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A GENERALIZATION OF THOM'S TRANSVERSALITY THEOREM

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Abstract. We prove a generalization of Thom's transversality theorem. It gives conditions under which the jet map $f_*|_Y\colon Y \subseteq J^r(D,M) \to J^r(D,N)$ is generically (for $f\colon M \to N$) transverse to a submanifold $Z \subseteq J^r(D,N)$. We apply this to study transversality properties of a restriction of a fixed map $g\colon M \to P$ to the preimage $(j^s f)^{-1}(A)$ of a submanifold $A \subseteq J^s(M,N)$ in terms of transversality properties of the original map f. Our main result is that for a reasonable class of submanifolds A and a generic map f the restriction $g|_{(j^s f)^{-1}(A)}$ is also generic. We also present an example of A where the theorem fails.

0. Introduction

We start by reminding that for smooth manifolds M and N the set $C^\infty(M,N)$ of smooth maps is endowed with two topologies called weak (compact-open) and strong (Whitney) topology. They agree when M is compact. We say that a subset of a topological space is *residual* if it contains a countable intersection of open dense subsets. The Baire property for $C^\infty(M,N)$ then guarantees that it is automatically dense. This holds for both topologies but is almost exclusively used for the strong one. Clearly every residual subset of $C^\infty(M,N)$ for the strong topology is also residual for the weak topology. The following is our main theorem in which we denote by $J^r_{\mathrm{imm}}(D,M)$ the subspace of all jets of immersions.

Theorem A. *Let D, M, N be manifolds, $Y \subseteq J^r_{\mathrm{imm}}(D,M)$ and $Z \subseteq J^r(D,N)$ submanifolds. Let us further assume that $\sigma_Y \pitchfork \sigma_Z$, where*

$$\sigma_Y = \sigma|_Y : Y \subseteq J^r(D,M) \to D \quad \text{and} \quad \sigma_Z = \sigma|_Z : Z \subseteq J^r(D,N) \to D$$

are the restrictions of the source maps. For a smooth map $f : M \to N$ let $f_*|_Y$ denote the map

$$Y \hookrightarrow J^r_{\mathrm{imm}}(D,M) \xrightarrow{f_*} J^r(D,N)\ .$$

Then the subset

$$\mathfrak{X} := \left\{ f \in C^\infty(M,N) \;\middle|\; f_*|_Y \pitchfork Z \right\}$$

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is residual in $C^\infty(M,N)$ with the strong topology, and open if Z is closed (as a subset) and the target map $\tau_Y : Y \to M$ is proper.

We are interested in the theorem mainly because of the following two applications, the first of which is the classical theorem of Thom.

Theorem B (Thom's Transversality Theorem)**.** *Let M, N be manifolds, $Z \subseteq J^r(M,N)$ a submanifold. Then the subset*

$$\mathfrak{X} := \{f \in C^\infty(M,N) \mid j^r f \pitchfork Z\}$$

is residual in $C^\infty(M,N)$. It is moreover open provided that Z is closed (as a subset).

To explain the second application we need to introduce a bit of notation. Let us consider the following diagram

$$\begin{tikzcd}
 & f^{-1}(A) \arrow[r] \arrow[d, hook, "j"] & A \arrow[d, hook] \\
g^{-1}(B) \arrow[r, hook] \arrow[d] & M \arrow[r, "f"] \arrow[d, "g"] & N \\
B \arrow[r, hook] & P &
\end{tikzcd}$$

of smooth manifolds and smooth maps between them where $\hookrightarrow$ indicates embeddings. We assume that $f \pitchfork A$ and $g \pitchfork B$ for the two pullbacks to be defined (which we emphasize by saying that they are transverse pullbacks) and also for some technical reasons. Suppose that we fix g and allow ourselves to change f (but only in such a way that $f \pitchfork A$). It is not hard to describe when the composition $gj = g|_{f^{-1}(A)}$ is transverse to B. A more general condition on gj is whether it satisfies some form of jet transversality. First we have to solve the problem of not having the source of $J^r(f^{-1}(A), P)$ fixed.

