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NEW SHEAF THEORETIC METHODS IN DIFFERENTIAL TOPOLOGY

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Abstract. The Mumford conjecture predicts the ring of rational characteristic classes for surface bundles with oriented connected fibers of large genus. The first proof in [11] relied on a number of well known but difficult theorems in differential topology. Most of these difficult ingredients have been eliminated in the years since then. This can be seen particularly in [7] which has a second proof of the Mumford conjecture, and in the work of Galatius [5] which is concerned mainly with a "graph" analogue of the Mumford conjecture. The newer proofs emphasize Tillmann's theorem [23] as well as some sheaf-theoretic concepts and their relations with classifying spaces of categories. These notes are an overview of the shortest known proof, or more precisely, the shortest known reduction of the Mumford conjecture to the Harer-Ivanov stability theorems for the homology of mapping class groups. Some digressions on the theme of classifying spaces and sheaf theory are included for motivation.

1. Introduction: Mapping class groups and the Mumford conjecture

1.1. **Introduction to the introduction.** These notes are about the Mumford conjecture on surface bundles and mapping class groups. They are a companion piece to my three talks at the Srni 2008 workshop. (An earlier version was handed out at the meeting.) Since the first proof of the Mumford conjecture appeared in [11], major simplifications have been made. It can be said that the resolution of the Mumford conjecture was a development which started in 1996-97 and has kept going until now. The important milestones in this were [23], [10], [11], [7] and [5]. The outline given here relies almost entirely on [23], [7] and [5]. It does not contribute anything new beyond what can be found in these papers. But I hope that it will be valuable as a guide.

The proof of the Mumford conjecture as sketched here carries over easily to the setting of graphs (in place of surfaces) and outer automorphism groups of free groups (in place of mapping class groups). For more details on this analogy, consult [5]. The search for a proof of the Mumford conjecture which carries over to the graph setting has to some extent driven the development after [11]. Certain arguments in [11] which relied heavily on differential topology were banished, and replaced largely by ideas from category theory and elementary sheaf theory. It is surprising

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that overall the proof became not only more generally applicable, but also much simpler.

1.2. **Mapping class groups.** Let's denote by $F_{g,b}$ an oriented smooth compact surface of genus g with b boundary circles. If $b = 0$, we also write F_g. Let

$$\mathrm{Diff}(F_{g,b}; \partial)$$

be the topological group of all diffeomorphisms $F_{g,b} \to F_{g,b}$ which respect the orientation and restrict to the identity on the boundary.

Definition 1.1. $\Gamma_{g,b} = \pi_0 \mathrm{Diff}(F_{g,b}; \partial)$. (That's a discrete group.)

1.3. **The Earle-Eells-Schatz theorem.**

Theorem 1.2. *The identity component of* $\mathrm{Diff}(F_{g,b}; \partial)$ *is contractible when* $g > 1$ *or* $b > 0$.

The proof in [3], [4] is a very beautiful tour through "classical" surface theory. In the case $b = 0$, it starts with the fact that the identity component of $\mathrm{Diff}(F_{g,b}; \partial)$ acts freely on the space of hyperbolic (constant curvature -1) riemannian metrics of $F_{g,b}$. One needs to know that the space of these metrics is contractible, and the orbit space (Teichmueller space) is also contractible. Showing that the space of these hyperbolic metrics is contractible has two hard steps: step 1, showing that it can be identified with the space of complex 1-manifold structures on $F_{g,b}$ (uniformization theorem); step 2, showing that the space of complex 1-manifold structures can be identified with the space of complex structures on the tangent bundle (Korn-Lichtenstein theorem). *However*, I was told by Allen Hatcher that there is also an easier elementary proof avoiding complex function theory. (I forgot the reference.)

1.4. **Classifying spaces of groups following Milnor.** Let G be a topological group (discrete groups allowed, of course).

Notation 1.3. A *reasonable space* is a space homotopy equivalent to a CW-space.

Definition 1.4. A space B with a principal G-bundle $E \to B$ is called a *classifying space* for G if, for every reasonable space X, the natural map

$$[X, B] \to \text{ isomorphism classes of principal } G\text{-bundles on } X$$

is a bijection. (Here $[X, B]$ denotes the set of homotopy classes of maps from X to B.) In such a case we say that $E \to B$ is a *universal* principal G-bundle and write $B = BG$ and $E = EG$. This characterizes BG up to *weak equivalence*, because it pins down the functor $X \mapsto [X, BG]$.

