundle $q\colon \mathbb{V} \to M$, then E is a finitely generated projective A-module ([30]). The algebra $\mathrm{End}_A(E)$ is the space of smooth sections of the vector bundle $\mathrm{End}(\mathbb{V})$ and its unit group $\mathrm{GL}_A(E) \cong \mathrm{Gau}\,(\mathbb{V})$ is the corresponding gauge group. We shall return to this class of examples below.

(b) We obtain a similar picture if A is the Banach algebra $C(X, \mathbb{K})$, where X is a compact space and E is the space of continuous sections of a finite-dimensional topological vector bundle over X. Then $\mathrm{End}_A(E)$ is a Banach algebra, so that its unit group $\mathrm{GL}_A(E)$ is a Banach–Lie group.

2. Semilinear automorphisms of finitely generated projective modules

In this section we take a closer look at the group $\Gamma\,\mathrm{L}(E)$ of semilinear automorphism of a right A-module E. One of our main results, proved in Section 3 below, asserts that if E is a finitely generated projective module of a CIA A, certain pull-backs of this group by a smooth action of a Lie group G on A lead to a Lie group extension $\widehat{G}$ of G by the Lie group $\mathrm{GL}_A(E)$ (cf. Proposition 1.4) acting smoothly on E.

The discussion in this section is very much inspired by Y. Kosmann's paper [19].

Definition 2.1. Let E be a topological right module of the CIA A, i.e., we assume that the module structure $E \times A \to E, (s, a) \mapsto s \cdot a =: \rho_E(a)s$ is a continuous bilinear map. We write $\mathrm{End}_A(E)$ for the algebra of continuous module endomorphisms of E and $\mathrm{GL}_A(E) := \mathrm{End}_A(E)^\times$ for its group of units, the module automorphism group of E. For $A = \mathbb{K}$ we have in particular $\mathrm{GL}(E) = \mathrm{GL}_{\mathbb{K}}(E)$. The group $\mathrm{GL}_A(E)$ is contained in the group

$$\begin{aligned}\Gamma\,\mathrm{L}(E) := \big\{\phi \in \mathrm{GL}_{\mathbb{K}}(E) : \big(\exists \phi_A \in \mathrm{Aut}(A)\big)(\forall s \in E)(\forall a \in A)\ \phi(s \cdot a) \\ = \phi(s) \cdot \phi_A(a)\big\} \\ = \big\{\phi \in \mathrm{GL}_{\mathbb{K}}(E) : \big(\exists \phi_A \in \mathrm{Aut}(A)\big)(\forall a \in A)\ \phi \circ \rho_E(a) \\ = \rho_E\big(\phi_A(a)\big) \circ \phi\big\}\,.\end{aligned}$$

of *semilinear automorphisms of* E, where we write $\mathrm{Aut}(A)$ for the group of topological automorphisms of A. We put

$$\widehat{\Gamma}\,\mathrm{L}(E) := \big\{(\phi, \phi_A) \in \mathrm{GL}_{\mathbb{K}}(E) \times \mathrm{Aut}(A) : (\forall a \in A)\ \ \phi \circ \rho_E(a) = \rho_E\big(\phi_A(a)\big) \circ \phi\big\}\,,$$

where the multiplication is componentwise multiplication in the product group. In [16], the elements of $\widehat{\Gamma}\,\mathrm{L}(E)$ are called semilinear automorphisms and, for A commutative, certain characteristic cohomology classes of E are constructed for this group with values in differential forms over A. If the representation of A on E is faithful, then ϕ_A is uniquely determined by ϕ, so that $\widehat{\Gamma}\,\mathrm{L}(E) \cong \Gamma\,\mathrm{L}(E)$.

The map $(\phi, \phi_A) \mapsto \phi_A$ defines a short exact sequence of groups

$$\mathbf{1} \to \mathrm{GL}_A(E) \to \widehat{\Gamma}\,\mathrm{L}(E) \to \mathrm{Aut}(A)_E \to \mathbf{1}\,,$$

where $\mathrm{Aut}(A)_E$ denotes the image of the group $\widehat{\Gamma}\,\mathrm{L}(E)$ in $\mathrm{Aut}(A)$.

Remark 2.2. (a) For each $\psi \in \mathrm{Aut}(A)$, we define the corresponding *twisted module* E^ψ by endowing the vector space E with the new A-module structure defined by $s *_\psi a := s \cdot \psi(a)$, i.e., $\rho_{E^\psi} = \rho_E \circ \psi$. Then a continuous linear map $\phi \colon E \to E^\psi$ is a morphism of A-modules if and only if $\phi(s \cdot a) = \phi(s) \cdot \psi(a)$ holds for all $s \in E$ and $a \in A$, i.e.,

$$\phi \circ \rho_E(a) = \rho_{E^\psi}(a) \circ \phi \quad \text{ for } \quad a \in A\,.$$

Therefore $(\phi, \psi) \in \widehat{\Gamma}\,\mathrm{L}(E)$ is equivalent to $\phi \colon E \to E^\psi$ being a module isomorphism. This shows that

$$\mathrm{Aut}(A)_E = \big\{\psi \in \mathrm{Aut}(A) : E^\psi \cong E\big\}\,.$$

(b) Let $\psi \in \mathrm{Diff}(M)$ and $q \colon \mathbb{V} \to M$ be a smooth vector bundle over M. We consider the pull-back vector bundle

$$\mathbb{V}^\psi := \psi^*\mathbb{V} := \big\{(x, v) \in M \times \mathbb{V} : \psi(x) = q_{\mathbb{V}}(v)\big\}$$

with the bundle projection $q_{\mathbb{V}}^\psi \colon \mathbb{V}^\psi \to M, (x, v) \mapsto x\,.$

If $s \colon M \to \mathbb{V}^\psi$ is a smooth section, then $s(x) = \big(x, s'(\psi(x))\big)$, where $s' \colon M \to \mathbb{V}$ is a smooth section, and this leads to an identification of the spaces of smooth sections of $\mathbb{V}$ and $\mathbb{V}^\psi$. For a smooth function $f \colon M \to \mathbb{K}$ and $s \in \Gamma(\mathbb{V}^\psi)$, we have $(s \cdot f)(x) = f(x)s(x) = \big(x, f(x)s'(\psi(x))\big)$, so that the corresponding right module structure on $E = \Gamma\mathbb{V}$ is given by $s' * f = s' \cdot (\psi \cdot f)$, where $\psi \cdot f = f \circ \psi^{-1}$. This shows that $E^\psi = (\Gamma\mathbb{V})^\psi \cong \Gamma(\mathbb{V}^\psi)$, i.e., the sections of the pull-back bundle form a twisted module.

(c) Let E be a finitely generated projective right A-module and $p \in \mathrm{Idem}(M_n(A))$ with $E \cong pA^n$. For $\psi \in \mathrm{Aut}(A)$ we write $\psi^{(n)}$ for the corresponding automorphisms of A^n, resp., $M_n(\psi)$ for the corresponding automorphism of $M_n(A)$. Then the map $\psi_n \colon M_n(\psi)^{-1}(p)A^n \to pA^n$ induces a module isomorphism $M_n(\psi)^{-1}(p)A^n \cong (pA^n)^\psi$.

(d) Let $\rho_E \colon A \to \mathrm{End}(E)$ denote the representation of A on E defining the right module structure. Then, for each $a \in A^\times$, we have $\big(\rho_E(a), c_a^{-1}\big) \in \widehat{\Gamma}\,\mathrm{L}(E)$ because $\rho_E(a)(s \cdot b) = s \cdot ba = (s \cdot a)(a^{-1}ba)$.

Definition 2.3. Let G be a group acting by automorphism on the group N via $\alpha \colon G \to \mathrm{Aut}(N)$. We call a map $f \colon G \to N$ a *crossed homomorphism* if

$$f(g_1 g_2) = f(g_1)\alpha(g_1)\big(f(g_2)\big) \quad \text{ for } \quad g_1, g_2 \in G\,.$$

Note that f is a crossed homomorphism if and only if $(f, \mathrm{id}_G) \colon G \to N \rtimes_\alpha G$ is a group homomorphism. The set of all crossed homomorphisms $G \to N$ is denoted by $Z^1(G, N)$. The group N acts naturally on $Z^1(G, N)$ by

$$(n * f)(g) := nf(g)\alpha(g)(n)^{-1}$$

and the set of N-orbits in $Z^1(G, N)$ is denoted $H^1(G, N)$. If N is not abelian, $Z^1(G, N)$ and $H^1(G, N)$ do not carry a group structure; only the constant map $\mathbf{1}$ is a distinguished element of $Z^1(G,N)$, and its class $[\mathbf{1}]$ is distinguished in $H^1(G,N)$. The crossed homomorphisms in the class $[\mathbf{1}]$ are called *trivial*. They are of the form $f(g) = n\alpha(g)(n)^{-1}$ for some $n \in N$.