Construction. Let D be a d-dimensional manifold and

$$\mathrm{Diff}(\mathbb{R}^d, 0) = \mathrm{inv}G_0(\mathbb{R}^d, \mathbb{R}^d)_0$$

the group[1] of germs at 0 of local diffeomorphisms $\mathbb{R}^d \to \mathbb{R}^d$ fixing 0. Define a principal $\mathrm{Diff}(\mathbb{R}^d, 0)$-bundle

$$\mathrm{Charts}_D = \mathrm{inv}G_0(\mathbb{R}^d, D) \xrightarrow{\mathrm{ev}_0} D$$

of germs at 0 of local diffeomorphisms $\mathbb{R}^d \to D$. If F is any manifold with a (smooth in some sense) action of $\mathrm{Diff}(\mathbb{R}^d, 0)$ then we can construct an associated bundle

$$D[F] := \mathrm{Charts}_D \times_{\mathrm{Diff}(\mathbb{R}^d,0)} F \longrightarrow D.$$

[1] If we wanted to give $\mathrm{Diff}(\mathbb{R}^d, 0)$ a topology we could do so by inducing the topology via the map $\mathrm{Diff}(\mathbb{R}^d, 0) \to J^\infty_0(\mathbb{R}^d, \mathbb{R}^d)_0$. Or one could replace $\mathrm{Diff}(\mathbb{R}^d, 0)$ by its image - the subspace of invertible ∞-jets.

Any bundle of this form is "local". Observe that this construction is functorial in D on the category of d-dimensional manifolds and local diffeomorphisms. As an example the bundle $J^r(D,P)$ of r-jets of maps $D \to P$ is a local bundle as

$$D[J_0^r(\mathbb{R}^d,P)] = \mathrm{Charts}_D \times_{\mathrm{Diff}(\mathbb{R}^d,0)} J_0^r(\mathbb{R}^d,P) \cong J^r(D,P)$$

where $J_0^r(\mathbb{R}^d,P)$ is the subspace of $J^r(\mathbb{R}^d,P)$ of r-jets with source 0. The bijection is provided by the map

$$[u,\alpha] \mapsto \alpha \circ j^r_{u(0)}(u^{-1}).$$

Having a $\mathrm{Diff}(\mathbb{R}^d,0)$-invariant submanifold $B \subseteq J_0^r(\mathbb{R}^d,P)$ we get an associated subbundle $D[B] \subseteq J^r(D,P)$ for any d-dimensional manifold D. This allows us to talk about jet transversality conditions on a map $D \to P$ without specifying what D (and hence also $J^r(D,P)$) is.

Let $A \subseteq J^s(M,N)$ be a submanifold and $j^s f \pitchfork A$. Then $f^*A := (j^s f)^{-1}(A)$ is a submanifold of M and for any $\mathrm{Diff}(\mathbb{R}^d,0)$-invariant submanifold $B \subseteq J_0^r(\mathbb{R}^d,P)$ and the associated submanifold

$$f^*A[B] \subseteq J^r(f^*A,P)$$

we might ask whether $j^r(g|_{f^*A})$ is transverse to $f^*A[B]$. To state the theorem we make the following notation: for a map $g\colon M \to P$ we write $g \pitchfork B$ in place of $g_* \pitchfork B$ where again

$$g_*\colon J^r_{0,\mathrm{imm}}(\mathbb{R}^d,M) \longrightarrow J_0^r(\mathbb{R}^d,P)\,.$$

This condition is satisfied e.g. by all submersions. Also consider the following map

$$\begin{aligned}(1)\qquad \kappa : J^{r+s}(M,N) \times_M J^r_{0,\mathrm{imm}}(\mathbb{R}^d,M) &\longrightarrow J_0^r(\mathbb{R}^d, J^s(M,N)),\\ (j_x^{r+s}\varphi, j_0^r\psi) &\longmapsto j_0^r(j^s(\varphi)\circ\psi)\,.\end{aligned}$$