Milnor [14] gave the following alternative characterization:

Theorem 1.5. *A principal G-bundle $E \to B$ is universal if and only if E is weakly contractible, that is, $[X, E]$ is a singleton for every reasonable space X.*

On the basis of that, he also gave a construction of EG and BG.

Definition 1.6. The join $X*Y$ of two spaces X and Y is often informally described as the space of "linear combinations" $sx+ty$ where $s,t\in[0,1]$ and $s+t=1$. It is understood that $1x+0y$ is the same for all y (depends only on x) and similarly $0x+1y$ is the same for all x (depends only on y). *Example*: $S^m * S^n \cong S^{m+n+1}$. *Important property:* If X is $(m-1)$-connected and Y is $(n-1)$-connected, then $X*Y$ is $(m+n)$-connected.

Example 1.7. Milnor's construction of EG: He defined

$$EG = G*G*G*G*\cdots\ .$$

This is the space of formal sums $\sum_{i=1}^{\infty} s_i g_i$ where $s_i\in[0,1]$ and $g_i\in G$, with $\sum s_i = 1$ and only finitely many nonzero s_i. (The order in which the terms $s_i g_i$ are listed matters. Terms $s_i g_i$ with $s_i=0$ can be replaced by a simple 0.) Reason:

– it is weakly contractible because it is n-connected for every n
– G acts *freely* on it (diagonally)
– so we let $BG=EG/G$ and we have a projection $EG\to BG$ which satisfies Milnor's criterion.

Example 1.8. Let $G=\mathrm{Diff}(F_g)$. For $k\gg 0$ let $E^{(k)}G$ be the space of smooth embeddings of F_g in $\mathbb{R}^k$. This comes with a free action of G. The orbit space $B^{(k)}G$ is the space of oriented smooth connected surfaces of genus g (with empty boundary) in $\mathbb{R}^k$.
We have $E^{(k)}G\subset E^{(k+1)}G$ and can pass to the limit $E^{(\infty)}G$. This is contractible ! We may therefore write

$$E^{(\infty)}G = EG\,.$$

Consequently $BG=B^{(\infty)}G$ can be thought of as the space of oriented smooth connected surfaces of genus g (with empty boundary) in $\mathbb{R}^k$, for "very large" k.
Further, by theorem 1.2, the projection homomorphism $G\to\pi_0 G = B\Gamma_g$ is a homotopy equivalence so that $B^{(\infty)}G = BG \simeq B\Gamma_g$ directly from Milnor's construction. Therefore: *Think of $B\Gamma_g$ as the space of oriented smooth connected surfaces of genus g (with empty boundary) in $\mathbb{R}^k$, for "very large" k.*

Remark 1.9. It is important to be aware that the functor $G\mapsto BG$ (from topological groups to spaces with base point) is inverse, in a weak sense, to the functor $X\mapsto\Omega X$ where

$$\Omega X = \text{space of base-point preserving maps from } S^1 \text{ to } X$$

is the *loop space* of X. It is possible to define ΩX in such a way that concatenation of loops makes ΩX into a topological group (where the inverse of a loop is the reversed loop). In that sense, G and $\Omega(BG)$ can be related by natural maps

$$G \leftarrow ? \rightarrow \Omega BG$$

which are both homomorphisms of topological groups and weak homotopy equivalences of spaces with base points.
In the light of Milnor's characterization of BG, this relationship should not come as a surprise because the fibration sequence $G\to EG\to BG$ with contractible EG

is so reminiscent of the fibration sequence $\Omega X \to PX \to X$ with contractible PX (the space of all paths $\gamma : [0,1] \to X$ with $\gamma(0)$ equal to the base point).

1.5. **Harer-Ivanov stability theorem.**

Theorem 1.10. *Suppose that one $F_{g,b}$ is contained in another, $F_{h,c}$. Then the induced homomorphism in homology,*

$$H_*(B\Gamma_{g,b};\mathbb{Z}) \longrightarrow H_*(B\Gamma_{h,c};\mathbb{Z})$$

is an isomorphism for $* < g/2 - 1$.

Remark. Harer's original statement [8] had a bound more like $* < g/3$. The stronger bound is due to N Ivanov [9]. It is said that the proofs by Harer and Ivanov were obtained independently.

Moral: for $g \gg k$ the homology $H_k(B\Gamma_{g,b};\mathbb{Z})$, and consequently also the cohomology $H^k(B\Gamma_{g,b};\mathbb{Z})$, are independent of b and also of g. We write

$$H_k(B\Gamma_\infty;\mathbb{Z})$$

etc. for this. This notation can also be justified by defining Γ_∞ as the direct limit of the groups $\Gamma_{g,1}$ using homomorphisms $\Gamma_{g,1} \to \Gamma_{g+1,1}$ as in Harer's theorem.