Proposition 2.4. *Let $\rho_E \colon A \to \mathrm{End}_{\mathbb{K}}(E)$ denote the action of A on the right A-module E and consider the action of the unit group $A^\times$ on the group $\Gamma\,\mathrm{L}(E)$ by $\widetilde{\rho}_E(a)(\phi) := \rho_E(a)^{-1}\phi\rho_E(a)$. To each $\psi \in \mathrm{Aut}(A)$ we associate the function*

$$C(\psi) \colon A^\times \to \Gamma\,\mathrm{L}(E)\,, \quad a \mapsto \rho_E(\psi(a)a^{-1})^{-1} = \rho_E(\psi(a))^{-1}\rho_E(a)\,.$$

Then $C(\psi) \in Z^1(A^\times, \Gamma\,\mathrm{L}(E))$, and we thus get an exact sequence of pointed sets

$$\mathbf{1} \to \mathrm{GL}_A(E) \to \widehat{\Gamma}\,\mathrm{L}(E) \to \mathrm{Aut}(A) \xrightarrow{\overline{C}} H^1\big(A^\times, \Gamma\,\mathrm{L}(E)\big)\,,$$

characterizing the subgroup $\mathrm{Aut}(A)_E$ as $\overline{C}^{-1}([\mathbf{1}])$.

Proof. That $C(\psi)$ is a crossed homomorphism follows from

$$\begin{aligned} C(\psi)(ab) &= \rho_E\big(\psi(ab)\big)^{-1}\rho_E(ab) = \rho_E\big(\psi(a)\big)^{-1}\rho_E\big(\psi(b)\big)^{-1}\rho_E(b)\rho_E(a) \\ &= C(\psi)(a)\rho_E(a)^{-1}C(\psi)(b)\rho_E(a) = C(\psi)(a)\widetilde{\rho}_E(a)\big(C(\psi)(b)\big)\,. \end{aligned}$$

That the crossed homomorphism $C(\psi)$ is trivial means that there is a $\phi \in \Gamma\,\mathrm{L}(E)$ with

$$C(\psi)(a) = \rho_E\big(\psi(a)\big)^{-1}\rho_E(a) = \phi\rho_E(a)^{-1}\phi^{-1}\rho_E(a)\,,$$

which means that $\rho_E\big(\psi(a)\big)\phi = \phi\rho_E(a)$ for $a \in A^\times$. Since each CIA A is generated by it unit group, which is an open subset, the latter relation is equivalent to $(\phi,\psi) \in \widehat{\Gamma}\,\mathrm{L}(E)$. We conclude that $\overline{C}^{-1}([\mathbf{1}]) = \mathrm{Aut}(A)_E$. □

Example 2.5. Let $q_{\mathbb{V}} \colon \mathbb{V} \to M$ be a smooth $\mathbb{K}$-vector bundle on the compact manifold M and $\mathrm{Aut}(\mathbb{V})$ the group of smooth bundle isomorphisms. Then each element ϕ of this group permutes the fibers of $\mathbb{V}$, hence induces a diffeomorphism ϕ_M of M. We thus obtain an exact sequence of groups

$$\mathbf{1} \to \mathrm{Gau}\,(\mathbb{V}) \to \mathrm{Aut}(\mathbb{V}) \to \mathrm{Diff}(M)_{[\mathbb{V}]} \to \mathbf{1}\,,$$

where $\mathrm{Gau}\,(\mathbb{V}) = \big\{\phi \in \mathrm{Aut}(\mathbb{V}) \colon \phi_M = \mathrm{id}_M\big\}$ is the gauge group of $\mathbb{V}$ and

$$\mathrm{Diff}(M)_{\mathbb{V}} = \big\{\psi \in \mathrm{Diff}(M) \colon \psi^*\mathbb{V} \cong \mathbb{V}\big\}$$

is the set of all diffeomorphisms ψ of M lifting to automorphisms of $\mathbb{V}$ (cf. Remark 2.2(b)). The group $\mathrm{Diff}(M)$ carries a natural Fréchet-Lie group structure for which $\mathrm{Diff}(M)_{\mathbb{V}}$ is an open subgroup, hence also a Lie group. Furthermore, it is shown in [1] that $\mathrm{Aut}(\mathbb{V})$ and $\mathrm{Gau}\,(\mathbb{V})$ carry natural Lie group structures for which $\mathrm{Aut}(\mathbb{V})$ is a Lie group extension of $\mathrm{Diff}(M)_{\mathbb{V}}$ by $\mathrm{Gau}\,(\mathbb{V})$.

Consider the CIA $A := C^\infty(M,\mathbb{K})$ and recall from Example 1.5(a) that the space $E := C^\infty(M,\mathbb{V})$ of smooth sections of $\mathbb{V}$ is a finitely generated projective A-module. The action of $\mathrm{Aut}(\mathbb{V})$ on $\mathbb{V}$ induces an action on E by

$$\phi_E(s)(x) := \phi \cdot s\big(\phi_M^{-1}(x)\big)\,.$$

For any smooth function $f\colon M \to \mathbb{K}$, we now have $\phi_E(fs)(x) := f\big(\phi_M^{-1}(x)\big) \cdot \phi_E(s)\big(\phi_M^{-1}(x)\big)$, i.e., $\phi_E(fs) = (\phi_M \cdot f) \cdot \phi_E(s)$. We conclude that $\mathrm{Aut}(\mathbb{V})$ acts on E by semilinear automorphisms of E and that we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Gau}\,(\mathbb{V}) \arrow[r] \arrow[d] & \mathrm{Aut}(\mathbb{V}) \arrow[r] \arrow[d] & \mathrm{Diff}(M)_{[\mathbb{V}]} \arrow[d] \\ \mathrm{GL}_A(E) \arrow[r] & \Gamma\,\mathrm{L}(E) \arrow[r] & \mathrm{Aut}(A)_E \end{tikzcd}$$

Next, we recall that

$$\mathrm{Aut}(A) = \mathrm{Aut}\,\big(C^\infty(M,\mathbb{K})\big) \cong \mathrm{Diff}(M) \tag{1}$$

(cf. [25, Thm. IX.2.1], [2], [13], [23]). Applying [19, Prop. 4] to the Lie group $G = \mathbb{Z}$, it follows that the map $\mathrm{Aut}(\mathbb{V}) \to \Gamma\,\mathrm{L}(E)$ is a bijective group homomorphism. Let us recall the basic idea of the argument.

First we observe that the vector bundle $\mathbb{V}$ can be reconstructed from the A-module E as follows. For each $m \in M$, we consider the maximal closed ideal $I_m := \big\{f \in A : f(m) = 0\big\}$ and associate the vector space $E_m := E/I_mE$. Using the local triviality of the vector bundle $\mathbb{V}$, it is easy to see that $E_m \cong \mathbb{V}_m$. We may thus recover $\mathbb{V}$ from E as the disjoint union

$$\mathbb{V} = \bigcup_{m\in M} E_m\,.$$

Any $\phi \in \Gamma\,\mathrm{L}(E)$ defines an automorphism ϕ_A of A, which we identify with a diffeomorphism ϕ_M of M via $\phi_A(f) := f \circ \phi_M^{-1}$. Then $\phi_A(I_m) = I_{\phi_M(m)}$ implies that ϕ induces an isomorphism of vector bundles

$$\mathbb{V} \to \mathbb{V}\,, \quad s + I_mE \mapsto \phi(s + I_mE) = \phi(s) + I_{\phi_M(m)}E\,.$$

Its smoothness follows easily by applying it to a set of sections $s_1, \ldots, s_n$ which are linearly independent in m. This implies that each element $\phi \in \Gamma\,\mathrm{L}(E)$ corresponds to an element of $\mathrm{Aut}(\mathbb{V})$, so that the vertical arrows in the diagram above are in fact isomorphisms of groups.

Finally, we take a look at the Lie structures on these groups. A priori, the automorphism group $\mathrm{Aut}(A)$ of a CIA carries no natural Lie group structure, but the group isomorphism $\mathrm{Diff}(M) \to \mathrm{Aut}(A)$ from (1) defines a smooth action of the Lie group $\mathrm{Diff}(M)$ on A. Indeed, this can be derived quite directly from the smoothness of the map

$$\mathrm{Diff}(M) \times C^\infty(M,\mathbb{K}) \times M \to \mathbb{K}\,, \quad (\phi, f, m) \mapsto f\big(\phi^{-1}(m)\big)$$

which is smooth because it is a composition of the smooth action map $\mathrm{Diff}(M) \times M \to M$ and the smooth evaluation map $A \times M \to M$ (cf. [27, Lemma A.2]).

Since the vector bundle $\mathbb{V}$ can be embedded into a trivial bundle $M \times \mathbb{K}^n$, we obtain a topological embedding of $\mathrm{Gau}\,(\mathbb{V})$ as a closed subgroup of $\mathrm{Gau}\,(M \times \mathbb{K}^n) \cong C^\infty\big(M, \mathrm{GL}_n(\mathbb{K})\big) \cong \mathrm{GL}_n(A)$. Accordingly, we obtain an embedding $E \hookrightarrow A^n$ and the preceding discussion yields an identification of $\mathrm{GL}_A(E)$ with

$\mathrm{Gau}(\mathbb{V})$ as the same closed subgroups of $\mathrm{GL}_n(A)$ (cf. Proposition 1.4). Since both groups are locally exponential Lie groups, the homeomorphism $\mathrm{Gau}(\mathbb{V}) \to \mathrm{GL}_A(E)$ is an isomorphism of Lie groups (cf. [25, Thm. IV.1.18] and [12] for more details on locally exponential Lie groups).