Theorem C. *Let*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"] & N \\ P & \end{tikzcd}$$

be a pair of smooth maps. Let $A \subseteq J^s(M,N)$ and $B \subseteq J_0^r(\mathbb{R}^d,P)$ be smooth submanifolds with $d = \dim M - \operatorname{codim} A$. Assuming that B is $\mathrm{Diff}(\mathbb{R}^d,0)$-invariant, $\kappa \pitchfork J_0^r(\mathbb{R}^d,A)$ and $g \pitchfork B$ the subset

$$\mathfrak{X} := \left\{ f \in C^\infty(M,N) \;\middle|\; j^s f \pitchfork A,\ j^r(g|_{f^*A}) \pitchfork f^*A[B] \right\}$$

is residual in $C^\infty(M,N)$ with the strong topology. If either $r=0$ or $s=0$ the transversality condition $\kappa \pitchfork J_0^r(\mathbb{R}^d,A)$ is automatically satisfied.

Remark. The transversality condition $\kappa \pitchfork J_0^r(\mathbb{R}^d,A)$ cannot be removed as we illustrate by an example in the next section. An interesting question arises whether there is a reasonable sufficient condition on A for which the transversality is automatic (an example of such is e.g. $s=0$).

Remark. It is also possible to state conditions under which $\mathfrak{X}$ is open.

1. An example

We describe here a family of examples. For $k = 1$ (and partially for $k = 2$) Theorem C can be applied whereas for $k \geq 3$ the theorem fails due to $\kappa \not\pitchfork J^r_0(\mathbb{R}^d, A)$. We will not be interested in a particular choice of B since the transversality condition in question depends only on A. Let us start with the following diagram

$$\begin{tikzcd} D^k \times M \arrow[r, "f"] \arrow[d] & \mathbb{R} \\ D^k & \end{tikzcd}$$

where we think of f as a family of functions $M \to \mathbb{R}$ parametrized by a disc D^k. To fit into our situation we should replace D^k by $\mathbb{R}^k$ but the boundary is not the issue in our example. Let $s = 1$ and let $A \subseteq J^1(D^k \times M, \mathbb{R})$ denote the subspace of all those jets which have zero derivative in the direction of M. Clearly A has codimension $\dim M$ and thus $d = k$.

For a function $\varphi\colon D^k \times M \to \mathbb{R}$ we have $j^1_{(x,y)}\varphi \in A$ iff the composition

$$T_yM \hookrightarrow T_xD^k \times T_yM \xrightarrow{\mathrm{d}\varphi} \mathbb{R}$$

is zero. We express this by saying that the map

$$\mathrm{d}|_{TM}\colon J^1(D^k \times M, \mathbb{R}) \to T^*M$$

describes A as the preimage of 0. We will now show how to describe $J^r_0(\mathbb{R}^k, A)$ (or f^*A in fact) in a similar way. Compose the defining equation with an immersion $\psi\colon \mathbb{R}^k \to D^k \times M$ as in (1) and differentiate to get

$$T_yM \otimes \mathbb{R}^k \xhookrightarrow{incl\otimes\psi_*} S^2(T_xD^k \times T_yM) \xrightarrow{\mathrm{d}^2\varphi} \mathbb{R}\ .$$

Differentiating further and putting all the maps together we get a single map

$$J^{r+1}(D^k \times M, \mathbb{R}) \times_M J^r_{0,\mathrm{imm}}(\mathbb{R}^k, D^k \times M) \xrightarrow{\chi} T^*M \otimes (\mathbb{R} \oplus \mathbb{R}^k \oplus \cdots \oplus S^r\mathbb{R}^k)^*$$

describing f^*A. Moreover χ is a submersion at f^*A iff $\kappa \pitchfork J^r_0(\mathbb{R}^k, A)$. Fixing $j^r_0\psi$ the $T^*M \otimes (S^i\mathbb{R}^k)^*$-coordinate is a sum of various restrictions of the derivatives of φ with only a single term involving the highest derivative $\mathrm{d}^{i+1}\varphi$, namely

$$T_yM \otimes S^i\mathbb{R}^k \xhookrightarrow{incl\otimes S^i\psi_*} S^{i+1}(T_xD^k \times T_yM) \xrightarrow{\mathrm{d}^{i+1}\varphi} \mathbb{R}\ .$$

This implies that if the image of the derivative of ψ_* at 0 intersects TM in a subspace of dimension at most 1, the highest order term is a surjective linear map and quite easily the whole map χ is a submersion (even for a fixed ψ). The opposite implication also holds as considerations at pairs with $j^{r+1}\varphi = 0$ show.