1.6. **The Mumford conjecture.** The Mumford conjecture as originally formulated by Mumford [18] was as follows:

Conjecture 1.11. (now proved):

$$H^*(B\Gamma_\infty;\mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \dots]$$

for certain classes κ_i which live in degree $2i$.

This is already a translation (due to Morita [16] and Miller [13]) of something which was meant for consumption by algebraic geometers, not topologists. Specific classes κ_i had been constructed by Mumford. Morita soon proved a part of the conjecture [17]: the Mumford classes κ_i are algebraically independent.

Madsen and Tillmann [10] gave a very illuminating reformulation and refinement of Mumford's conjecture. This uses the Pontryagin-Thom construction.

Construction 1.12. Let M be a d-dimensional smooth compact oriented manifold without boundary embedded in $\mathbb{R}^k$. Let W be a tubular neighborhood of M, so $W \subset \mathbb{R}^k$ as an open set and there is a projection $W \to M$ with a vector bundle structure. As a vector bundle, W is identified with the normal bundle of M in $\mathbb{R}^k$:

$$j_x \colon W_x \cong T_xM^\perp$$

for $x \in M$. This linear isomorphism j_x is the derivative at x of the composition

$$W_x \subset W \subset \mathbb{R}^k \xrightarrow{\text{projection}} T_xM^\perp$$

A point y in $W_x \subset W$ determines a pair $(T_xM, j_x(y)) \in G_d(\mathbb{R}^k) \times T_xM^\perp$, where $G_d(\mathbb{R}^k)$ is the Grassmannian of oriented d-planes in $\mathbb{R}^k$. This amounts to a map

$$W \to V(d,k)$$

where $V(d,k) \to G_d(\mathbb{R}^k)$ is one of the two canonical vector bundles on $G_d(\mathbb{R}^k)$, the one with fibers of dimension $k-d$. The map extends to a continuous map

$$\wp_M : \mathbb{R}^k \cup \infty \to V(d,k) \cup \infty$$

where "$\cup\infty$" stands for the one-point compactification(s). The extension is defined by $\wp_M(z) = \infty$ if $z \notin W$.

Therefore we have

$$\boxed{M \mapsto \wp_M \in \Omega^k\big(\mathrm{Th}\,(V(d,k))\big)}$$

where

- the prefix Ω^k means *space of continuous base-point-preserving maps from* $\mathbb{R}^k \cup \infty$ *to something* (that's also the k-fold iteration of $\Omega = \Omega^1$) ;
- $\mathrm{Th}\,(V(d,k))$ is the Thom space $V(d,k) \cup \infty$ of the vector bundle

$$V(d,k) \to G_d(\mathbb{R}^k)\,.$$

Taking $d = 2$ and letting M vary among connected closed oriented surfaces of genus g in $\mathbb{R}^k$ gives us a map

$$\boxed{\begin{array}{c}\text{space of compact connected oriented genus } g \text{ surfaces without boundary in } \mathbb{R}^k \\ \Big\downarrow \\ \Omega^k(\mathrm{Th}\,(V(2,k)))_g\end{array}}$$

where the subscript g picks out the "right" connected component of $\Omega^k(\mathrm{Th}\,(V(2,k)))$.

Conjecture 1.13. (Madsen-Tillmann's integral form of the Mumford conjecture, now proved): *In the limit $k \to \infty$ and $g \to \infty$ this map induces an isomorphism in integer (co-)homology.*

Remark 1.14. • For $k \to \infty$, g fixed, the source of the map becomes $B\Gamma_g$, as seen earlier.
• The rational cohomology of $\Omega^k\big(\mathrm{Th}\,(V(2,k)\big)$ is easy to calculate, especially when $k \to \infty$. Reasons: the effect of Ω^k on rational cohomology is well understood and the effect of taking Thom spaces on cohomology is well understood (Thom isomorphism). So it boils down to calculating the rational cohomology of $G_2(\mathbb{R}^k)$ for large k. In the limit, $k \to \infty$, the Grassmannian $G_2(\mathbb{R}^k)$ becomes homotopy equivalent to $\mathbb{C}P^\infty$. Therefore

$$H^*(\Omega^k(\mathrm{Th}\,(V(2,k)))_g;\mathbb{Q}) \;\cong_{k\to\infty}\; \mathbb{Q}[\kappa_1,\kappa_2,\dots]$$

for any $g \in \mathbb{Z}$, and the classes κ_i are really just "made from" the Chern classes c_{i+1} of the universal complex line bundle on $\mathbb{C}P^\infty$.
• Galatius has also calculated the cohomology of $\Omega^k(\mathrm{Th}\,(V(2,k))$ (in the limit $k = \infty$) with finite field coefficients [6]. This is quite a bit harder than the rational case.