3. Lie group extensions associated to projective modules

In this section we consider a Lie group G, acting smoothly by automorphisms on the CIA A. We write $\mu_A\colon G \to \mathrm{Aut}(A)$ for the corresponding homomorphism. For each right A-module E, we then consider the subgroup

$$G_E := \{g \in G : E^g \cong E\} = \mu_A^{-1}\big(\mathrm{Aut}(A)_E\big)\,,$$

where we write $E^g := E^{\mu_A(g)}$ for the corresponding twisted module (cf. Remark 2.2(a)). The main result of this section is Theorem 3.3 which asserts that for $G = G_E$, the pull-back of the group extension $\widehat{\Gamma}\,\mathrm{L}(E)$ of $\mathrm{Aut}(A)_E$ by $\mathrm{GL}_A(E)$ yields a Lie group extension $\widehat{G}$ of G by $\mathrm{GL}_A(E)$.

Proposition 3.1. *If E is a finitely generated projective right A-module, then the subgroup G_E of G is open. In particular, we have $\mu_A(G) \subseteq \mathrm{Aut}(A)_E$ if G is connected.*

Proof. Since E is finitely generated and projective, it is isomorphic to an A-module of the form pA^n for some idempotent $p \in M_n(A)$. We recall from Remark 2.2(c) that for any automorphism $\psi \in \mathrm{Aut}(A)$ and $\gamma \in \mathrm{GL}_n(A)$ with $M_n(\psi)^{-1}(p) = \gamma^{-1}p\gamma$, the maps

$$M_n(\psi)^{-1}(p)A^n \to (pA^n)^\psi, x \mapsto \psi^{(n)}(x)\,, \quad M_n(\psi)^{-1}(p)A^n \to pA^n, s \mapsto \gamma\cdot s$$

are isomorphisms of A-modules. According to Proposition 1.1, all orbits of the group $\mathrm{GL}_n(A)_0$ in $\mathrm{Idem}\big(M_n(A)\big)$ are connected open subsets of $\mathrm{Idem}\big(M_n(A)\big)$, hence coincide with its connected components. Therefore the subset $\{g \in G\colon c\cdot p \in \mathrm{GL}_n(A)_0\cdot p\}$ of G is open. In view of Lemma 1.3(2), this open subset is contained in the subgroup G_E, hence G_E is open. □

From now on we assume that $G = G_E$. Then we obtain a group extension

$$\mathbf{1} \to \mathrm{GL}_A(E) \to \widehat{G} \xrightarrow{q} G \to \mathbf{1}\,,$$

where $q(\phi, g) = g$, and

$$\widehat{G} := \big\{(\phi, g) \in \Gamma\,\mathrm{L}(E)\times G\colon \big(\phi, \mu_A(g)\big) \in \widehat{\Gamma}\,\mathrm{L}(E)\big\} \cong \mu_A^*\widehat{\Gamma}\,\mathrm{L}(E) \tag{2}$$

acts on E via $\pi(\phi, g)\cdot s := \phi(s)$ by semilinear automorphisms. The main result of the present section is that $\widehat{G}$ carries a natural Lie group structure and that it is a Lie group extension of G by $\mathrm{GL}_A(E)$. Let us make this more precise:

Definition 3.2. An *extension of Lie groups* is a short exact sequence

$$\mathbf{1} \to N \xrightarrow{\iota} \widehat{G} \xrightarrow{q} G \to \mathbf{1}$$

of Lie group morphisms, for which $\widehat{G}$ is a smooth (locally trivial) principal N-bundle over G with respect to the right action of N given by $(\widehat{g}, n) \mapsto \widehat{g}n$. In the following, we identify N with the subgroup $\iota(N) \trianglelefteq \widehat{G}$.

Theorem 3.3. *If A is a CIA, G is a Lie group acting smoothly on A by $\mu_A\colon G \to \mathrm{Aut}(A)$, and E is a finitely generated projective right A-module with $\mu_A(G) \subseteq \mathrm{Aut}(A)_E$, then $\mathrm{GL}_A(E)$ and $\widehat{G}$ carry natural Lie group structures such that the short exact sequence*

$$\mathbf{1} \to \mathrm{GL}_A(E) \to \widehat{G} \xrightarrow{\ q\ } G \to \mathbf{1}$$

defines a Lie group extension of G by $\mathrm{GL}_A(E)$.

Proof. In view of Proposition 1.4, $\mathrm{End}_A(E)$ is a CIA and its unit group $\mathrm{GL}_A(E)$ is a Lie group. The assumption $\mu_A(G) \subseteq \mathrm{Aut}(A)_E$ implies that $G = G_E$, so that the group $\widehat{G}$ is indeed a group extension of G by $\mathrm{GL}_A(E)$.

Choose n and $p \in \mathrm{Idem}\big(M_n(A)\big)$ with $E \cong pA^n$ and let U_p be as in Proposition 1.1. In the following we identify E with pA^n. We write $g{*}a := M_n\big(\mu_A(g)\big)(a)$ for the smooth action of G on the CIA $M_n(A)$ and $g\sharp x := \mu_A(g)^{(n)}(x)$ for the action of G on A^n, induced by the smooth action of G on A. Then $U_G := \{g \in G : g{*}p \in U_p\}$ is an open neighborhood of the identity in G, and we have a map

$$\gamma\colon U_G \to \mathrm{GL}_n(A)\,, \quad g \mapsto s_{g*p} := p \cdot g * p + (\mathbf{1} - p)\cdot(\mathbf{1} - g * p)\,,$$

which, in view of Proposition 1.1, satisfies

$$\gamma(g)(g * p)\gamma(g)^{-1} = p \quad \text{for all} \quad g \in U_G\,. \tag{3}$$

This implies in particular that the natural action of the pair

$$\big(\gamma(g), \mu_A(g)\big) \in \mathrm{GL}_n(A) \rtimes \mathrm{Aut}(A) \cong \mathrm{GL}_A(A^n) \rtimes \mathrm{Aut}(A)$$

on A^n by $\big(\gamma(g), \mu_A(g)\big)(x) = \gamma(g)\cdot(g\sharp x)$ preserves the submodule $pA^n = E$, and that we thus get a map

$$S_E\colon U_G \to \Gamma\,\mathrm{L}(E) = \Gamma\,\mathrm{L}(pA^n)\,, \quad S_E(g)(s) := \gamma(g)\cdot(g\sharp s)\,.$$

For $g \in G$, $s \in E$ and $a \in A$, we then have

$$S_E(g)(s\cdot a) = \gamma(g)\cdot(g\sharp(sa)) = \gamma(g)\cdot(g\sharp s)\cdot(g * a) = S_E(g)(s)\cdot(g\cdot a)\,,$$

which shows that $\sigma\colon U_G \to \widehat{G}$, $g \mapsto \big(S_E(g), g\big)$ is a section of the group extension $q\colon \widehat{G} \to G$. We now extend σ in an arbitrary fashion to a map $\sigma\colon G \to \widehat{G}$ with $q \circ \sigma = \mathrm{id}_G$.

Identifying $\mathrm{GL}_A(E)$ with the kernel of the factor map $q\colon \widehat{G} \to G$, we obtain for $g, g' \in G$ an element

$$\omega(g, g') := \sigma(g)\sigma(g')\sigma(gg')^{-1} \in \mathrm{GL}_A(E)\,,$$

and this element is given for $g, g' \in U_G$ by

$$\omega(g, g') = \gamma(g)\cdot g * \big(\gamma(g')\cdot g^{-1} * \gamma(gg')^{-1}\big) = \gamma(g)\cdot\big(g * \gamma(g')\big)\cdot\gamma(gg')^{-1}\,.$$

Since all the maps involved are smooth, the preceding formula shows immediately that ω is smooth on the open identity neighborhood

$$\{(g, g') \in U_G \times U_G : g, g', gg' \in U_G\}$$

of $(\mathbf{1}, \mathbf{1})$ in $G \times G$.

Next we observe that the map

$$S\colon G \to \mathrm{Aut}\,\big(\mathrm{GL}_A(E)\big), \quad S(g)(\phi) := \sigma(g)\phi\sigma(g)^{-1}$$

has the property that the corresponding map

$$U_G \times \mathrm{GL}_A(E) \to \mathrm{GL}_A(E)\,, \quad (g,\phi) \mapsto S(g)(\phi) = \gamma(g)(g*\phi)\gamma(g)^{-1}$$

is smooth because γ and the action of G on $M_n(A)$ are smooth.

In the terminology of [26, Def. I.1], this means that $\omega \in C^2_s\big(G, \mathrm{GL}_A(E)\big)$ and that $S \in C^1_s\big(G, \mathrm{Aut}(\mathrm{GL}_A(E))\big)$. We claim that we even have $\omega \in C^2_{ss}\big(G, \mathrm{GL}_A(E)\big)$, i.e., for each $g \in G$, the function

$$\omega_g\colon G \to \mathrm{GL}_A(E), \quad x \mapsto \omega(g,x)\omega(gxg^{-1},g)^{-1} = \sigma(g)\sigma(x)\sigma(g)^{-1}\sigma(gxg^{-1})^{-1}$$

is smooth in an identity neighborhood of G.