For $k = 1$ we have shown that $\kappa \pitchfork J^r_0(\mathbb{R}, A)$ and therefore for a generic map f the fibrewise singularity set $\Sigma_f = f^*A$ is a 1-dimensional submanifold for which the projection $\Sigma_f \to D^1$ is generic. This means that only regular points and folds appear. The regular points of the projection are exactly the fibrewise Morse singularities of f while folds correspond to the fibrewise A_3-singularities (those of the form $x_1^3 \pm x_2^2 \pm \cdots \pm x_m^2$).

For $k \geq 2$ it is *not* the case that $\kappa \pitchfork J_0^r(\mathbb{R}^k, A)$, namely at pairs $(j_x^{r+s}\varphi, j_0^r\psi)$ where the image of the tangential map ψ_* at 0 intersects the tangent space of the fibre M in a subspace of dimension at least 2.

Nevertheless for $k = 2$ the conclusion of Theorem C still holds. This is because the condition $\kappa \pitchfork J_0^r(\mathbb{R}^k, A)$ is only required at those $(j_x^{r+s}\varphi, j_0^r\psi)$ for which $\psi(\mathbb{R}^d) \subseteq f^*A$ and for a generic map f such ψ cannot be tangent to the fibre M (if that was the case then the rank of the fibrewise Hessian of f at x would drop by 2 and this does not happen generically). Therefore again Σ_f is a 2-dimensinal submanifold of $D^2 \times M$ partitioned into three parts: the cusps of the projection $\Sigma_f \to D^2$ (where f attains an isolated fibrewise A_4-singularity); the folds form a 1-dimensional submanifold (where the fibrewise A_3-singularities occur); and the fibrewise Morse singularities.

For $k = 3$ the conclusion of Theorem C fails. This is due to the fact that for a generic map between 3-manifolds the rank drops at most by 1 whereas for a generic 3-parameter family of functions $M \to \mathbb{R}$ the fibrewise Hessian can drop rank by 2.

2. Proofs

First we will show how to translate transversality conditions for a restriction of a fixed map $g\colon M \to P$ to the preimage $f^{-1}(A)$ of a submanifold along a map $f\colon M \to N$ in terms of transversality conditions on f itself. At the end we prove that such properties are generic in the sense that maps satisfying them form a residual subset of $C^\infty(M, N)$. First version does not involve any jet conditions.

Lemma 1. *Let*

$$\begin{tikzcd} & f^{-1}(A) \arrow[r] \arrow[d, hook, "j"] & A \arrow[d, hook] \\ g^{-1}(B) \arrow[r, hook, "i"] \arrow[d] & M \arrow[r, "f"] \arrow[d, "g"] & N \\ B \arrow[r, hook] & P & \end{tikzcd}$$

be a diagram where we assume that $f \pitchfork A$ and $g \pitchfork B$. Then the following conditions are equivalent:

(i) $gj \pitchfork B$,

(ii) $fi \pitchfork A$,

(iii) $f^{-1}(A) \pitchfork g^{-1}(B)$.

Proof. Since (iii) is symmetric it is enough to show the equivalence of (i) and (iii). But (i) is equivalent to the map

$$g_* : T_x f^{-1}(A) \longrightarrow T_{g(x)}P/T_{g(x)}B$$

induced by the derivative of g being surjective for every $x \in f^{-1}(A) \cap g^{-1}(B)$. Because of the assumption $g \pitchfork B$, we have a commutative diagram

$$\begin{tikzcd} T_x f^{-1}(A) \arrow[r] \arrow[d] & T_{g(x)}P/T_{g(x)}B \\ T_xM/T_xg^{-1}(B) \arrow[ur, "\cong"'] & \end{tikzcd}$$

and so (i) is equivalent to (iii). □

Now we will generalize the lemma to jet transversality conditions. Let us recall that for a $\mathrm{Diff}(\mathbb{R}^d, 0)$-invariant submanifold $B \subseteq J^r_0(\mathbb{R}^d, P)$ we have constructed an associated subbundle $D[B] \subseteq J^r(D, P)$ for any d-dimensional manifold D.