Remark 1.15. The Harer-Ivanov stability theorem is in some sense a prerequisite for the formulation of the Mumford conjecture. But it is also a very important ingredient in all the currently known proofs of the Mumford conjecture. If the proof of the Mumford conjecture outlined below seems very easy, then that is a deception. It only looks so easy because it uses the Harer-Ivanov stability theorem as a black box (in the proof of theorem 4.2). In other words, it is the reduction of the Mumford conjecture to the Harer-Ivanov stability theorem which, over the last few years, has turned out to be surprisingly easy.

2. Classifying spaces and sheaf theory

2.1. **The bar construction.** From Milnor's description of EG and BG (for a topological group G), it was only a small step to the description which is now standard, the *bar construction.* The "modern" EG has elements

$$\sum_{i=1}^{\infty} s_i g_i$$

like Milnor's, but with the convention that zero terms ($s_i g_i$ with $s_i = 0$) can be discarded and the other terms re-numbered.[1] This is still contractible and the diagonal action of G is still free. An element of the new EG can be imagined as a partition of the interval $(0, 1]$ with labels, as in the following picture:

$$\begin{array}{ccccccccc} & g_1 & & g_2 & & g_3 & & g_4 & \\ ] & \text{———} &] & \text{—————} &] & \text{——} &] & \text{———} &] \\ 0 & & s_1 & & s_1+s_2 & & s_1+s_2+s_3 & & 1 \end{array}$$

The lengths of the partition intervals are s_1, s_2, s_3 and $s_4 = 1 - s_3 - s_2 - s_1$ in this picture. The G-orbit of this element (an element of BG) is best represented by the diagram

$$\begin{array}{ccccccccc} & & h_1 & & h_2 & & h_3 & & \\ | & \text{———} & | & \text{—————} & | & \text{——} & | & \text{———} & | \\ 0 & & s_1 & & s_1+s_2 & & s_1+s_2+s_3 & & 1 \end{array}$$

where $h_3 = g_3 g_4^{-1}$, $h_2 = g_2 g_3^{-1}$ and $h_1 = g_1 g_2^{-1}$. In this description of BG, we "see" the topology by moving partition points around and using certain simple rules for re-labelling after collisions: For example, we can make a *continuous* path from

$$\begin{array}{ccccccccc} & & h_1 & & h_2 & & h_3 & & \\ | & \text{———} & | & \text{—————} & | & \text{——} & | & \text{————} & | \\ 0 & & 0.2 & & 0.6 & & 0.7 & & 1 \end{array}$$

[1] Actually there is another elimination convention which deals with cases where $g_i = g_{i+1}$ for some i. But that is not essential for the following and I shall not use it.

to

$$\begin{array}{ccccccc} & & h_1 & & h_2h_3 & & \\ | & \text{———} & | & \text{——————} & | & \text{————} & | \\ 0 & & 0.2 & & 0.7 & & 1 \end{array}$$

by letting the partition point with label h_2 move towards the next partition point to the right. Note how the group structure is used! (It was not used in the construction of EG.) When partition points collide with the endpoints 0 or 1 of the interval, their labels in G can be erased.

2.2. **Categories and their classifying spaces.** Grothendieck realized (late 1950's) that this construction of BG does not use the existence of inverses and, moreover, does not strictly speaking require that products are always defined. In short he saw that the bar construction BG for a group G generalizes to a bar construction $B\mathcal{C}$ for a small category $\mathcal{C}$.

An element of $B\mathcal{C}$ is a partition of the interval $[0, 1]$ into finitely many subintervals, a labelling of the subintervals by objects of $\mathcal{C}$ and a labelling of the partition points by morphisms in $\mathcal{C}$, as in the following example:

$$\begin{array}{ccccccccc} & a_0 & h_1 & a_1 & h_2 & a_2 & h_3 & a_3 & \\ | & \text{———} & | & \text{————} & | & \text{——} & | & \text{———} & | \\ 0 & & s_1 & & s_1+s_2 & & s_1+s_2+s_3 & & 1 \end{array}$$

Here $h_3 \in \mathrm{mor}(a_3, a_2)$ and $h_2 \in \mathrm{mor}(a_2, a_1)$ and $h_1 \in \mathrm{mor}(a_1, a_0)$. The lengths of the subintervals in this example are s_1, s_2, s_3 and $s_4 = 1 - s_1 - s_2 - s_3$. We can make a *continuous* path from this element of $B\mathcal{C}$ to the element

$$\begin{array}{ccccccc} & a_0 & h_1 & a_1 & h_2h_3 & a_3 & \\ | & \text{———} & | & \text{——————} & | & \text{———} & | \\ 0 & & s_1 & & s_1+s_2+s_3 & & 1 \end{array}$$

by letting the partition point with label h_2 move towards the next partition point to the right. Note how the composition of morphisms is used!