Case 1: First we consider the case $g \in U'_G := \{h \in G : h*p \in \mathrm{GL}_n(A)\cdot p\}$. The map $\gamma\colon U_G \to \mathrm{GL}_n(A)$ extends to a map $\gamma\colon U'_G \to \mathrm{GL}_n(A)$ satisfying $g*p = \gamma(g)^{-1}p\gamma(g)$ for each $g \in U'_G$. Then $S'_E(g)(s) := \gamma(g)\cdot(g\sharp s)$ defines an element of $\Gamma\,\mathrm{L}(E)$, and we have $\sigma(g) = \big(\phi(g)S'_E(g), g\big)$ for some $\phi(g) \in \mathrm{GL}_A(E)$. For $x \in U_G \cap g^{-1}U_G g$ we now have

$$\begin{aligned}\omega_g(x)s &= \sigma(g)\sigma(x)\sigma(g)^{-1}\sigma(gxg^{-1})^{-1}s \\ &= \phi(g)\gamma(g)\cdot g\sharp\Big(\gamma(x)\cdot x\sharp\Big(g^{-1}\sharp\big(\gamma(g)^{-1}\phi(g)^{-1}\cdot\big[(gx^{-1}g^{-1})\sharp\gamma(gxg^{-1})^{-1}s\big]\big)\Big)\Big) \\ &= \phi(g)\gamma(g)(g*\gamma(x))\Big((gxg^{-1})*\big(\gamma(g)^{-1}\cdot\phi(g)^{-1}\big)\Big)\cdot\gamma(gxg^{-1})^{-1}s\,.\end{aligned}$$

We conclude that

$$\omega_g(x) = \phi(g)\gamma(g)(g*\gamma(x))\Big((gxg^{-1})*\big(\gamma(g)^{-1}\cdot\phi(g)^{-1}\big)\Big)\cdot\gamma(gxg^{-1})^{-1}\,.$$

If g is fixed, all the factors in this product are smooth $\mathrm{GL}_n(A)$-valued functions of x in the identity neighborhood $U_G \cap g^{-1}U_G g$, hence ω_g is smooth on this set.

Case 2: Now we consider the case $g*p \notin \mathrm{GL}_n(A)\cdot p$. Since $g*p$ corresponds to the right A-module $(g*p)A^n \cong (pA^n)^{\mu_A(g)^{-1}} = E^{\mu_A(g)^{-1}}$ and $E^{\mu_A(g)^{-1}} \cong E$ follows from $g \in G_E = G$, there exists an element $\eta(g) \in \mathrm{GL}_{2n}(A)$ with $\eta(g)(g*\widetilde{p})\eta(g)^{-1} = \widetilde{p}$ for $\widetilde{p} = \begin{pmatrix} p & 0 \\ 0 & 0\end{pmatrix} \in M_{2n}(A)$ (Lemma 1.3). We now have $pA^n \cong \widetilde{p}A^{2n}$, and $g*\widetilde{p} \in \mathrm{GL}_{2n}(A)\cdot\widetilde{p}$, so that the assumptions of Case 1 are satisfied with $2n$ instead of n. Therefore ω_g is smooth in an identity neighborhood.

The multiplication in $\widehat{G} = \mathrm{GL}_A(E)\sigma(G)$ is given by the formula

$$n\sigma(g)\cdot n'\sigma(g') = \big(nS(g)(n')\omega(g,g')\big)\sigma(gg')\,, \tag{4}$$

so that the preceding arguments imply that (S,ω) is a smooth factor system in the sense of [26, Def. II.6]. Here the algebraic conditions on factor systems follow from the fact that (4) defines a group multiplication. We now derive from [26, Prop. II.8] that $\widehat{G}$ carries a natural Lie group structure for which the projection map $q\colon \widehat{G} \to G$ defines a Lie group extension of G by $\mathrm{GL}_A(E)$. $\square$

Proposition 3.4. (a) *The group $\widehat{G}$ acts smoothly on E by $(\phi, g).s := \phi(s)$.*

(b) *Let $p_1 \colon \widehat{G} \to \Gamma\mathrm{L}(E)$ denote the projection to the first component. For a Lie group H, a group homomorphism $\Phi \colon H \to \widehat{G}$ is smooth if and only if $q \circ \Phi \colon\ H \to G$ is smooth and the action of H on E, defined by $p_1 \circ \Phi \colon H \to \Gamma\mathrm{L}(E)$, is smooth.*

(c) *The Lie group extension $\widehat{G}$ of G splits if and only if there is a smooth action of G on E by semilinear automorphisms which is compatible with the action of G on A in the sense that the corresponding homomorphism $\pi_E \colon G \to \Gamma\mathrm{L}(E)$ satisfies*

$$\pi_E(g) \circ \rho_E(a) = \rho_E(\mu_A(g)a) \circ \pi_E(g) \quad \text{for} \quad g \in G\,. \tag{5}$$

Proof. (a) Since $\widehat{G}$ acts by semilinear automorphisms of E which are continuous and hence smooth, it suffices to see that the action map $\widehat{G} \times E \to E$ is smooth on a set of the form $U \times E$, where $U \subseteq \widehat{G}$ is an open $\mathbf{1}$-neighborhood. With the notation of the proof of Theorem 3.3, let

$$U := \mathrm{GL}_A(E) \cdot \sigma(U_G),$$

which is diffeomorphic to $\mathrm{GL}_A(E) \times U_G$ via the map $(\phi, g) \mapsto \phi \cdot \sigma(g)$. Now it remains to observe that the map

$$\mathrm{GL}_A(E) \times U_G \times E \to E\,, \quad (\phi, g, s) \mapsto \phi\big(S_E(g)s\big) = \phi\big(\gamma(g) \cdot (g\sharp s)\big)$$

is smooth, which follows from the smoothness of the action of $\mathrm{GL}_A(E)$ on E, the smoothness of $\gamma \colon U_G \to \mathrm{GL}_n(A)$ and the smoothness of the action of G on A^n.

(b) If Φ is smooth, then $q \circ \Phi$ is smooth and (a) implies that $f := p_1 \circ \Phi$ defines a smooth action of H on E. Suppose, conversely, that $q \circ \Phi$ is smooth and that f defines a smooth action on E. Let U_G and U be as in (a) and put $W := \Phi^{-1}(U)$. Since $q \circ \Phi$ is continuous,

$$W = \Phi^{-1}\big(q^{-1}(U_G)\big) = (q \circ \Phi)^{-1}(U_G)$$

is an open subset of H. Since Φ is a group homomorphism, it suffices to verify its smoothness on W. We know from (a) that the map

$$\widetilde{S}_E \colon U_G \times E \to E\,, \quad (g, s) \mapsto S_E(g)s$$

is smooth. For $h \in W$ we have $\Phi(h) = f_1(h)\sigma\big(f_2(h)\big)$, where $f_1(h) \in \mathrm{GL}_A(E)$ and $f_2 := q \circ \Phi \colon W \to U_G$ is a smooth map. Therefore the map

$$W \times E \to E\,, \quad (h, s) \mapsto f(h)\big(\sigma(f_2(h))^{-1} \cdot s\big) = f(h) S_E\big(f_2(h)\big)^{-1} \cdot s = f_1(h) \cdot s$$

is smooth. If $e_1, \ldots, e_n \in A^n$ denote the canonical basis elements of the right A-module A^n, then we conclude that all maps

$$W \to \mathrm{GL}_A(E)\,, \quad h \mapsto f_1(h) \cdot e_i = f_1(h) p e_i$$

are smooth because $pe_i \in E$. Hence all columns of the matrix $f_1(h)$ depend smoothly on h, and thus $f_1 \colon W \to \mathrm{GL}_A(E) \subseteq M_n(A)$ is smooth. This in turn implies that $\Phi(h) = f_1(h)\sigma\colon (f_2(h))$ is smooth on W, hence on H because it is a group homomorphism.

(c) First we note that any homomorphism $\Phi\colon G \to \widehat{G}$ is of the form $\widehat{\sigma}(g) = (f(g), g)$, where $f\colon G \to \Gamma\mathrm{L}(E)$ is a homomorphism satisfying $\big(f(g), \mu_A(g)\big) \in \widehat{\Gamma}\,\mathrm{L}(E)$.

If the extension $\widehat{G}$ of G by $\mathrm{GL}_A(E)$ splits, then there is such a smooth Φ, and then (a) implies that $\pi_E = f$ defines a smooth action of G on E, satisfying all requirements.

If, conversely, $\pi_E\colon G \to \Gamma\mathrm{L}(E)$ defines a smooth action with (5), then the map $\Phi = (\pi_E, \mathrm{id}_G)\colon G \to \widehat{G}$ is a group homomorphism whose smoothness follows from (b), and therefore the Lie group extension $\widehat{G}$ splits. □

Examples 3.5. (a) If A is a Banach algebra, then $\mathrm{Aut}(A)$ carries a natural Banach–Lie group structure (cf. [17], [24, Prop. IV.14]). For each finitely generated projective module pA^n, $p \in \mathrm{Idem}(M_n(A))$, the subgroup $G := \mathrm{Aut}(A)_E$ is open (Proposition 3.1), and we thus obtain a Lie group extension

$$\mathbf{1} \to \mathrm{GL}_A(E) \hookrightarrow \widehat{G} \twoheadrightarrow G = \mathrm{Aut}(A)_E \to \mathbf{1}\,.$$

(b) For each CIA A, the Lie group $G := A^\times$ acts smoothly by conjugation on A and $g \mapsto \rho_E(g^{-1})$ defines a smooth action of G on E by semilinear automorphisms. This leads to a homomorphism

$$\sigma\colon A^\times \to \widehat{G} = \big\{(\phi, g) \in \Gamma\mathrm{L}(E) \times G\colon \big(\phi, \mu_A(g)\big) \in \widehat{\Gamma}\,\mathrm{L}(E)\big\},\ g \mapsto \big(\rho_E(g^{-1}), g\big)\,,$$

splitting the Lie group extension $\widehat{G}$ (Proposition 3.4).