Lemma 2. *For $h : D \to P$ the following conditions are equivalent*

(i) $h_* \colon J^r_{0,\mathrm{imm}}(\mathbb{R}^d, D) \to J^r_0(\mathbb{R}^d, P)$ *is transverse to* B,

(ii) $j^r(h) \colon D \to J^r(D, P)$ *is transverse to* $D[B]$.

Proof. Taking associated bundles (i) is clearly equivalent to the transversality of

$$h_* : J^r_{\mathrm{imm}}(D, D) \to J^r(D, P)$$

to $D[B]$. Let $j^r_x(k) \in J^r_{x,\mathrm{imm}}(D, D)_y$ be an r-jet of a diffeomorphism $k : V \to W$ between open subsets of D. Then we have a diagram

$$\begin{tikzcd} J^r_{\mathrm{imm}}(V, D) \arrow[r, "h_*"] \arrow[d, "k_*", "\cong"'] & J^r(V, P) \arrow[d, "k_*", "\cong"'] & V[B] \arrow[l, hook'] \arrow[d, "k_*", "\cong"'] \\ J^r_{\mathrm{imm}}(W, D) \arrow[r, "h_*"] & J^r(W, P) & W[B] \arrow[l, hook'] \end{tikzcd}$$

Now $j^r_x(k)$ in the top left corner is mapped by k_* down to $j^r_y(\mathrm{id})$. Hence we see that it is enough (equivalent) to check the transversality only at $j^r_y(\mathrm{id})$'s for all $y \in D$ for which $h_*(j^r_y(\mathrm{id})) = j^r_y(h) \in D[B]$. For such y the same diagram shows that every $j^r_x(k)$ with target y is mapped by h_* to $D[B]$. Thus the whole fibre over y of the target map

$$J^r_{\mathrm{imm}}(D, D) \xrightarrow{\tau} D$$

is mapped to $D[B]$. The target map τ has a section

$$j^r(\mathrm{id}) : D \to J^r_{\mathrm{imm}}(D, D)$$

and so (i) is finally equivalent to the composite

$$D \xrightarrow{j^r(\mathrm{id})} J^r_{\mathrm{imm}}(D, D) \xrightarrow{h_*} J^r(D, P)$$

being transverse to $D[B]$. This is (ii). □

We say that a map $g : M \to P$ is transverse to a $\mathrm{Diff}(\mathbb{R}^d, 0)$-invariant submanifold $B \subseteq J^r_0(\mathbb{R}^d, P)$, denoted $g \pitchfork B$, if

$$g_* : J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M) \to J^r_0(\mathbb{R}^d, P)$$

is transverse to B. When $r = 0$ this is equivalent to the usual transversality of a map to a submanifold. Let $f \pitchfork A$ where $f : M \to N$ and $A \subseteq N$ is a submanifold. Then we have the following diagram

$$\begin{tikzcd} J^r_{0,\mathrm{imm}}(\mathbb{R}^d, f^{-1}(A)) \arrow[r, hook] \arrow[d, "j_*"] & J^r_0(\mathbb{R}^d, f^{-1}(A)) \arrow[r] \arrow[d, "j_*"] & J^r_0(\mathbb{R}^d, A) \arrow[d] \\ J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M) \arrow[r, hook] & J^r_0(\mathbb{R}^d, M) \arrow[r, "f_*"] & J^r_0(\mathbb{R}^d, N) \end{tikzcd}$$

where both squares are transverse pullbacks. This can be easily seen in local coordinates. Combining Lemma 1 with Lemma 2 we get:

Lemma 3. *Given a diagram*

$$\begin{tikzcd} f^{-1}(A) \arrow[r] \arrow[d, hook, "j"] & A \arrow[d, hook] \\ M \arrow[r, "f"] \arrow[d, "g"] & N \\ P & \end{tikzcd}$$

assume that $f \pitchfork A$ *and* $g \pitchfork B$, *where* $B \subseteq J^r_0(\mathbb{R}^d, P)$ *is a* $\mathrm{Diff}(\mathbb{R}^d, 0)$*-invariant submanifold with* $d = \dim M + \dim A - \dim N$. *Then the following conditions are equivalent:*