(This explicit description of $B\mathcal{C}$ is taken from a recent short paper by Drinfeld [2]. Drinfeld also refers to similar and simultaneous papers by Besser and Grayson. But I have tried to motivate it through Milnor's construction of BG.)

2.3. **Segal's characterization of the classifying space of a category.** The question now arose what $B\mathcal{C}$ could possibly be good for. (That was *never* a question for BG.) It seems that Grothendieck did not pursue the matter. It was taken up again in 1968 by Segal [20]. Segal had an answer, and it is roughly this:

Theorem 2.1. *Homotopy classes of maps from $B\mathcal{C}$ to a space X are in a natural bijection with* equivalence classes *of pairs (F, g) where F is a functor from $\mathcal{C}$ to contractible spaces and $g: F \Rightarrow X_{\mathcal{C}}$ is a natural transformation (and $X_{\mathcal{C}}$ is the constant functor from $\mathcal{C}$ to spaces with constant value X).*

Here we need to define when two such natural transformations, $F_1 \Rightarrow X_{\mathcal{C}}$ and $F_2 \Rightarrow X_{\mathcal{C}}$, are equivalent. They are equivalent if there exists a third, $F_3 \Rightarrow X_{\mathcal{C}}$, and natural transformations $F_1 \Rightarrow F_3 \Leftarrow F_2$ respecting the natural transformations to $X_{\mathcal{C}}$.

Example 2.2. Let $\mathcal{C} = G$ be a discrete group. As observed before, a functor from $\mathcal{C}$ to spaces is just a G-space. A natural transformation between two such functors is just a G-map. The functor $X_{\mathcal{C}}$ is simply the space X with the trivial G-action. Among the G-spaces with contractible underlying space, there are two extremes: EG, which has a free G-action, and a single point with the trivial G-action. However, the equivalence relation allows us to represent any of the equivalence classes by a G-map from EG to X, where X has the trivial G-action. That is the same as a map from $EG/G = BG$ to X. So we have confirmed Segal's description of $[B\mathcal{C}, X]$ in a special case.

Segal's characterization of $B\mathcal{C}$ paved the way for an invasion of category theory into the land of algebraic topology, occasionally causing displeasure and despair to those who felt that algebraic topology should ally itself more with geometric analysis.

2.4. **What the classifying space of a category classifies: Modern answer.** Let's note that theorem 2.1 says nothing about maps $X \to B\mathcal{C}$, despite the fact that we know and care what they mean in the case where $\mathcal{C}$ is a group G.

To remedy this, following Moerdijk [15] and [24], we need to have some sheaf vocabulary. For some people, a sheaf on a space X is a rule Γ which to every open set U in X associates a set $\Gamma(U)$, to every inclusion $U \to V$ of open sets, a restriction map $\Gamma(V) \to \Gamma(U)$, and then there are some "gluing conditions". That is the definition I prefer, but for the sake of brevity I want to use the other one which says that a sheaf on X consists of a space E and a continuous map

$$p : E \to X$$

which is *étale*. That is, every point z in E has an open neighbourhood U which p maps homeomorphically to an open neighbourhood of $p(z)$ in X. Three remarks are in order:

(i) The fibers of an étale map are *sets* (= discrete spaces).

(ii) Very often X is Hausdorff but E is not Hausdorff. Example: There is an étale map $E \to \mathbb{R}$ (unique up to isomorphism over $\mathbb{R}$) which, over every open interval $U \subset \mathbb{R}$ containing 0, has exactly two sections and which over every open interval not containing 0 has exactly one section.

(iii) An étale map $E \to X$ need not be a covering projection (even in those cases where E and X are both Hausdorff). To make examples, start with a covering projection $p : E \to X$, and restrict that to an open subset U of E. The restriction is still an étale map $U \to X$.

