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METRIZATION PROBLEM FOR LINEAR CONNECTIONS AND HOLONOMY ALGEBRAS

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Abstract. We contribute to the following: given a manifold endowed with a linear connection, decide whether the connection arises from some metric tensor. Compatibility condition for a metric is given by a system of ordinary differential equations. Our aim is to emphasize the role of holonomy algebra in comparison with certain more classical approaches, and propose a possible application in the Calculus of Variations (for a particular type of second order system of ODE's, which define geodesics of a linear connection, components of a metric compatible with the connection play the role of variational multipliers).

1. Motivation, geometry of paths

The metrizability problem for linear connections can be formulated as follows: given an affine manifold (M,∇), consisting of an n-dimenisonal manifold M endowed with a torsion-free linear connection ∇, under what conditions is ∇ metrizable; that is, when is there a (pseudo-)Riemannian (=non-degenerate) metric g such that ∇ is just the Levi-Civita (Riemannian) connection of (M,g). A quite natural motivation comes from theoretical physics ([2] etc.). As well known, the Riemannian connection is uniquely determined by zero torsion and $\nabla g = 0$, its components are related to components of the metric by the well-known formula $\Gamma^{\ell}_{ik} = \frac{1}{2} g^{\ell j}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j}\right)$, $g^{is}g_{sj} = \delta^i_j$. In [3], geometries were studied which arise on an analytic n-manifold when a system of curves called "paths" is given as the family of solutions of the system of differential equations $\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0$, $i,j,k \in \{1,\dots,n\}$ where $\Gamma^i_{jk}(x)$ are analytic functions of the coordinates in the manifold. As a motivation coming from gravitation theory, let us mentione free-fall trajectories as example of a "preferred family" of curves. The question formulated in [3] was in fact a bit more general. In a free paraphrase: On (M,∇), find a covariantly constant symmetric type $(0,2)$ tensor field g (which might not be a metric if no condition concerning its rank is required). The approach used in [3] was of analytic character, possible solutions of the system corresponding to the condition $\nabla g = 0$, $\frac{\partial g_{ij}}{\partial x^k} - g_{sj}\Gamma^s_{ik} - g_{is}\Gamma^s_{jk} = 0$ (for unknowns g_{ij}) were discussed under the implicite assumption $\det(g_{ij}) \neq 0$. Applying higher order covariant derivatives, the

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integrability conditions were derived, and necessary conditions for metrizability were given in the form of an infinite homogeneous system of linear equations in g_{ij} (coefficients being functions in $\Gamma's$ and their partial derivatives), which reads (in coordinate-free form)

$$g(R(X,Y)Z,W)+g(Z,R(X,Y)W)=0\,, \tag{1}$$

$$g(\nabla^r R(X,Y;Z_1;\dots;Z_r)(Z),W)+g(Z,\nabla^r R(X,Y;Z_1;\dots;Z_r)(W))=0 \tag{2}$$

for all $X,Y,Z,W,Z_1,\dots,Z_r\in\mathcal{X}(M)$, $1\le r<\infty$. In the "matrix form", $g\circ\nabla^r R+(\nabla^r R)^T\circ g^T=0$, $r=0,1,\dots<\infty$, [13]. Obviously, any flat connection ($R=0$) is metrizable.

In the case of a metrizable connection, the above linear conditions must stabilize for some positive integer N (in the sense that from the $(N+1)^{\text{th}}$ stage, the conditions are algebraic consequences of the previous ones). For any $n\ge 2$, there exist non-metrizable n-dimensional affine spaces (and there are in fact more non-metrizable examples than metrizable ones). In general, $0\le\operatorname{rank}(g_{ij})\le n$ holds, but it might happen that the maximum q of ranks of all possible solutions of (1)–(2) is less than n, then the affine space is non-metrizable (Example 3, 4; even the case $q=0$ might come). The procedure was described in [3] and applied by various authors later on, e.g. [4], [5] ($n=2$), [6], [7] ($n=4$), [11, p. 75] ($n=2$) etc. For 2-manifolds, the metrization problem was solved e.g. in [2], [13], [14].

Proposition 1 ([3, p. 23], a free paraphrase)**.** *An affine manifold (M,∇) with the curvature tensor R is metrizable if and only if the system*

$$g_{sj}R^s_{ik\ell}+g_{is}R^s_{jk\ell}=0 \tag{3}$$

has at least one-dimensional solution, and any solution of (3) *satisfies*

$$g_{sj}R^s_{ik\ell;m}+g_{is}R^s_{jk\ell;m}=0\,,\quad i,j,k,\ell,m\in\{1,\dots,n\}\,. \tag{4}$$