(i) $j^r(gj) \pitchfork (f^{-1}(A))[B]$, *where*

$$j^r(gj) : f^{-1}(A) \to J^r(f^{-1}(A), P)$$

is the jet prolongation,

(ii) $f_*|_Y \pitchfork J^r_0(\mathbb{R}^d, A)$, *where* $Y = (g_*)^{-1}(B)$ *is defined by a pullback diagram*

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M) \arrow[d, "g_*"] \\ B \arrow[r, hook] & J^r_0(\mathbb{R}^d, P) \end{tikzcd}$$

Proof. Applying Lemma 1 to the diagram

$$\begin{tikzcd} & J^r_{0,\mathrm{imm}}(\mathbb{R}^d, f^{-1}(A)) \arrow[r] \arrow[d, "j_*"] & J^r_0(\mathbb{R}^d, A) \arrow[d] \\ Y \arrow[r, hook] \arrow[d] & J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M) \arrow[r, "f_*"] \arrow[d, "g_*"] & J^r_0(\mathbb{R}^d, N) \\ B \arrow[r, hook] & J^r_0(\mathbb{R}^d, P) & \end{tikzcd}$$

gives an equivalence of (ii) with the transversality of

$$(gj)_* : J^r_{0,\mathrm{imm}}(\mathbb{R}^d, f^{-1}(A)) \longrightarrow J^r_0(\mathbb{R}^d, P)$$

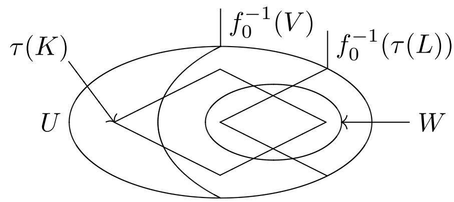
to B. By Lemma 2 this is equivalent to (i). $\square$

Now that we know how f controls the transversality of a map defined on the preimage $f^{-1}(A)$ of some submanifold, we would like to see that this transversality condition (any of the two equivalent conditions in Lemma 3) is generic. This is indeed the case. We first prove a more general result which at the same time happens to generalize the Thom Transversality Theorem.

Proof of Theorem A. This is an application of Lemma 5 from Section 3. We have a map

$$\alpha : C^\infty(M, N) \to C^\infty(Y, J^r(D, N))$$

sending f to $f_*|_Y$. This map is continuous for the weak topology on the target and clearly $\mathfrak{X} = \{f \in C^\infty(M, N) \mid \alpha(f) \pitchfork Z\}$. We have to verify the conditions of Lemma 5.



Let $f_0 \in C^\infty(M, N)$ and $K \subseteq Y$, $L \subseteq Z$ compact discs. We can assume that $\tau(K)$ lies in a coordinate chart $\mathbb{R}^m \cong U \subseteq M$ and that $\tau(L)$ lies in a coordinate chart $\mathbb{R}^n \cong V \subseteq N$. We use these charts to identify U with $\mathbb{R}^m$ and V with $\mathbb{R}^n$ when needed. Let $\lambda : U \to \mathbb{R}$ be a compactly supported function such that $\lambda = 1$ on a neighborhood of $\tau(K) \cap f_0^{-1}(\tau(L))$ and such that $\lambda = 0$ on $U - f_0^{-1}(V)$. This is summarized in the picture above where we put $W = \operatorname{int} \lambda^{-1}(1)$.