Definition 2.3. A *$\mathcal{C}$-sheaf* on a space X is a contravariant functor E from $\mathcal{C}$ to the category of sheaves on X. (So we get, for every object c of $\mathcal{C}$, a sheaf or étale map $E(c) \to X$, and for every morphism $u : c \to d$ in $\mathcal{C}$, a map $E(u) : E(d) \to E(c)$ which is "over" X ; we have $E(uv) = E(v)E(u)$ and so on.)

Definition 2.4. A $\mathcal{C}$*-sheaf* E on a space X is *principal* if, for every $x \in X$, the fiber E_x of E over x (which is a contravariant functor from $\mathcal{C}$ to sets) is isomorphic to

$$b \mapsto \mathrm{mor}(b, c)$$

for a fixed object c in $\mathcal{C}$.

Definition 2.5. Two principal $\mathcal{C}$-sheaves E_0 and E_1 on X are *concordant* if there exists a principal $\mathcal{C}$-sheaf on $X \times [0,1]$ whose restrictions to $X \times 0 \cong X$ and $X \times 1 \cong X$ are isomorphic to E_0 and E_1, respectively.

Example 2.6. If $\mathcal{C} = G$ for a (discrete) group G, then a G-sheaf on X is a sheaf $E \to X$ with a fiberwise action of G on E. The G-sheaf $E \to X$ is principal if and only if each fiber E_x is a principal G-set (that is, the action of G on E_x is free and transitive). If that is the case, then the G-sheaf $E \to X$ is automatically a bundle (covering space), and therefore a principal G-bundle.
It is well known, but not completely trivial, that two principal G-bundles on a reasonable space X are concordant if and only if they are isomorphic.

Theorem 2.7 ([15], [24])**.** *For a reasonable space X, homotopy classes of maps from X to $B\mathcal{C}$ are in a natural bijective correspondence with concordance classes of principal $\mathcal{C}$-sheaves on X.*

This gives a nice, modern, memorizable answer to the question *What does the classifying space of a small category classify.* Moerdijk has various generalizations of this to topological categories. Let's not try to discuss a proof here.

Remark 2.8. In general, for an arbitrary (small, discrete) category $\mathcal{C}$, two principal $\mathcal{C}$-sheaves on X which are concordant need not be isomorphic. It is easy to produce (counter)examples where $X = [0,1]$ and $\mathcal{C}$ is the monoid of non-negative integers with addition, viewed as a category with one object.

2.5. **What the classifying space of a category classifies: Older answer.** Now let's go back in time to about 1970 and look for older answers to the same question, *What does the classifying space of a category classify.* This is like the step backwards from the "global" definition of a principal G-bundles to Steenrod's definition in terms of bundle charts. According to Steenrod, a principal G-bundle on X consists of an open covering $(U_i)_{i \in J}$ of X and maps $U_i \cap U_j \to G$, and so on. (This is absolutely old-fashioned, but everybody knows that it has some advantages in some situations, and if Steenrod were around he would surely defend it.)

So let's start with an open covering $\mathcal{U} = (U_i)_{i \in J}$ of X. I also want to assume that $\mathcal{U}$ is locally finite. Segal [20] observed that this determines a partially ordered topological space $X_{\mathcal{U}}$ with underlying space

$$\coprod_{\substack{S \subset J \\ S \neq \emptyset}} \bigcap_{i \in S} U_i \,.$$

Elements in this space are pairs (S, x) with nonempty $S \subset J$ and $x \in \bigcap_{i \in S} U_i$ (hence S finite). We decree that $(S, x) \leq (T, y)$ iff $x = y$ and $S \subset T$. A topological poset is a topological category, and so we have a classifying space $BX_{\mathcal{U}}$ with an "obvious" map

$$BX_{\mathcal{U}} \to X\,.$$

The fiber of $BX_{\mathcal{U}} \to X$ over $x \in X$ is the classifying space of the poset of all nonempty subsets of the finite set $S_x = \{i \in J \mid U_i \ni x\}$. So each fiber is a simplex. It follows (almost) that $BX_{\mathcal{U}} \to X$ is a homotopy equivalence. Consequently any *continuous* functor

$$\varphi\colon X_{\mathcal{U}} \to \mathcal{C}$$

induces a map φ_* from $X \simeq BX_{\mathcal{U}}$ to $B\mathcal{C}$. (Since we are assuming that $\mathcal{C}$ is discrete, the continuity condition simply means that φ is *locally constant.*) We call such a functor a $\mathcal{C}$-cocycle on X. This leads us to the following:

Theorem 2.9. *$B\mathcal{C}$ classifies concordance classes of $\mathcal{C}$-cocycles.*

More precisely, the claim is that $[X, B\mathcal{C}]$ is in a natural bijection with the set of concordance classes of pairs $(\mathcal{U}, \varphi)$ where $\mathcal{U}$ is a locally finite open covering of X and φ is a continuous functor $X_{\mathcal{U}} \to \mathcal{C}$. We should insist that X is reasonable. We have already proved one half of the theorem, i.e. we know how to turn a $\mathcal{C}$-cocycle on X into a map $X \to B\mathcal{C}$. But I'm not going to prove the other half here. Credits: [11] and a set of unpublished 1971 lecture notes by Tom Dieck, shown to me by R. Vogt.