Note that for any CIA A, we have $Z(A)^\times = Z(A^\times)$ because $A^\times$ is an open subset of A, so that its centralizer coincides with the center $Z(A)$ of A. We also note that $\rho_E(Z(A)) \subseteq \mathrm{End}_A(E)$ and that the direct product group $\mathrm{GL}_A(E) \times A^\times$ acts on E by $(\phi, g) \cdot s := \phi \circ \rho_E(g^{-1})s$, where the pairs $\big(\rho_E(z), z^{-1}\big)$, $z \in Z(A^\times)$, act trivially.

If, in addition, A is Mackey complete, the Lie group $\mathrm{GL}_A(E) \times A^\times$ and both factors are locally exponential, the subgroup

$$\Delta_Z := \big\{\big(\rho_E(z), z^{-1}\big) : z \in Z(A^\times)\big\}$$

is a central Lie subgroup and the Quotient Theorem in [12] (see also [25, Thm. IV.2.9]) implies that $(\mathrm{GL}_A(E) \times A^\times)/Z(A^\times)$ carries a locally exponential Lie group structure.

If, in addition, E is a faithful A-module, then Δ_Z coincides with the kernel of the action of $\mathrm{GL}_A(E) \times A^\times$ on E, so that the Lie group $\big(\mathrm{GL}_A(E) \times A^\times\big)/Z(A^\times)$ injects into $\Gamma\mathrm{L}(E)$. If, moreover, all automorphisms of A are inner, we have $\mathrm{Aut}(A) \cong A^\times/Z(A^\times)$, which carries a locally exponential Lie group structure ([25, Thm. IV.3.8]). We obtain

$$\Gamma\mathrm{L}(E) \cong (\mathrm{GL}_A(E) \times A^\times)/Z(A^\times)\,,$$

and a Lie group extension

$$\mathbf{1} \to \mathrm{GL}_A(E) \to \Gamma\mathrm{L}(E) \twoheadrightarrow \mathrm{Aut}(A) \to \mathbf{1}\,.$$

(c) (Free modules) Let $n \in \mathbb{N}$ and let $p = \mathbf{1} \in M_n(A)$. Then $pA^n = A^n$, $\mathrm{GL}_A(E) \cong \mathrm{GL}_n(A)$ acts by left multiplication, and

$$\Gamma\mathrm{L}(E) \cong \mathrm{GL}_n(A) \rtimes \mathrm{Aut}(A)$$

is a split extension. For any smooth Lie group action $\mu_A \colon G \to \mathrm{Aut}(A)$, we accordingly get $G_E = G$ and a split extension $\widehat{G} \cong \mathrm{GL}_n(A) \rtimes G$.

(d) Let $A = B(X)$ denote the Banach algebra of all bounded operators on the complex Banach space X. If $p \in A$ is a rank-1-projection,

$$pA \cong X' = \mathrm{Hom}(X, \mathbb{C})$$

is the dual space, considered as a right A-module, the module structure given by $\phi \cdot a := \phi \circ a$. In this case $pAp \cong \mathbb{C}$, $\mathrm{GL}_A(E) \cong \mathbb{C}^\times$, and for the group $G := \mathrm{PGL}(X) := \mathrm{GL}(X)/\mathbb{C}^\times$, acting by conjugation on A, we obtain the central extension

$$\mathbf{1} \to \mathbb{C}^\times \to \widehat{G} \cong \mathrm{GL}(X) \to G = \mathrm{PGL}(X) \to \mathbf{1}\,.$$

4. The corresponding Lie algebra extension

We now determine the Lie algebra of the Lie group $\widehat{G}$ constructed in Theorem 3.3. This will lead us from semilinear automorphisms of a module to derivative endomorphisms. The relations to connections in the context of non-commutative geometry will be discussed in Section 5 below.

Definition 4.1. We write $\mathfrak{gl}_A(E)$ for the Lie algebra underlying the associative algebra $\mathrm{End}_A(E)$ and

$$\mathrm{DEnd}(E) := \{\phi \in \mathrm{End}_{\mathbb{K}}(E) : (\exists D_\phi \in \mathrm{der}(A))(\forall a \in A)\ \ [\phi, \rho_E(a)] = \rho_E(D_\phi(a))\}\,.$$

for the Lie algebra of *derivative endomorphisms of* E (cf. [19]). We write

$$\widehat{\mathrm{DEnd}}(E) := \big\{(\phi, D) \in \mathrm{End}_{\mathbb{K}}(E) \times \mathrm{der}(A) : (\forall a \in A)\ [\phi, \rho_E(a)] = \rho_E(D \cdot a)\big\}\,.$$

We then have a short exact sequence

$$\mathbf{0} \to \mathfrak{gl}_A(E) \to \widehat{\mathrm{DEnd}}(E) \to \mathrm{der}(A)_E \to \mathbf{0}$$

of Lie algebras, where

$$\mathrm{der}(A)_E = \{D \in \mathrm{der}(A) : (\exists \phi \in \mathrm{DEnd}(E))\ D_\phi = D\}$$

is the image of the homomorphism $\mathrm{DEnd}(E) \to \mathrm{der}(A)$.

Example 4.2. If $E = C^\infty(M, \mathbb{V})$ is the space of smooth sections of the vector bundle $\mathbb{V}$ with typical fiber V on the compact manifold M, then it is interesting to identify the Lie algebra $\mathrm{DEnd}(E)$. From the short exact sequence

$$\mathbf{0} \to \mathfrak{gl}_A(E) = C^\infty\big(\mathrm{End}(\mathbb{V})\big) \hookrightarrow \mathrm{DEnd}(E) \twoheadrightarrow \mathcal{V}(M) = \mathrm{der}(A) \to \mathbf{0}\,,$$

it easily follows that the Lie algebra $\mathrm{DEnd}(E)$ can be identified with the Lie algebra $\mathcal{V}(\mathrm{Fr}\,\mathbb{V})^{\mathrm{GL}(V)}$ of $\mathrm{GL}(V)$-invariant vector fields on the frame bundle $\mathrm{Fr}\,\mathbb{V}$ (cf. [19] for details).

Lemma 4.3. *For each $a \in A$ we have $(\rho_E(a), -\operatorname{ad} a) \in \widehat{\mathrm{DEnd}}(E)$ and in particular $\rho_E(A) \subseteq \mathrm{DEnd}(E)$.*

Proof. For each $b \in A$ we have $\rho_E(a)(s \cdot b) - \rho_E(a)(s) \cdot b = s \cdot (ba - ab) = s \cdot \big(- \operatorname{ad} a(b)\big)$. $\square$

Lemma 4.4. *Let $p \in \operatorname{Idem}(M_n(A))$ and $E := pA^n$. We define*

$$\gamma\colon \operatorname{der}(A) \to M_n(A)\,, \quad \gamma(D) := (2p - \mathbf{1}) \cdot (D \cdot p)\,.$$

Then $[p, \gamma(D)] = D \cdot p$ and the operator

$$\nabla_D s := \gamma(D)s + D \cdot s$$

on A^n preserves $E = pA^n$ and $(\nabla_D, D) \in \widehat{\operatorname{DEnd}}(E)$.

Proof. From $p^2 = p$ we immediately get $D \cdot p = D \cdot p^2 = p \cdot (D \cdot p) + (D \cdot p) \cdot p$, showing that $p \cdot (D \cdot p) = p \cdot (D \cdot p) + p \cdot (D \cdot p) \cdot p$, and therefore $p \cdot (D \cdot p) \cdot p = 0$. We likewise obtain $(\mathbf{1} - p) \cdot (D \cdot p) \cdot (\mathbf{1} - p) = 0$, so that $D \cdot p \in pA(\mathbf{1} - p) + (\mathbf{1} - p)Ap$. This leads to

$$\begin{aligned}[\gamma(D), p] &= (2p - \mathbf{1}) \cdot (D \cdot p) \cdot p - p(2p - \mathbf{1}) \cdot D \cdot p \\ &= -(D \cdot p) \cdot p - p \cdot D \cdot p = -D \cdot (p^2) = -D \cdot p\,.\end{aligned}$$

For any $s \in pA^n$ we have $ps = s$ and therefore

$$\begin{aligned}p(\gamma(D)s + D \cdot s) &= [p, \gamma(D)]s + \gamma(D)ps + p(D \cdot s) = (D \cdot p)s + \gamma(D)s + p(D \cdot s) \\ &= D \cdot (ps) + \gamma(D)s = D \cdot s + \gamma(D)s\,.\end{aligned}$$

This implies that $\nabla_D \cdot s \in pA^n = \{x \in A^n : px = x\}$.