We set $Q := J^r_0(\mathbb{R}^m, \mathbb{R}^n)$ and identify it both with $J^r_*(U, V)$, where $*$ stands for an arbitrary point in U, and also with the space of polynomial mappings $U \to V$. Then we get a map

$$\beta : Q \to C^\infty(M, N)$$

sending q to the function $f_0 + \lambda q$ where the operations are interpreted inside V via the chart. It is continuous (in the strong topology) and the adjoint map

$$\gamma = (\alpha\beta)^\sharp : Q \times Y \to J^r(D, N) \tag{2}$$

is smooth. Thus it is enough to show that (after a suitable restriction) $\gamma \pitchfork Z$. Clearly γ sends $(q, j^r_x(h))$ to $j^r_x((f_0 + \lambda q)h)$. Suppose now that $h(x) \in W$ so that this equals to $j^r_x(f_0 h + qh)$. By restriction we get a map

$$\delta\colon Q \cong Q \times \{j^r_x(h)\} \xrightarrow{\gamma} J^r_x(D, V)\,.$$

In the affine structure on $J^r_x(D, V)$ inherited from the chart, δ is clearly affine. Identifying Q with $J^r_{h(x)}(U, V)$ the linear part of δ is just a precomposition with h

$$h^* : J^r_{h(x)}(U, V) \to J^r_x(D, V). \tag{3}$$

The map h, being an immersion, has (locally - near x) a left inverse π which then gives a right inverse π^* of h^* and so the linear part of δ is surjective and hence it is a submersion.

In the horizontal direction our transversality condition $\sigma_Y \pitchfork \sigma_Z$ applies and so $\gamma \pitchfork Z$ on $Q \times \tau_Y^{-1}(W)$. If $f : M \to N$ is close enough to f_0 then

$$\tau(K) \cap f^{-1}(\tau(L)) \subseteq W$$

(equivalently $f(\tau(K) - W) \subseteq N - \tau(L)$ which is one of the basic open sets for the compact-open topology) and in particular there is a neighbourhood Q' of 0 in Q such that $\beta(Q')$ consists only of such maps. Therefore the restriction of γ to

$$Q' \times K \longrightarrow J^r(D, N)$$

is transverse to L and hence the same is also true in some neighbourhoods of K and L which then form the coverings required in Lemma 5.

If τ_Y happens to be proper then α is continuous even in strong topologies and $\mathfrak{X}$ is a preimage of the open subset of maps $f : Y \to J^r(D, N)$ transverse to Z. $\square$

Now we prove two corollaries of the previous theorem.

Proof of Theorem B. We apply Theorem A to $D = M$ and

$$Y = M \xhookrightarrow{j^r(\mathrm{id})} J^r(M, M) .$$

As $\sigma_Y = \mathrm{id} = \tau_Y$ it is both proper and transverse to σ_Z for any Z. $\square$

Corollary 4. *Let M, N be manifolds, $Y \subseteq J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M)$ and $Z \subseteq J^r_0(\mathbb{R}^d, N)$ submanifolds. Then the subset*

$$\{f \in C^\infty(M, N) \mid f_*|_Y \pitchfork Z\}$$

is residual in $C^\infty(M, N)$ with the strong topology, and open if Z is closed (as a subset) and $\tau_Y : Y \to M$ proper.

Proof. Under the natural identification $\mathbb{R}^d \times J^r_0(\mathbb{R}^d, N) \cong J^r(\mathbb{R}^d, N)$ we can apply Theorem A to $D = \mathbb{R}^d$, the same M, N and Y but with $\mathbb{R}^d \times Z \subseteq J^r(\mathbb{R}^d, N)$ in place of Z. $\square$

Now we are finally able to prove our main theorem.

Proof of Theorem C. Consider first the special case $s = 0$ and

$$A = M \times A_0 \subseteq M \times N = J^0(M, N) .$$

Then $j^0 f \pitchfork A$ iff $f \pitchfork A_0$ and under this assumption Lemma 3 provides a translation between $g|_{f^*A} \pitchfork f^*A[B]$ and $f_*|_Y \pitchfork Z$, where $Z = J^r_0(\mathbb{R}^d, A_0) \subseteq J^r_0(\mathbb{R}^d, N)$. The genericity of $f \pitchfork A_0$ is the usual transversality theorem while the genericity of $f_*|_Y \pitchfork Z$ is the last corollary.