Let's explore the relationship between theorem 2.9 and theorem 2.7. Suppose that $(\mathcal{U}, \varphi)$ is a $\mathcal{C}$-cocycle on X, where $\mathcal{U} = (U_i)_{i\in J}$. We try to make a principal $\mathcal{C}$-sheaf $E \to X$ out of that. The idea is that for $x \in X$, the fiber E_x of $E \to X$ over x (which is a contravariant functor from $\mathcal{C}$ to sets) should be given by

$$c \mapsto \mathrm{mor}(c, \varphi(S_x, x))$$

for $c \in \mathrm{ob}(\mathcal{C})$, where $S_x = \{i \in J \mid U_i \ni x\}$ as before. This works very nicely; details omitted. I do not claim to have a good way to proceed in the other direction, from a principal $\mathcal{C}$-sheaf as in theorem 2.7 to a $\mathcal{C}$-cocycle as in Theorem 2.9.

3. Cobordism categories and submersions

Here we introduce various categories $\mathcal{C}_k^i$ for $k \geq 3$ and $0 \leq i < k$ whose objects are certain oriented smooth 1-manifolds and whose morphisms are certain oriented smooth 2-manifolds. Then we use the results from the previous chapter to "understand" the homotopy types of their classifying spaces. The main results say that

$$B\mathcal{C}_k^i \simeq \Omega B\mathcal{C}_k^{i+1}$$

for $0 \leq i < k-1$, and that there is a homotopy equivalence

$$B\mathcal{C}_k^{k-1} \to \mathrm{Th}\big(V(2,k)\big)$$

where $\mathrm{Th}\big(V(2,k)\big)$ is the Thom space from the statement of the integral Mumford conjecture. The next chapter will outline the construction of a map

$$\mathbb{Z} \times B\Gamma_\infty \longrightarrow \Omega B\mathcal{C}_\infty^0$$

which induces an isomorphism in homology. This uses Harer stability and some hands-on differential topology. Putting all that together, we have a proof (in outline!) of the integral Mumford conjecture. This organisation of the proof is heavily influenced by [5]. A rigorous implementation seems to require a certain amount of sheaf language, as in [5] and [11]. This is suppressed here and consequently the level of rigour in this section, even in definitions, is not very high.

3.1. **Submersions.**

Definition 3.1. Let M and N be smooth manifolds without boundary. A smooth map $p\colon M \to N$ is a submersion if, for every $x \in M$, the differential $dp\colon T_xM \to T_{f(x)}N$ is a surjective (linear) map.

By the implicit function theorem, a submersion $p\colon M \to N$ (where $\dim(N) = n$ and $\dim(M) = n+d$) looks "locally" like a linear projection: i.e., for every $x \in M$ there exist coordinate charts $\alpha\colon U_1 \to \mathbb{R}^{n+d}$ around x and $\beta\colon U_2 \to \mathbb{R}^n$ around $p(x)$ such that $\beta p \alpha^{-1}$ is a linear map of rank n (where defined). It follows that the fibers of p are smooth submanifolds of M, of dimension d. But the "locality" is in M, not in N. As a consequence, submersions are, generally speaking, not fiber bundles. It is very easy to illustrate that by examples:

– Let $M = \mathbb{R}^2 \smallsetminus (0,0)$ and $N = \mathbb{R}$ and define $p\colon M \to N$ by $p(x_1, x_2) = x_1$. Then p is a submersion. It is not a fibration since the fiber over 0 is non-connected while all other fibers are connected.
– For any smooth manifold N and open subset $W \subset N$, the inclusion $W \to N$ is a submersion.
– Generalizing the previous two examples: If M and N are smooth manifolds and $p\colon M \to N$ is a smooth fiber bundle, and U is any open subset of M, then $p|U$ from U to N is still a submersion.

And yet — there is a simple "sufficient condition" which ensures that a submersion is a bundle. Recall that a map between locally compact spaces is *proper* if preimages of compact sets under the map are compact.