The remaining assertion follows from the fact that left and right multiplications commute and $D \cdot (s \cdot a) = (D \cdot s) \cdot a + s \cdot (D \cdot a)$. $\square$

Remark 4.5. To calculate the Lie algebra $\mathbf{L}(G)$ of a Lie group G, one may use a local chart $\phi\colon U \to \mathbf{L}(G)$, $U \subseteq G$ an open $\mathbf{1}$-neighborhood and $\phi(\mathbf{1}) = 0$, and consider the Taylor expansion of order 2 of the multiplication

$$x * y := \phi(\phi^{-1}(x)\phi^{-1}(y)) = x + y + b_{\mathfrak{g}}(x, y) + \cdots\,.$$

Then the Lie bracket in $\mathbf{L}(G)$ satisfies

$$[x, y] = b_{\mathfrak{g}}(x, y) - b_{\mathfrak{g}}(y, x)$$

(cf. [22], [12]).

Suppose that a Lie group extension $N \hookrightarrow \widehat{G} \twoheadrightarrow G$ is given by a pair (S, ω) via the multiplication on the product set $N \times G$:

$$(n, g)(n', g') = \big(nS(g)(n')\omega(g, g'), gg'\big)\,,$$

where the maps $S\colon G \to \operatorname{Aut}(N)$ and $\omega\colon G \times G \to N$ satisfy:

(a) the map $G \times N \to N$, $(g, n) \mapsto S(g)n$ is smooth on a set of the form $U \times N$, where U is an identity neighborhood of G.

(b) ω is smooth in an identity neighborhood with $\omega(g, \mathbf{1}) = \omega(\mathbf{1}, g) = \mathbf{1}$.

Let $\mathfrak{g} = \mathbf{L}(G)$, $\mathfrak{n} = \mathbf{L}(N)$ and $\widehat{\mathfrak{g}} = \mathbf{L}(\widehat{G})$ be the corresponding Lie algebras. Then we may use a product chart of $\widehat{G}$ in some sufficiently small $\mathbf{1}$-neighborhood. Write $\mathbf{L}(S(g)) \in \mathrm{Aut}(\mathfrak{n})$ for the Lie algebra automorphism induced by $S(g) \in \mathrm{Aut}(N)$ and put $\mathbf{L}(S)(g, v) := \mathbf{L}(S(g))(v)$. Then we define $DS\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{n})$ by

$$DS(x)(v) := T_{(\mathbf{1},v)}\big(\mathbf{L}(S)\big)(x, 0)\,.$$

We further define

$$D\omega(y, y') := d^2\omega(\mathbf{1},\mathbf{1})\big((y,0),(0,y')\big) - d^2\omega(\mathbf{1},\mathbf{1})\big((y',0),(0,y)\big)$$

and note that this is well-defined, i.e., independent of the chart, because (b) implies that ω vanishes of order 1 in $(\mathbf{1},\mathbf{1})$. We thus obtain the second order Taylor expansion

$$\begin{aligned}(x,y) * (x',y') = \big(x + DS(y)(x') + b_{\mathfrak{n}}(x,x') + d^2\omega(\mathbf{1},\mathbf{1})\big((y,0),(0,y')\big) + \cdots\,,\\ y + y' + b_{\mathfrak{g}}(y,y') + \cdots\big)\,,\end{aligned}$$

which provides the following formula for the Lie bracket in $\widehat{\mathfrak{g}}$, written as the product set $\mathfrak{n} \times \mathfrak{g}$:

$$\big[(x,y),(x',y')\big] = \big([x,x'] + DS(y)(x') - DS(y')(x) + D\omega(y,y'), [y,y']\big)\,.$$

To show that the Lie algebra of the group $\widehat{G}$, constructed in Theorem 3.3, is the corresponding pull-back on the Lie algebra level, we need the following lemma:

Lemma 4.6. *Let G and H be Lie groups with Lie algebra $\mathfrak{g}$, resp., $\mathfrak{h}$, $U \subseteq G$ an open $\mathbf{1}$-neighborhood and $\sigma\colon U \to H$ a smooth map with $\sigma(\mathbf{1}) = \mathbf{1}$. For the map*

$$\omega\colon U \times U \to H, \quad (g, g') \mapsto \sigma(g)\sigma(g')\sigma(gg')^{-1}\,,$$

we then have

$$\begin{aligned}d^2\omega(\mathbf{1},\mathbf{1})\big((x,0),(0,x')\big) - d^2\omega(\mathbf{1},\mathbf{1})\big((x',0),(0,x)\big)\\ = \big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - T_{\mathbf{1}}(\sigma)[x,x']\,.\end{aligned}$$

Proof. For the function $\omega_g(g') := \omega(g, g')$, we directly obtain with respect to the group structure on the tangent bundle TH:

$$\begin{aligned}T_{\mathbf{1}}(\omega_g)x' &= \sigma(g)\cdot T_{\mathbf{1}}(\sigma)x'\cdot\sigma(g)^{-1} + \sigma(g)\big(-\sigma(g)^{-1}T_g(\sigma)(g\cdot x')\sigma(g)^{-1}\big)\\ &= \sigma(g)\cdot T_{\mathbf{1}}(\sigma)x'\cdot\sigma(g)^{-1} - T_g(\sigma)(g\cdot x')\sigma(g)^{-1}\\ &= \mathrm{Ad}\,\big(\sigma(g)\big)T_{\mathbf{1}}(\sigma)x' - \delta^r(\sigma)(x'_l)(g)\,,\end{aligned}$$

where $\delta^r(\sigma) \in \Omega^1(G, \mathfrak{h})$ is the right logarithmic derivative of σ and $x'_l(g) = g\cdot x'$ is the left invariant vector field on G, corresponding to x'. Taking derivatives in $g = \mathbf{1}$ with respect to x, this in turn leads to

$$(d\omega)(\mathbf{1},\mathbf{1})(x,x') = \big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - x_l\big(\delta^r(\sigma)(x'_l)\big)(\mathbf{1})\,.$$

Using the Maurer–Cartan equation ([20])

$$\mathsf{d}\delta^r(\sigma)(X,Y) = \big[\delta^r(\sigma)(X), \delta^r(\sigma)(Y)\big]\,,$$

we now obtain

$$\begin{aligned}
&d^2\omega(\mathbf{1},\mathbf{1})\big((x,0),(0,x')\big) - d^2\omega(\mathbf{1},\mathbf{1})\big((x',0),(0,x)\big)\\
&\quad = 2\big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - x_l\big(\delta^r(\sigma)(x_l')\big)(\mathbf{1}) + x_l'\big(\delta^r(\sigma)(x_l)\big)(\mathbf{1})\\
&\quad = 2\big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - \mathtt{d}\delta^r(\sigma)(x_l,x_l')(\mathbf{1}) - \delta^r(\sigma)\big([x_l,x_l']\big)(\mathbf{1})\\
&\quad = 2\big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - \big[\delta^r(\sigma)(x_l),\delta^r(\sigma)(x_l')\big](\mathbf{1}) - T_{\mathbf{1}}(\sigma)\big([x,x']\big)\\
&\quad = 2\big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - \big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - T_{\mathbf{1}}(\sigma)\big([x,x']\big)\\
&\quad = \big[T_{\mathbf{1}}(\sigma)x, T_{\mathbf{1}}(\sigma)x'\big] - T_{\mathbf{1}}(\sigma)\big([x,x']\big)\,.
\end{aligned}$$

□

Since, in general, $\mathrm{Aut}(A)$ does not carry a natural Lie group structure, the Lie algebra $\mathrm{der}(A)$ is not literally the Lie algebra of $\mathrm{Aut}(A)$, but μ_A leads to a homomorphism $\mathbf{L}(\mu_A)\colon \mathfrak{g}\to\mathrm{der}(A)$ of Lie algebras, given by

$$x\cdot a := \mathbf{L}(\mu_A)(x)(a) := (T\mu_A)(\mathbf{1},a)(x,0)\,.$$

(cf. [12, App. E] for a discussion of these subtle points in the infinite-dimensional context; see also [25, Remark II.3.6(a)]).

Proposition 4.7. *The Lie algebra* $\widehat{\mathfrak{g}} := \mathbf{L}(\widehat{G})$ *of the Lie group* $\widehat{G}$ *from* Theorem 3.3 *is isomorphic to*

$$\mathbf{L}(\mu_A)^*\widehat{\mathrm{DEnd}}(E) \cong \big\{(x,\phi)\in\mathfrak{g}\times\mathrm{DEnd}(E) : (\phi,\mathbf{L}(\mu_A)x)\in\widehat{\mathrm{DEnd}}(E)\big\}\,.$$

Proof. First we note that

$$\begin{aligned}
\mathbf{L}(\mu_A)^*\widehat{\mathrm{DEnd}}(E) &= \big\{(x,(\phi,D))\in\mathfrak{g}\times\widehat{\mathrm{DEnd}}(E) : \mathbf{L}(\mu_A)x = D\big\}\\
&= \big\{(x,(\phi,\mathbf{L}(\mu_A)x))\in\mathfrak{g}\times\mathrm{DEnd}(E)\times\mathrm{der}(A)\colon (\phi,\mathbf{L}(\mu_A)x)\in\widehat{\mathrm{DEnd}}(E)\big\}\\
&\cong \widetilde{\mathfrak{g}} := \big\{(x,\phi)\in\mathfrak{g}\times\mathrm{DEnd}(E) : (\phi,\mathbf{L}(\mu_A)x)\in\widehat{\mathrm{DEnd}}(E)\big\}\,.
\end{aligned}$$