In examples, compatible metrics can be found using certain steps from the proof. Suppose that the system (3) is solvable, and that any solution of (3) satisfies (4). Let $\langle G^{(1)},\dots,G^{(p)}\rangle$ be a basis of the solution space. Then any solution g can be written in the form $g=\sum_{\alpha=1}^p\varphi^{(\alpha)}G^{(\alpha)}$ where $\varphi^{(\alpha)}$ are some functions on M. Due to (4), $G^{(\alpha)}_{sj;m}R^s_{ik\ell}+G^{(\alpha)}_{is;m}R^s_{jk\ell}=0$ holds, $\alpha=1,\dots,p$. That is, the covariant derivatives $G^{(\alpha)}_{sj;m}$ satisfy (3), too, and hence can be expressed as $G^{(\alpha)}_{ij;k}=\sum_{\beta=1}^p\mu_k^{(\alpha\beta)}G^{(\beta)}_{ij}$. Since second covariant derivatives satisfy the (so-called Ricci) indentity $G^{(\alpha)}_{ij;k\ell}-G^{(\alpha)}_{ij;\ell k}=G^{(\alpha)}_{sj}R^s_{ik\ell}+G^{(\alpha)}_{is}R^s_{jk\ell}$, and the right hand side vanishes for our $G^{(\alpha)}_{ij}$, we get $G^{(\alpha)}_{ij;k\ell}-G^{(\alpha)}_{ij;\ell k}=0$, and further (after some evaluations)

$$\frac{\partial\mu_k^{(\alpha\beta)}}{\partial x^\ell}-\frac{\partial\mu_\ell^{(\alpha\beta)}}{\partial x^k}+\sum_{\gamma=1}^p\Big(\mu_k^{(\alpha\gamma)}\mu_\ell^{(\gamma\beta)}-\mu_\ell^{(\alpha\gamma)}\mu_k^{(\gamma\beta)}\Big)=0\,. \tag{5}$$

If g shall satisfy $\nabla g=0$ then the φ's must satisfy the equations

$$\frac{\partial\varphi^{(\alpha)}}{\partial x^k}+\sum_{\beta=1}^p\varphi^{(\beta)}\mu_k^{(\alpha\beta)}=0\,,\quad\alpha=1,\dots,p\,. \tag{6}$$

But according to (5), the system (6) is completely integrable, hence there exist functions $\varphi^{(1)}, \dots, \varphi^{(p)}$ which determine a compatible (pseudo-)Riemannian metric. Let us demonstrate the method presented above on a simple example.

Example 1 ([15])**.** On the manifold $(\mathbb{R}^2, \mathrm{id})$ with (global) coordinates (x, y), consider the symmetric linear connection ∇ with the only non-zero components $\Gamma^1_{11} = x/(x^2+1)$, $\Gamma^2_{22} = y/(y^2+1)$. The curvature tensor vanishes identically, $R \equiv 0$, the corresponding system of equations is empty (the connection is surely metrizable since flat). Anyway, let us find the metrics to demonstrate the method. The solution space can be given e.g. as a span of independent (global analytic) type $(0,2)$ symmetric tensor fields $G^{(1)} = \mathrm{d}x \otimes \mathrm{d}x$, $G^{(2)} = \mathrm{d}y \otimes \mathrm{d}y$, $G^{(3)} = \mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x$. Their covariant derivatives, which must satisfy the system (3), can be expressed as combinations of the generators (with coefficients which are at most functions of x). We get $G^{(1)}_{ij;1} = -\frac{2x}{x^2+1} G^{(1)}_{ij}$, $G^{(1)}_{ij;2} = G^{(2)}_{ij;1} = 0$, $G^{(2)}_{ij;1} = -\frac{2y}{y^2+1} G^{(2)}_{ij}$, $G^{(3)}_{ij;1} = -\frac{x}{x^2+1} G^{(3)}_{ij}$, $G^{(3)}_{ij;2} = -\frac{y}{y^2+1} G^{(3)}_{ij}$. Hence $\mu^{(11)}_1 = -\frac{2x}{x^2+1}$, $\mu^{(22)}_1 = -\frac{2y}{y^2+1}$, $\mu^{(33)}_1 = -\frac{x}{x^2+1}$, $\mu^{(33)}_2 = -\frac{y}{y^2+1}$ are just the non-zero coefficients, and all compatible metrics are of the form $g = \varphi^{(1)} G^{(1)} + \varphi^{(2)} G^{(2)} + \varphi^{(3)} G^{(3)}$ where the functions $\varphi' s$ solve the system (6); we get $\varphi^{(1)} = -\frac{x}{x^2+1}$ etc. All compatible metrics g are described explicitely in Example 5 below.