The next step is to replace the transversality to $A_0 \subseteq N$ by a jet transversality condition. Let $A \subseteq J^s(M,N)$ be a submanifold and consider

$$\begin{tikzcd} f^*A \arrow[r] \arrow[d, hook, "j"] & A \arrow[d, hook] \\ M \arrow[r, "j^s f"] \arrow[d, "g"] & J^s(M,N) \\ P & \end{tikzcd}$$

It is possible to apply Lemma 3 to this situation and translate $g|_{f^*A} \pitchfork f^*A[B]$ to the transversality of the composition

$$Y \hookrightarrow J^r_{0,\mathrm{imm}}(\mathbb{R}^d, M) \xrightarrow{(j^s f)_*} J^r_0(\mathbb{R}^d, J^s(M,N))$$

to $J^r_0(\mathbb{R}^d, A)$. We cannot however apply Theorem A directly since we are not interested in all maps $M \to J^s(M,N)$ but only in the holonomic sections (those of the form $j^s f$). This means that in our proof of Theorem A, $Q = J^r_*(\mathbb{R}^m, J^s(\mathbb{R}^m, \mathbb{R}^n))$ has to be replaced by its subspace $J^{r+s}_*(\mathbb{R}^m, \mathbb{R}^n)$ and in general there is no guarantee that the new map γ (see (2)) will be transverse (after restriction) to $Z = J^r_0(\mathbb{R}^d, A)$. This is however easily implied by $\kappa \pitchfork J^r_0(\mathbb{R}^d, A)$ since again we can arrive at the linear part of γ being a composition map (as in (3)) which is then easily identified with κ. $\square$

3. A general transversality lemma

We will formulate a basic lemma for deciding whether a given family of maps contains a dense subset of maps with a particular transversality property. In a sense this is just the essence of any proof of such a statement. We will be considering maps $\varphi : R \to C^\infty(S,T)$. We denote by $\varphi^\sharp$ its adjoint

$$\varphi^\sharp : R \times S \to T.$$

Lemma 5. *Let S, T be smooth manifolds and $Z \subseteq T$ a submanifold. Let there be given two open coverings: $\mathcal{U}$ of S and $\mathcal{V}$ of T. Let R be a topological space and $\varphi : R \to C^\infty(S,T)$ a continuous map where $C^\infty(S,T)$ is given the weak topology. Assume that for every $r_0 \in R$ and every $U \in \mathcal{U}$, $V \in \mathcal{V}$ there is a finite dimensional manifold Q and a continuous map $k : Q \to R$ with r_0 in its image such that*

$$Q \times U \xrightarrow{k \times incl} R \times S \xrightarrow{\varphi^\sharp} T$$

is smooth and transverse to V. Then the subset

$$\mathfrak{X} := \left\{ r \in R \;\middle|\; \varphi(r) \pitchfork Z \right\} \subseteq R$$

is residual in R.

Proof. Following the proof of the Theorem 4.9. of Chapter 4 of [1], let us cover S by a countable family of compact discs K_i that have a neighbourhood $U_i \in \mathcal{U}$ and at the same time we choose a covering of Z by a countable family of compact discs

L_j that have a neighbourhood $V_j \in \mathcal{V}$. Then the set $\mathfrak{X}$ is a countable intersection of the sets

$$\mathfrak{X}_{ij} := \{r \in R \mid \varphi(r) \pitchfork L_j \text{ on } K_i\}$$

and it is enough to show that each $\mathfrak{X}_{ij}$ is open and dense. The set $\hat{\mathfrak{X}}_{ij}$ of maps $S \to T$ transverse to L_j on K_i is open in $C^\infty(S,T)$ and $\mathfrak{X}_{ij} = \varphi^{-1}(\hat{\mathfrak{X}}_{ij})$ so it is also open.

To prove the denseness we fix $r_0 \in R$ and choose a map $k : Q \to R$ with $r_0 = k(q_0)$ such that the map

$$l \colon Q \times U_i \to T$$

from the statement is smooth and transverse to V_j. By the parametric transversality theorem (see e.g. Theorem 2.7, Chapter 3 in [2]) the points $q \in Q$ for which $l(q,-) \pitchfork V_j$ is dense in Q. In particular q_0 lies in the closure of this set and hence r_0 lies in the closure of its image in R. But this image certainly lies in $\mathfrak{X}_{ij}$. $\square$

References

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