Recall from the proof of Theorem 3.3 the maps

$$\gamma(g) = p\cdot(g\cdot p) + (\mathbf{1}-p)\cdot\big(\mathbf{1}-(g\cdot p)\big) \quad\text{ and }\quad S_E(g)(s) := \gamma(g)\cdot(g\cdot s)\,.$$

Taking derivatives, we get

$$\dot\gamma(x) := T_{\mathbf{1}}(\gamma)(x) = (2p-\mathbf{1})\mathbf{L}(\mu_A)(x)\cdot p = (2p-\mathbf{1})\cdot(x\cdot p)$$

and with Lemma 4.4 we obtain $[\dot\gamma(x),p] = -x\cdot p$. We further derive

$$T_{\mathbf{1}}(S_E)(x)\cdot s := \dot\gamma(x)\cdot s + x\cdot s\in E\,,$$

and the linear map $T_{\mathbf{1}}(S_E)\colon\mathfrak{g}\to\mathrm{DEnd}(E)$ satisfies

$$\big[T_{\mathbf{1}}(S_E)(x),\rho_E(a)\big] = \rho_E(x\cdot a)$$

for each $x\in\mathfrak{g}$, which means that $\big(T_1(S_E)(x),x\big)\in\widetilde{\mathfrak{g}}$. Now

$$\Gamma\colon \widehat{\mathfrak{g}} = \mathfrak{gl}_A(E)\oplus\mathfrak{g}\to\widetilde{\mathfrak{g}}\,,\quad (\phi,x)\mapsto\big(\phi+T_{\mathbf{1}}(S_E)(x),x\big)$$

is a linear isomorphism with

$$\begin{aligned}&\big[\Gamma(\phi,x),\Gamma(\phi',x')\big] =\\ &\big([\phi,\phi'] + \big[T_{\mathbf{1}}(S_E)(x),\phi'\big] - \big[T_{\mathbf{1}}(S_E)(x'),\phi\big] + \big[T_{\mathbf{1}}(S_E)(x),T_{\mathbf{1}}(S_E)(x')\big],[x,x']\big)\,.\end{aligned}$$

On the other hand, we have seen in Remark 4.5 that the Lie bracket in $\widehat{\mathfrak{g}} = \mathfrak{gl}_A(E)\times\mathfrak{g}$ is given by

$$\big[(\phi,x),(\phi,x')\big] = \big([\phi,\phi'] + DS(x)(\phi') - DS(x')(\phi) + D\omega(x,x'),[x,x']\big)\,.$$

From $S(g)(\phi) = \mathrm{Ad}(\gamma(g))(g\cdot\phi)$ we derive that

$$DS(x)(\phi) = \big[\dot\gamma(x),\phi\big] + x\cdot\phi = \big[T_{\mathbf{1}}(S_E)(x),\phi\big]\,.$$

To show that Γ is an isomorphism of Lie algebras, it therefore suffices to show that

$$\big[T_{\mathbf{1}}(S_E)(x),T_{\mathbf{1}}(S_E)(x')\big] = D\omega(x,x') + T_{\mathbf{1}}(S_E)\big([x,x']\big) \tag{6}$$

holds for $x,x'\in\mathfrak{g}$.

To verify this relation, we first observe that the smoothness of the action of G on $\mathrm{GL}_n(A)$ implies that $\mathrm{GL}_n(A)\rtimes G$ is a Lie group, acting smoothly on A^n by $(a,g)\cdot s := a(g\sharp s)$. Then we consider the smooth map

$$\widetilde{S}_E\colon U_G\to \mathrm{GL}_n(A)\rtimes G\,,\quad g\mapsto \big(\gamma(g),g\big)\,,$$

also satisfying $\omega(g,g') = \widetilde{S}_E(g)\widetilde{S}_E(g')\widetilde{S}_E(gg')^{-1}$ and

$$\widetilde{S}_E(g)\cdot s = S_E(g)\cdot s \quad\text{for}\quad s\in E = pA^n\,. \tag{7}$$

Lemma 4.6 provides the identity

$$D\omega(x,x') = \big[T_{\mathbf{1}}(\widetilde{S}_E)x,T_{\mathbf{1}}(\widetilde{S}_E)x'\big] - T_{\mathbf{1}}(\widetilde{S}_E)[x,x']\,,$$

in $\mathfrak{gl}_A(E)$, so that (7) leads to

$$D\omega(x,x') = \big[T_{\mathbf{1}}(S_E)x,T_{\mathbf{1}}(S_E)x'\big] - T_{\mathbf{1}}(S_E)[x,x']\,,$$

as linear operators on E, and this is (6). □

The following general lemma prepares the discussion in Example 4.9 below, which exhibits the Lie group $\mathrm{Aut}(\mathbb{V})$ of automorphisms of a vector bundle as one of the Lie group extensions from Theorem 3.3.

Lemma 4.8. *If $\phi\colon G\to H$ is a bijective morphism of regular connected Lie groups and $\mathbf{L}(\phi)\colon \mathbf{L}(G)\to\mathbf{L}(H)$ is an isomorphism of locally convex Lie algebras, then ϕ is an isomorphism of Lie groups.*

Proof. Let $q_G\colon \widetilde{G}\to G$ and $q_H\colon\widetilde{H}\to H$ denote simply connected universal covering groups with $\mathbf{L}(q_G) = \mathrm{id}_{\mathbf{L}(G)}$ and $\mathbf{L}(q_H) = \mathrm{id}_{\mathbf{L}(H)}$. Then the induced morphims $\widetilde{\phi}\colon\widetilde{G}\to\widetilde{H}$ is the unique morphism of Lie groups with $\mathbf{L}(\widetilde{\phi}) = \mathbf{L}(\phi)$, hence an isomorphism, whose inverse is the unique morphism $\psi\colon\widetilde{H}\to\widetilde{G}$ with $\mathbf{L}(\psi) = \mathbf{L}(\phi)^{-1}$ ([25, Thm. IV.1.19]). Since $\widetilde{\phi}$ is an isomorphism, ϕ is a local isomorphism of Lie groups, so that its bijectivity implies that it is an isomorphism. □

Example 4.9. We continue the discussion of Example 2.5 in the light of Theorem 3.3. Recall that $q_{\mathbb{V}}\colon \mathbb{V} \to M$ denotes a smooth $\mathbb{K}$-vector bundle on the compact manifold M and $\mathrm{Aut}(\mathbb{V})$ its group of smooth bundle isomorphisms.

Let $G := \mathrm{Diff}(M)_{[\mathbb{V}]}$ and recall that this group acts smoothly on the CIA $A = C^\infty(M, \mathbb{K})$, preserving the equivalence class of the projective module $E = \Gamma\mathbb{V}$. Let $\widehat{G}$ be the Lie group extension of G by $\mathrm{GL}_A(E)$ from Theorem 3.3. In view of Proposition 2.4, we have a smooth representation $\pi\colon \widehat{G} \to \Gamma\,\mathrm{L}(E)$ of $\widehat{G}$ on E whose range is a subgroup of $\Gamma\,\mathrm{L}(E)$ containing $\mathrm{GL}_A(E)$ and projecting onto $\mathrm{Diff}(M)_{[\mathbb{V}]} \cong \mathrm{Aut}(A)_E$, which implies that the representation π is a bijection. In Example 2.5 we have seen that $\widehat{G} \cong \mathrm{Aut}(\mathbb{V})$ as abstract groups.

Next we recall that G is a regular Lie group ([20, Thm. 38.6]), and that we have seen in Example 2.5 that $\mathrm{GL}_A(E) \cong \mathrm{Gau}\,(\mathbb{V})$ as Lie groups, which implies that $\mathrm{GL}_A(E)$ is regular ([20, Thm. 38.6]). Hence $\widehat{G}$ is an extension of a regular Lie group by a regular Lie group and therefore regular (cf. [20], [12]). For similar reasons, the Lie group $\mathrm{Aut}(\mathbb{V})$ is regular. To see that $\mathrm{Aut}(\mathbb{V}) \cong \widehat{G}$ as Lie groups, it therefore suffices to show that the canonical isomorphism $\phi\colon\ \mathrm{Aut}(\mathbb{V}) \to \Gamma\,\mathrm{L}(E) \cong \widehat{G}$ is smooth and that $\mathbf{L}(\phi)$ is an isomorphism of topological Lie algebras (Lemma 4.8).

In view of Proposition 3.4(b), the smoothness of ϕ follows from the smoothness of the action of $\mathrm{Aut}(\mathbb{V})$ on $\Gamma\mathbb{V}$. To verify this smoothness, let $P = \mathrm{Fr}\,\mathbb{V}$ denote the frame bundle of $\mathbb{V}$ and recall that

$$\Gamma\mathbb{V} \cong \left\{ f \in C^\infty(P, V) : (\forall p \in P)(\forall k \in \mathrm{GL}(V))\ f(p \cdot k) = k^{-1} f(p) \right\}.$$

Since the evaluation map of $C^\infty(P, V)$ is smooth, the smoothness of the action of $\mathrm{Aut}(\mathbb{V}) \cong \mathrm{Aut}(P)$ on $\Gamma\mathbb{V}$ now follows from the smoothness of the action of $\mathrm{Aut}(P)$ on P (cf. [1]).