2. Metrization via holonomy algebras

2.1. **Holonomies.** The method described above gives a very little insight into a geometric meaning of the integrability conditions and the restrictions imposed on the connection. A more geometric and sophisticated approach to the interpretation of necessary and sufficient metrization conditions (1), (2) can be given using parallel transport and holonomy groups. The *holonomy* of (M, ∇) at $x \in M$ along a piecewise-differentiable[1] loop (i.e. closed curve with x as starting point as well as endpoint; loops are taken with usual composition, [8]) is an automorphism of the tangent space $T_x M$ which is given by parallel propagation of vectors[2] along the given loop. Due to properties of the parallel transport along curves[3], all holonomies at x together with composition form the so-called (full linear) *holonomy group* $\Phi(x) := \mathrm{Hol}^{\nabla}_x$ of (M, ∇) at x, which appears to be a Lie transformation group; $\Phi(x)$ is a subgroup of $GL(T_x M)$. As well known, on a connected manifold, holonomy groups at different points are isomorphic. If we restrict ourselves onto loops which are homotopic to zero (contractible to a point), a similar construction gives rise to the (linear) *restricted holonomy group* $\Phi^0(x)$ of ∇ with the reference point x, which is a connected Lie transformation group, and plays the role of component of unit in $\Phi(x)$. Denote by $\underline{h}(x) := \underline{\mathrm{Hol}}^{\nabla}_x$ the corresponding Lie algebra of $\Phi(x)$. Particular Lie subgroups in $\Phi(x)$, namely a (linear) *local holonomy group* $\Phi^*(x)$ and an *infinitesimal holonomy group* $\Phi'(x)$ can be introduced, $\Phi'(x) \subset \Phi^*(x) \subset \Phi^0(x)$ holds, hence the corresponding holonomy Lie algebras satisfy $\underline{h}'(x) \subset \underline{h}^*(x) \subset \underline{h}(x)$

[1] The class C^1 is sufficient, [8, I, p. 85, Th. 7.2].

[2] In [8], parallel transport of frames is used.

[3] $\tau_{\mu^{-1}} = \tau_\mu^{-1}$ and $\tau_\mu \circ \tau_\eta = \tau_{\eta\mu}$

(the inclusions might be sharp in general), [8, I, Ch. II, p. 94, 95]. For a smooth connection, the infinitesimal holonomy algebra $\underline{h}'(x)$ can be calculated from the curvature tensor and its covariant derivatives:

Lemma 1 ([8, Ch. III, p. 152, Lemma 1, Th. 9.2]). *For smooth (C^∞) connections, the Lie algebra $\underline{h}'(x)$, as a vector space, is a span of the linear maps*

$$\nabla^k R(X,Y;Z_1,\dots,Z_k)\,,\qquad X,Y,Z_1,\dots,Z_k\in T_xM\,,\quad 0\le k<\infty\,. \tag{7}$$

Lemma 2 ([8, Ch. II, p. 101, Th. 10.8]). *Holonomy groups of a real analytic connection on a real analytic manifold satisfy $\Phi'(x)=\Phi^*(x)=\Phi^0(x)$, $x\in M$.*

Corollary 1. *In the real analytic case, the holonomy algebra coincides with the infinitesimal holonomy algebra, $\underline{h}(x)=\underline{h}'(x)$, hence the component of unit $\Phi^0(x)$ can be retrieved from $\underline{h}'(x)$. Moreover, the restricted holonomy group of a connected real analytic manifold (M,∇) with an analytic connection is fully determined by the curvature tensor R and its iterated covariant derivatives $\nabla^k R$, $k\in\mathbb{N}$.*

2.2. Hol_x^∇**-invariant bilinear forms.** The possibility to employ holomomy groups in order to solve the metrization problem for linear connections was discussed e.g. in [12], [1]. The holonomy group "decides" whether a connection is metrizable or not. Given a connection on a connected simply connected manifold, $\Phi(x)$ is a connected Lie subgroup of the automorphism (transformation) group $GL(T_xM)$ of the fibre, therefore it is uniquely determined by its Lie algebra. If a connection is induced by a certain metric, then the scalar product defined by g on particular tangent spaces is preserved by parallel translations, therefore elements of the holonomy group are isometries of the tangent space, and $\Phi^0(x)$ identifies with a subgroup of $SO(T_xM)$, i.e., according to the signature of the metric, with a subgroup of $SO(n)$ or of $SO(p,q)$, $p+q=n$, respectively. On the other hand, if $\Phi^0(x)$ is a subgroup of the special orthogonal group of the fibre at one[4] point then we can define scalar product on this particular fibre[5] T_xM, and create a compatible metric using parallel transport, [1], [12], [10]. In simple examples, it works, [15]. The following shows how to characterize quadratic forms invariant under the holonomy group in terms of the holonomy algebra.

Lemma 3. *Let (M,∇) be a simply connected smooth manifold with ∇ torsion-free, $x\in M$ a fixed point. Given a symmetric bilinear form G on T_xM, $G\in S^2(T_x^*M)$, then the following holds: G is invariant by $\Phi(x)$ if and only if*

$$G(AX,Y)+G(X,AY)=0\quad \textit{for all}\quad A\in\underline{h}(x),\ X,Y\in T_xM. \tag{8}$$

Proof. We check here that elements of the holonomy algebra satisfy (8). The other implication also holds but the proof is not so trivial. If $A\in\underline{h}(x)$ consider the corresponding one-parameter subgroup $s^A:\mathbb{R}\to\Phi(x)$, $t\mapsto s^A(t)$ uniquely determined by the initial data $s^A(0)=1$, $(s^A)'(0):=(\frac{d}{dt})_{t=0}\,s^A(t)=A$. Let G be

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[4] we mean here a "nice", generic point, cf. the definition below

[5] It can be done e.g. using a fixed chart about x: we can suppose that the tangent space is isomorphic to $(R^n,\langle\,,\,\rangle)$, endowed with a standard scalar product, [1].

invariant under the holonomy group[6]. Then we get $G(s^A(t)X, s^A(t)Y) = G(X,Y)$ for $X, Y \in T_xM$. Let us differentiate with respect to t, making use of the formula for scalar product[7], and consider $t \to 0$:

$$G((s^A)'(0)(X), s^A(0)(Y)) + G(s^A(0)(X), (s^A)'(0)(Y)) = 0\,.$$

That is, (8) is satisfied. $\square$

In general, we can not calculate the holonomy group from the curvature tensor and its covariant derivatives. It might be difficult to find the holonomy group and a quadratic form invariant under it. The situation is easier in the real analytic case: the assumptions on $\Phi(x)$ can be reformulated as assumptions on $\underline{h}(x)$. The above gives us a quite natural motivation for introducing the vector subspace $H(x)$, $x \in M$, of all symmetric bilinear forms satisfying the condition from Lemma 3,

$$\text{(9)}\quad H(x) := \{G_x \in S^2(T_x^*M) \mid G_x(AX,Y) + G_x(X,AY) = 0, A \in \underline{h}(x),\ X, Y \in T_xM\}\,.$$

Theorem 1 ([12], a free paraphrase)**.** *Let (M,∇) be connected and let there exist $G_{x_0} \in H(x_0)$*[8]*. Then ∇ is the Levi-Civita connection of a metric on M which has the same signature as G_{x_0}.*

If ∇ is Riemannian (comes from a positive definite metric) then for every $x \in M$, $H(x)$ includes a positive definite form; under additional assumptions, the converse also holds.

Theorem 2 ([10, Prop. 1], [12])**.** *Given a connected simply connected (M,∇) and $x \in M$, let there be a positive definite form $G_{x_0} \in H(x)$. Then ∇ is Riemannian.*

It might be a problem to check whether there is a positive definite form in $H(x)$. Since no direct decision algorithm based on linear algebra only is available, an algorithm using geometric properties of the Levi-Civita conection, particularly the canonical (de Rham) decomposition of the tangent space T_xM of a Riemannian manifold (M,g) with respect to $\Phi(x)$ was developed, [10].

3. Decision algorithm

In [10], O. Kowalski proposed an algorithm based on the holonomy algebra which enables to decide effectively whether a given (analytic) connection on an analytic manifold satisfying additional conditions is a Riemannian one, and suggested a method for constructing all corresponding Riemannian metrics in the affirmative case. Let (M,∇) be connected, simply connected and analytic. Denote

$$\text{(10)}\quad \underline{h}_r(x) = \operatorname{span}\{\nabla^k R(X,Y;Z_1,\dots,Z_k), X, Y, Z_1,\dots,Z_k \in T_xM,\ x \in M,\ 0 \le k \le r\}\,.$$

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[6]That is, $G(\tau X, \tau Y) = G(X,Y)$ for any $\tau \in \Phi(x)$.

[7]$G'(u(t), v(t)) = G(u'(t), v(t)) + G(u(t), v'(t))$.

[8]Originally: "Let $\Phi(x_0)$ keeps a non-degenerate quadratic form G_{x_0} (on $T_{x_0}M$) invariant"

A point x will be called Φ-*regular* (regular, generic) if for each $r \in \mathbb{N}$, the dimension $\dim \underline{h}_r(x)$ of the subspace reaches its maximum in a neighborhood $U_x \ni x$. The set of Φ-regular points is a dense open subset in M. At a regular point, the sequence of subspaces stabilizes for some N, $\underline{h}_N(x) = \underline{h}_{N+1}(x)$, and the same must hold in a neighborhood. We get[9] $\underline{h}(y) = \underline{h}_N(y)$ at any point $y \in U_x$. Hence if we find the subspaces $\underline{h}_r(y)$ in a coordinate neighborhood of a point x we are able to decide whether x is regular or not; about a regular point, we can calculate the algebra $\underline{h}(y)$ in the given coordinate system. Let us give here a shortened version of the decision process (according to which a computer program can be constructed).

Algorithm.

We choose local coordinates in an open subset $U \subset M$ about x, and calculate the curvature tensor, its covariant derivatives at x, and the subspaces $\underline{h}_o(x) \subset \underline{h}_1(x) \subset \dots$ at x (step by step). If there is a natural number N such that $\underline{h}_N(x) = \underline{h}_{N+1}(x)$ then x is Φ-regular. If not we try another point.