To see that $\widehat{\mathfrak{g}} \cong \mathbf{L}(\mathrm{Aut}(\mathbb{V})) \cong \mathcal{V}(\mathrm{Fr}\,\mathbb{V})^{\mathrm{GL}(V)}$, we first recall from Example 4.2 that $\mathrm{DEnd}(\mathbb{V}) \cong \mathcal{V}(\mathrm{Fr}\,\mathbb{V})^{\mathrm{GL}(V)}$. Hence both $\widehat{\mathfrak{g}}$ and $\mathbf{L}(\mathrm{Aut}(\mathbb{V}))$ are Fréchet-Lie algebras and $\mathbf{L}(\phi)$ is a continuous homomorphism inducing bijections $\Gamma(\mathrm{End}(\mathbb{V})) \to \mathfrak{gl}_A(E)$ and $\mathcal{V}(M) \to \mathfrak{g}$. Hence $\mathbf{L}(\phi)$ is bijective, and the Open Mapping Theorem ([28, Thm. 2.11]) implies that it is an isomorphism of topological Lie algebras. This completes the proof that $\widehat{G} \cong \mathrm{Aut}(\mathbb{V})$ as Lie groups.

Remark 4.10. That the extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{gl}_A(E)$ splits is equivalent to the existence of a continuous linear map $\alpha\colon \mathfrak{g} \to \mathfrak{gl}_A(E)$ for which

$$T_{\mathbf{1}}(S_E) + \alpha\colon \mathfrak{g} \to \mathrm{DEnd}(E)$$

is a homomorphism of Lie algebras (cf. Proposition 3.4 for the corresponding group analog). This is equivalent to the existence of a $\mathfrak{g}$-module structure on E, lifting the action of $\mathfrak{g}$ on A, given by $\mathbf{L}(\mu_A)$.

If such a homomorphism exists and the group $\widehat{G}$ is regular, then there exists a morphism of Lie groups $\widetilde{G}_0 \to \widehat{G}$, splitting the pull-back extension $q_G^*\widehat{G}$ of $\widetilde{G}_0$ by $\mathrm{GL}_A(E)$.

5. Covariant derivatives

In this short final section, we briefly explain the connections between linear splittings of the Lie algebra extension from Proposition 4.7 and covariant derivatives, resp., connections, as they occur in non-commutative geometry.

Definition 5.1. (a) There are many ways to construct "differential forms" for a non-commutative algebra (cf. [5]). One approach, which is closest to our construction, is the one described by Dubois-Violette in [7] (cf. [6] and [9]): First, one considers A as a module of the Lie algebra $\mathrm{der}(A)$, and since the multiplication on A is $\mathrm{der}(A)$-invariant, the algebra multiplication provides on the Chevalley-Eilenberg complex $\big(C(\mathrm{der}\, A, A), \mathtt{d}_A\big)$ the structure of a differential graded algebra. The differential subalgebra generated by $A \cong C^0(\mathrm{der}\, A, A)$ and $\mathtt{d}_A(A)$ is denoted $\Omega_D(A)$. From the inclusion of A as a subalgebra, $\Omega_D(A)$ inherits a natural A-bimodule structure. In particular, $E \otimes_A \Omega^1_D(A)$ is defined and a right A-module.

(b) Let E be a right A-module. A *connection* on E is a linear map $\nabla \colon E \to E \otimes_A \Omega^1_D(A)$ satisfying

$$\nabla(s \cdot a) = \nabla(s)a + s \otimes \mathtt{d}_A a \quad \text{for} \quad s \in E,\ a \in A\,. \tag{8}$$

Since the elements of $\Omega^1_D(A)$ are linear maps $\mathrm{der}(A) \to A$, each element of $E \otimes_A \Omega^1_D(A)$ defines a linear map $\mathrm{der}(A) \to E$. If $i_D \colon \Omega^1_D(A) \to A$, $\alpha \mapsto \alpha(D)$, denotes the evaluation map, we thus obtain for each derivation $D \in \mathrm{der}\, A$ a linear map, the corresponding *covariant derivative*,

$$\nabla_D := (\mathrm{id}_E \otimes i_D) \circ \nabla \colon E \to E\,,$$

where we identify $E \otimes_A A$ with E. The covariant derivative satisfies

$$\nabla_D(sa) = \nabla_D(s)a + sDa \quad \text{for} \quad s \in E,\ a \in A\,.$$

Remark 5.2. (a) For any connection ∇ and $D \in \mathrm{der}(A)$, we have $(\nabla_D, D) \in \widehat{\mathrm{DEnd}}(E)$. In particular, we have $\mathrm{der}(A)_E = \mathrm{der}\, A$ whenever a connection exists, and in this case any connection ∇ defines a splitting of the Lie algebra extension $\widehat{\mathrm{DEnd}}(E) \to \mathrm{der}(A)$ of $\mathrm{der}(A)$ by $\mathfrak{gl}_A(E)$ (cf. Definition 4.1). In this sense, we call any linear section of this Lie algebra extension a *covariant derivative on E*.

(b) (Covariant coordinates) We have already seen in Lemma 4.3 that for each $a \in A$, the operator $\rho_E(a)$ is contained in $\mathrm{DEnd}(E)$ and satisfies $D_{\rho_E(a)} = -\operatorname{ad} a$. If ∇ is a connection, we therefore have

$$\widehat{\rho}_E(a) := \rho_E(a) + \nabla_{\operatorname{ad} a} \in \mathfrak{gl}_A(E), \quad \text{i.e.,} \quad \big[\widehat{\rho}_E(A), \rho_E(A)\big] = \{0\}\,.$$

In the context of non-commutative geometry, the operators $\widetilde{\rho}_E(a)$ are called *covariant coordinates* because they commute with all "coordinate operators" $\rho_E(a)$, $a \in A$ (cf. [29], [18]).

Remark 5.3. (a) If a E is a finitely generated projective module, then it is of the form pA^n for some idempotent $p \in M_n(A)$. In this case we have the *Levi–Civita connection*, given by

$$\nabla(s) := p \cdot \mathtt{d}_{A^n}(s)\,,$$

where $\mathtt{d}_{A^n} \colon A^n \to A^n \otimes_A \Omega^1_D(A) \cong \Omega^1_D(A)^n, (a_i) \mapsto \big(\mathtt{d}_A(a_i)\big)$ is the canonical connection of the free right module A^n. Note that the Levi–Civita connection is not an intrinsic object, it depends on the embedding $E \hookrightarrow A^n$ and the module complement, all of which is encoded in the choice of the idempotent p. For a derivation $D \in \operatorname{der}(A)$, the operator $\nabla_D \in \operatorname{DEnd}(A^n)$, defined by

$$\nabla_D(s) = p \cdot D \cdot s$$

is the covariant derivative corresponding to the Levi–Civita connection.

In Lemma 4.4 we have seen that $\nabla'_D s := (2p - \mathbf{1})(D \cdot p)s + D \cdot s$ also defines a covariant derivative on E. In view of $ps = s$ for $s \in E$, we have $p(D \cdot p)s = p(D \cdot p)(ps) = 0$ (see the proof of Lemma 4.4), so that

$$\nabla'_D s = -(D \cdot p)s + D \cdot s = D \cdot (ps) - (D \cdot p)s = p(D \cdot s) = \nabla_D s\,.$$

(b) For $E = pA^n$ as above, any connection ∇' on E is of the form

$$\nabla_\alpha = p\mathtt{d}_{A^n} + \alpha\,,$$

where $\alpha \in M_n\big(\Omega^1_D(A)\big)$ satisfies $\alpha = p\alpha p$, i.e., $\alpha \in pM_n\big(\Omega^1_D(A)\big)p$. Here we use that for any other connection ∇' we have

$$\begin{aligned}\nabla' - \nabla \in \operatorname{Hom}\big(E, E \otimes_A \Omega^1_D(A)\big) &= \operatorname{Hom}\big(pA^n, pA^n \otimes_A \Omega^1_D(A)\big)\\ &\cong p\operatorname{Hom}\big(A^n, A^n \otimes_A \Omega^1_D(A)\big)p = pM_n\big(\Omega^1_D(A)\big)p\,.\end{aligned}$$

For any gauge transformation $g \in \mathrm{GL}_A(E)$, we then have

$$\nabla(g \cdot s) = p\mathtt{d}_{A^n}(g \cdot s) = p(\mathtt{d}_{M_n(A)}g \cdot s + g \cdot \mathtt{d}_{A^n}s) = g \cdot \big(\nabla(s) + g^{-1} \cdot \mathtt{d}_{M_n(A)}(g) \cdot s\big)\,,$$

which for $\nabla^g(s) := g^{-1}\nabla(g.s)$ and the left logarithmic derivative $\delta(g) := g^{-1} \cdot \mathtt{d}_{M_n(A)}(g) \in pM_n\big(\Omega^1_D(A)\big)p$ leads to

$$\nabla^g = \nabla + \delta(g)\,.$$

More generally, we get

$$\nabla^g_\alpha = \nabla + \delta(g) + \operatorname{Ad}(g^{-1}) \cdot \alpha = \nabla_{\alpha'} \quad \text{for} \quad \alpha' = \delta(g) + \operatorname{Ad}(g^{-1}) \cdot \alpha\,.$$

From that we derive in particular that $\nabla^g_\alpha = \nabla_\alpha$ is equivalent to $\delta(g) = \alpha - \operatorname{Ad}(g^{-1}) \cdot \alpha$.

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