If x is regular we calculate the space $H(x)$ as the solution space of the system of homogeneous equations obtained from (9) when we put successively $A = \nabla^r R$, $r = 0, \dots, N$. In fact, we find solution of the equations from (1), (2) corresponding to $r \leq N$. Find $\dim H(x)$. If $p = \dim H(x) = 0$, ∇ is not Riemannian[10], and we STOP the process. If $p = \dim H(x) \geq 1$ we choose a basis $\langle G^1, \dots, G^p \rangle$ of $H(x)$; any element of $H(x)$ has a unique expression in the form $G = \lambda_1 G^1 + \dots + \lambda_p G^p$. Local components $G^m_{k\ell}$ of the base forms are rational functions of the components of covariant derivatives of the curvature tensor.

To decide whether there is a regular form in $H(x)$ or not, compute the determinant $\det(\sum_m \lambda_m G^m_{k\ell})$, $k, \ell \in \{1, \dots, n\}$, viewed as a polynomial of independent variables λ_ℓ. If the resulting polynomial is non-zero, ∇ is not Riemannian[11], STOP. If we get a non-zero polynomial then there is a regular form in $H(x)$, and we can continue our search. Step by step, we choose integers $\hat{\lambda}_1, \dots, \hat{\lambda}_p$ so that to obtain a particular (regular) form $\hat{h} \in H(x)$.

In our local coordinates, let us calculate the linear operators[12] $S^1, \dots, S^p$ corresponding to respective base elements $G^1, \dots, G^p$ via the regular form $\hat{h}$ according to the formulae

$$\hat{h}(S^\ell X, Y) = G^\ell(X, Y), \quad X, Y \in T_x M, \quad \ell = 1, \dots, p. \tag{11}$$

Let us evaluate a span in $\mathrm{End}\,(T_x M)$ of all commutators of the above endomorphisms, the commutant $C_x = \mathrm{span}\,\{[S^\ell, S^k]; \ell, k = 1, \dots, p\}$ of the set $\{S^1, \dots, S^p\}$. Find the common null-space N_x of C_x. If N_x is not invariant under $S^1, \dots, S^p$, or if the restriction $\hat{h}|N_x$ is not regular then ∇ is not Riemannian, STOP. Otherwise continue.

[9] using covariant differentiation

[10] and is not metrizable, either.

[11] and is not metrizable, either.

[12] Each S^ℓ is an symmetric endomorphism w.r.t. $\hat{h}$ of the space $(T_x M, \hat{h})$.

If $C_x \neq (0)$ find the orthogonal complement $\hat{N}_x$ of N_x in T_xM with respect to $\hat{h}$. Calculate the restrictions $G^\ell|\hat{N}_x$, $\ell = 1, \dots, p$. If they do not generate the space $S^2(\hat{N}_x^*)$ of all symmeric bilinear forms on $\hat{N}_x$ then ∇ is not Riemannian, STOP. If they generate $S^2(\hat{N}_x^*)$ we continue; if $C_x = (0)$ we continue directly.

In the family of the restrictions $S^1|N_x, \dots, S^p|N_x$, find a set of independent generators $S^{(1)}, \dots, S^{(s)}$ of the space $H(x)|N_x$. Calculate all eigenspaces of $S^{(1)}, \dots, S^{(s)}$ and all possible intersections $Z^{(1)_\alpha} \cap \cdots \cap Z^{(s)_\gamma}$ of various eigenspaces of $S^{(1)}, \dots, S^{(s)}$. Let $(0), L_1, \dots, L_r$ be just the set of all intersections. Then the necessary conditions for ∇ be Riemannian are:

- $r = s$, and $N_x = L_1 \oplus \cdots \oplus L_r$ (the orthogonal decomposition w.r.t. $\hat{h}$).

If the above conditions are not satisfied, STOP. If they are satisfied continue.

If each of the restrictions $\hat{h}|L_j$ is either positive or negative definite, ∇ is Riemannian (and $\hat{N}_x := T_x^0 \subset T_xM$ is the subspace on which $\Phi(x)$ acts trivially). If this is not the case then ∇ is not Riemannian, STOP. □

Note that if $n = 2$, and the manifold (M_2, ∇) is real analytic connected and simply connected then the decision procedure can be simplified; ∇ is Riemannian only in two cases, namely, either at the given Φ-regular point x, $p = \dim H(x) = 1$ and the space $H(x)$ is generated by a positive definite form, or $p = 3$ (which happens if and only if $R = 0$), and then the connection ∇ is Euclidean.

Example 2. To demonstrate the above Algorithm let us solve Example 1 by the above method. Choose an arbitrary point $x{=}(x_0, y_0)$; in our case, it is surely regular; all the objects involved will be calculated in fact not only about x, but on the entire manifold. Independently of the point, the holonomy algebra (the restricted holonomy group, respectively) is trivial, $\underline{h} = (0)$, $\Phi^0(x) = (\mathrm{id}\,)$. Consequently, since only the zero morphism $A = 0$ is to be considered in (9), the condition holds for any symmetric bilinear form on the tangent space at x. Hence the subspace $H(x) \subset S^2(T_x^*\mathbb{R}^2)$, $x \in \mathbb{R}^2$, satisfying (9) is of maximal dimension $p = 3$, and can be given e.g. as $H(x) = \operatorname{span}\{G^{(1)}(x), G^{(2)}(x), G^{(3)}(x)\}$ where $G^{(i)}(x)$ (satisfying (9) for any x in the manifold) are the same as in Example 1. Obviously, $G^{(1)} + G^{(2)} \in H(x)$ is positive definite. Nevertheless, for illustration choose the regular form $\hat{g} = G^{(3)}$ which is not positive definite (and has constant components on the manifold). At any $x \in \mathbb{R}^2$, the symmetric operators from (11) and the commutant of the set $\{S^1, S^2, S^3\}$ have matrix representations

$$S^1_{ij} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S^2_{ij} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^3_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_x = \operatorname{span}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

The commutant is non-trivial, so let us determine the orthogonal complement $\hat{N}_x$ of its nullspace $N_x = \{(0,0)\}$ with respect to our form $\tilde{g} = G^{(3)}$, which should coincide with the maximal subspace on which Hol_x acts trivially: $T^0 = \hat{N}_x = T_x\mathbb{R}^2$. The space $S^2(T_x^*\mathbb{R}^2)$ is a span of the restrictions $G^{(\alpha)}|T^0 = G^{(\alpha)}$. The remaining conditions are automatically satisfied due to triviality of N_x. Hence the Riemannian metrics do exist.

4. Construction of metrics

The following results enable us to construct the metric explicitely.

Theorem 3 ([10, p. 8], a free paraphrase). *On a connected simply connected analytic manifold with an analytic connection ∇, let U_x be an open neighborhood of $x \in M$ formed exclusively by Φ-regular points. Let ∇ be Riemannian (metrizable and positive definite), and let $\hat{g} \in H(x)$ be regular on U. Let $H^{(1)}, \dots, H^{(t)}$ be analytic tensor fields on U such that for any $y \in U$, $H^{(1)}_y, \dots, H^{(t)}_y$ are linearly independent symmetric bilinear forms on T_yM, with the same null-space equal to N_y, and let the restrictions $H^{(1)}|\hat{N}_y, \dots, H^{(t)}|\hat{N}_y$ to the complement $\hat{N}_y$ of N_y span the space $S^2(\hat{N}^*_y)$. Then there exist 1-forms ω^i_j on U such that $H^{(i)} = \sum \omega^i_j \otimes H^{(j)}$, $1 \leq i, j \leq t$. Moreover, the system of linear homogeneous PDE's $\mathrm{d}\lambda_i + \lambda_k \omega^k_i = 0$, $1 \leq i \leq t$, is completely integrable.*

Theorem 4 ([10, p. 9], a free paraphrase). *Under the same assumptions and notation as above, suppose that for any $y \in U_x$, $N_y = L_{1,y} \oplus \cdots \oplus L_{s,y}$ is the orthogonal decomposition w.r.t. $\hat{g}$. Let h_i denote the tensor field on U such that its null-space at $y \in U$ coincides with the orthogonal complement of $L_{i,y}$ in T_yM w.r.t. $\hat{g}$, and which coincides with $\hat{g}$ on $L_{i,y}$ for any $y \in U$. Then there exist exact 1-forms ω_i (first integrals, $\omega_i = \mathrm{d}f_i$), such that $\nabla h_i = \omega_i \otimes h_i$, $1 \leq i \leq s$ (i.e. h_i are recurrent).*

Theorem 5. *Under the same assumptions as above, with $H^{(i)}$ and h_i analytic on U, all admissible Riemannian metrics are of the form*

$$g = \sum_{i,k=1}^{t} b_i \lambda^i_k H^{(k)} + \sum_{k=1}^{s} c_k e^{-f_k} h_k,$$

where f_j is some primitive function of the exact diferential form ω_j, $1 \leq j \leq s$, the functions $(\lambda^i_1, \dots, \lambda^i_t)$, $1 \leq i \leq t$ form a basis of the solution space of the completely integrable system from Theorem 3, and the real parameters b_i, c_k are chosen in such a way that g is positive definite.

5. Application

Given a system of second order ODE's of a particular form

$$\ddot{x}^k + \Gamma^k_{rs}(x)\dot{x}^r\dot{x}^s = 0\,, \qquad k = 1, \dots, n\,, \tag{12}$$

that is, second derivatives can be expressed as quadratic forms in first derivatives (where $\Gamma^k_{rs}(x)$ are smooth functions), we can use the above theory for deciding whether the system is derivable from a Lagrangian. Namely, we can assume that the functions $\Gamma^k_{rs}(x)$ are components of a symmetric linear connection ∇ on some neighborhood $U \subset \mathbb{R}^n$, and curves solving the equations (12) are geodesics of ∇. If ∇ is (locally) metrizable, $g_{ij}(x)$ (with $\det(g_{ij}(x)) \neq 0$ at any $x \in U$) being components of some non-degenerate metric g compatible with ∇ on U then the system of equations (12) is equivalent to the system

$$g_{ik}\left(\ddot{x}^k + \Gamma^k_{rs}(x)\dot{x}^r\dot{x}^s\right) = 0, \qquad k = 1, \dots, n, \tag{13}$$

hence the functions[13] $g_{ik}(x)$ can be taken as variational multipliers, and the corresponding Lagrangian (kinetic energy) is

$$T = \frac{1}{2} g_{ij}(x)\dot{x}^i \dot{x}^j \,. \tag{14}$$

In simple cases we can decide about solution of $\nabla g = 0$ directly:

Example 3 ([3, p. 122])**.** On $\mathbb{R}^2$ with coordinates $x = (x^1, x^2)$, assume the system of ODE's (ordinary differential equations)

$$\begin{aligned} (\ddot{x}^1)^2 + (x^1 - x^2)(\dot{x}^1)^2 = 0, \\ (\ddot{x}^2)^2 + (x^1 - x^2)(\dot{x}^2)^2 = 0. \end{aligned} \tag{15}$$

Curves $c(s) : I \to \mathbb{R}^2$ (parametrized by arc length), which are solutions of the system, represent the family of geodesics of a (symmetric) linear connection ∇ with components $\Gamma^1_{11} = \Gamma^2_{22} = x^1 - x^2$, $\Gamma^i_{jk} = 0$ otherwise, or equivalently,

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1} = (x^1 - x^2)\frac{\partial}{\partial x^1} \,, \\ &\nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^2} = (x^1 - x^2)\frac{\partial}{\partial x^2} \,, \quad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0 \quad \text{otherwise.} \end{aligned}$$

Let us solve directly the system of equations arising from the condition $\nabla g = 0$ for (smooth) functions $g_{ij}(x^1, x^2)$, which should be components of a symmetric non-singular functional matrix $G = (g_{ij})$ (in short we write ∂_k instead of $\frac{\partial}{\partial x_k}$):

$$\begin{aligned} &\partial_1 g_{11} = (x^1 - x^2) g_{11} \,, && \partial_2 g_{11} = 0 \,, \\ &\partial_1 g_{12} = 0 \,, && \partial_2 g_{12} = (x^1 - x^2) g_{12} \,, \quad g_{ij} = 0 \text{ for all } \ i, j \,, \\ &\partial_1 g_{22} = 0 \,, && \partial_2 g_{22} = (x^1 - x^2) g_{22}; \quad \text{max rank} = q = 0 \,. \end{aligned}$$

Hence $G = 0$, the only solution is trivial, the connection is not metrizable, and there are no variational multipliers of the form $g_{ij}(x)$.

Example 4. The system

$$\ddot{x}^1 = 0, \quad \ddot{x}^2 = -2\dot{x}^1 \dot{x}^2 \tag{16}$$

defines on $\mathbb{R}^2$ (or on the cylinder $\mathbb{R} \times \mathbb{S}^1$, or on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$) a symmetric linear connection ∇ which is not a metric one. In fact, the only non-zero Christoffels are $\Gamma^2_{12} = \Gamma^2_{21} = 1$, i.e. ∇ can be introduced also by

$$\nabla_{X_1} X_1 = \nabla_{X_2} X_2 = 0, \quad \nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_2, \quad X_i = \frac{\partial}{\partial x_i} \,.$$

Solving directly the corresponding system of differential equations we get

$$\begin{aligned} &\partial_1 g_{11} = 0 \,, && \partial_2 g_{11} = 2 g_{12} \,, \quad G = (g_{ij}) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \ a \in \mathbb{R} \,, \\ &\partial_1 g_{12} = g_{12} \,, && \partial_2 g_{12} = g_{22} \,, \\ &\partial_1 g_{22} = 2 g_{22} \,, && \partial_2 g_{22} = 0 \,, \qquad \text{max rank} = q = 1 \,, \end{aligned}$$

[13] In general, there might exist multipliers of a more general form $g_{ik}(t, x, \dot{x})$, depending on "time, positions and velocities", which might bring more complicated Lagrangians.

the corresponding tensor is degenerate; no multipliers of the form $g_{ij}(x)$ can be found.

Example 5. Assume the system

$$\ddot{x}+\frac{x}{x^2+1}\dot{x}^2=0\,,\qquad \ddot{y}+\frac{y}{y^2+1}\dot{y}^2=0\,. \tag{17}$$

The corresponding connection ∇ (on $\mathbb{R}^2$) was described in Example 1, and metrizability was confirmed in Example 2. Variational multipliers $g_{ij}(x)$ for the system can be found as components of a non-degenerate metric g compatible with ∇. According to the Theorems 3, 4, 5 the metrics we are searching for are in fact of the form $g=b_i\lambda^i_k H^{(k)}$, $1\le i,k\le 3$ (the second summand vanishes due to triviality of N_y on the neighborhood). As tensor fields $H^{(i)}$, we can choose $H^{(1)}=\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x$, $H^{(2)}=\mathrm{d}x\otimes\mathrm{d}x+\mathrm{d}y\otimes\mathrm{d}y$, $H^{(3)}=\mathrm{d}x\otimes\mathrm{d}x-\mathrm{d}y\otimes\mathrm{d}y$. Their covariant derivatives are

$$\begin{aligned}
\nabla H^{(1)} &= -\tfrac{x}{x^2+1}(\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x)\otimes\mathrm{d}x-\tfrac{y}{y^2+1}(\mathrm{d}x\otimes\mathrm{d}y+\mathrm{d}y\otimes\mathrm{d}x)\otimes\mathrm{d}y\,,\\
\nabla H^{(2)} &= -\tfrac{2x}{x^2+1}\mathrm{d}x\otimes\mathrm{d}x\otimes\mathrm{d}x-\tfrac{2y}{y^2+1}\mathrm{d}y\otimes\mathrm{d}y\otimes\mathrm{d}y\,,\\
\nabla H^{(3)} &= -\tfrac{2x}{x^2+1}\mathrm{d}x\otimes\mathrm{d}x\otimes\mathrm{d}x+\tfrac{2y}{y^2+1}\mathrm{d}y\otimes\mathrm{d}y\otimes\mathrm{d}y\,.
\end{aligned}$$

Hence the forms satisfy $\nabla H^{(i)}=\omega^i_j\otimes H^{(i)}$ with

$$\begin{array}{ll}
\omega^1_1=-\frac{x}{x^2+1}\mathrm{d}x-\frac{y}{y^2+1}\mathrm{d}y, & \omega^1_2=\omega^1_3=\omega^2_1=\omega^3_1=0,\\
\omega^2_2=-\frac{x}{x^2+1}\mathrm{d}x-\frac{y}{y^2+1}\mathrm{d}y, & \omega^2_3=-\frac{x}{x^2+1}\mathrm{d}x+\frac{y}{y^2+1}\mathrm{d}y,\\
\omega^3_2=-\frac{x}{x^2+1}\mathrm{d}x+\frac{y}{y^2+1}\mathrm{d}y, & \omega^3_3=-\frac{x}{x^2+1}\mathrm{d}x-\frac{y}{y^2+1}\mathrm{d}y.
\end{array}$$

The solution space of the system of linear PDE's

$$\begin{aligned}
\mathrm{d}\lambda_1 &= \lambda_1\frac{x}{x^2+1}\mathrm{d}x+\lambda_1\frac{y}{y^2+1}\mathrm{d}y,\\
\mathrm{d}\lambda_2 &= (\lambda_2+\lambda_3)\frac{x}{x^2+1}\mathrm{d}x+(\lambda_2-\lambda_3)\frac{y}{y^2+1}\mathrm{d}y,\\
\mathrm{d}\lambda_3 &= (\lambda_2+\lambda_3)\frac{x}{x^2+1}\mathrm{d}x-(\lambda_2-\lambda_3)\frac{y}{y^2+1}\mathrm{d}y
\end{aligned}$$

has a basis $\langle\lambda^1,\lambda^2,\lambda^3\rangle$ formed by triples of functions $\lambda^i=(\lambda^i_1,\lambda^i_2,\lambda^i_3)$, $1\le i\le 3$,

$$\lambda^1=(\sqrt{x^2+1}\sqrt{y^2+1},0,0)\,,\quad \lambda^2=(0,x^2+1,x^2+1)\,,\quad \lambda^3=(0,y^2+1,y^2+1)\,.$$

We can see the desired multipliers from the matrix representation of the compatible Riemannian metrics:

$$(g_{ij})=\begin{pmatrix} 2b_2(x^2+1) & b_1\sqrt{x^2+1}\sqrt{y^2+1}\\ b_1\sqrt{x^2+1}\sqrt{y^2+1} & 2b_3(y^2+1)\end{pmatrix}$$

where the (real) parameters b_1,b_2,b_3 should be chosen so that g be positive definite. In tensor notation, all compatible tensors are $g=2b_2(x^2+1)\mathrm{d}x\otimes\mathrm{d}x+b_1\sqrt{x^2+1}\sqrt{y^2+1}\mathrm{d}x\otimes\mathrm{d}y+b_1\sqrt{x^2+1}\sqrt{y^2+1}\mathrm{d}y\otimes\mathrm{d}x+2b_3(y^2+1)\mathrm{d}y\otimes\mathrm{d}y$, i.e. in the classical notation, all admissible Riemannian metrics are

$$\mathrm{d}s^2=2b_2(x^2+1)\mathrm{d}x^2+2b_1\sqrt{x^2+1}\sqrt{y^2+1}\mathrm{d}x\mathrm{d}y+2b_3(y^2+1)\mathrm{d}y^2\,.$$

Using (14) we get (some of) the Lagrangians.

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