Theorem 3.2 (Ehresmann's fibration theorem)**.** *Let $p\colon M \to N$ be a smooth submersion. If p is proper, then it is a fiber bundle projection.*

The proof is a straightforward application of integration of vector fields. Without loss of generality, $N = \mathbb{R}^n$, and by integrating a lift to M of one of the standard vector fields on $\mathbb{R}^n$ we achieve a dimension reduction of 1 in source and target. Properness alias compactness comes in because it guarantees the existence of undisrupted integral curves. The details can be seen for example in the second (English, CUP) edition of [1], but curiously not in the first (German, Springer) edition.

3.2. **Cobordism categories.** We fix an integer $d > 0$. (For the Mumford conjecture, $d = 2$ is the right choice.)

Definition 3.3. An object of $\mathcal{C}_k^0$ is a smooth, compact, oriented $(d-1)$-manifold L without boundary in $\mathbb{R}^{k-1}$. A (non-identity) morphism $L \to L'$ is a smooth, compact, oriented d-manifold M in $\mathbb{R}^{k-1} \times [0,a]$, for some $a > 0$ in $\mathbb{R}$, such that M meets the boundary of $\mathbb{R}^{k-1} \times [0,a]$ perpendicularly, and the intersection is equal to

$$\partial M = L \times 0 \ \cup \ L' \times a\,.$$

Composition is concatenation (stacking d-manifolds on top of each other).

Remark 3.4. Many details are missing: e.g., identity morphisms. If precision were required, one option would be to say that the space of objects and the space of (all) morphisms of $\mathcal{C}_k^0$ are both infinite-dimensional smooth manifolds. A smooth map from a smooth manifold N to the space of objects of $\mathcal{C}_k^0$ should be imagined as a smooth submanifold $L \subset N \times \mathbb{R}^{k-1}$ such that the projection $L \to N$ is a smooth fiber bundle with $(d-1)$-dimensional oriented fibers, etc. A smooth map from N to the space of all morphisms of $\mathcal{C}_k^0$ (source and target unspecified) should be imagined as a smooth submanifold $M \subset N \times \mathbb{R}^k$, with boundary, such that the projection $M \to N$ is a smooth fiber bundle with d-dimensional fibers, etc. (The topology on object space and morphism space of $\mathcal{C}_k^0$ influence the topology on $B\mathcal{C}_k^0$.)

For the next definition, we write $\mathbb{R}^{k-1} = \mathbb{R}^i \times \mathbb{R}^{k-1-i}$.

Definition 3.5. An object of $\mathcal{C}_k^i$ is a smooth oriented $(d-1)$-manifold L without boundary in $\mathbb{R}^{k-1}$ such that the projection from L to $\mathbb{R}^i$ is *proper*. A morphism $L \to L'$ is a smooth oriented d-manifold M in $\mathbb{R}^{k-1} \times [0,a]$, for some $a > 0$, such that the projection from M to $\mathbb{R}^i$ is proper, M meets the boundary of $\mathbb{R}^{k-1} \times [0,a]$ perpendicularly, and the intersection is equal to

$$\partial M = L \times 0 \ \cup L' \times a\,.$$

Composition is concatenation.

Remark 3.6. A smooth map from a smooth manifold N (without boundary) to the space of objects of $\mathcal{C}_k^i$ should be imagined as a smooth submanifold $L \subset N \times \mathbb{R}^{k-1}$ such that the projection $L \to N$ is a smooth *submersion* with $(d-1)$-dimensional oriented fibers, and the projection $L \to N \times \mathbb{R}^i$ is proper. A smooth map from N to the space of all morphisms of $\mathcal{C}_k^i$ should be imagined as a smooth submanifold with boundary $M \subset N \times \mathbb{R}^k$ such that the projection $M \to N$ is a smooth *submersion* with d-dimensional fibers, etc., and the projection $M \to N \times \mathbb{R}^i$ is proper.

3.3. **Cobordism categories and submersions.** Let's fix N and let's look at smooth manifolds $M \subset N \times \mathbb{R}^k$ such that the projection $p : M \to N$ is a smooth submersion with d-dimensional *oriented* fibers, and the projection $M \to N \times \mathbb{R}$ (using the last coordinate of points in $\mathbb{R}^k$) is proper.

Theorem 3.7. *Concordance classes of such $M \subset N \times \mathbb{R}^k$ are in a natural bijection with homotopy classes of maps $N \to B\mathcal{C}_k^0$.*

Proof. The main argument comes from theorem 2.9. We start with $M \subset N \times \mathbb{R}^k$ satisfying all those conditions. (We aim to make a map $N \to B\mathcal{C}_k^0$ out of that.) Let

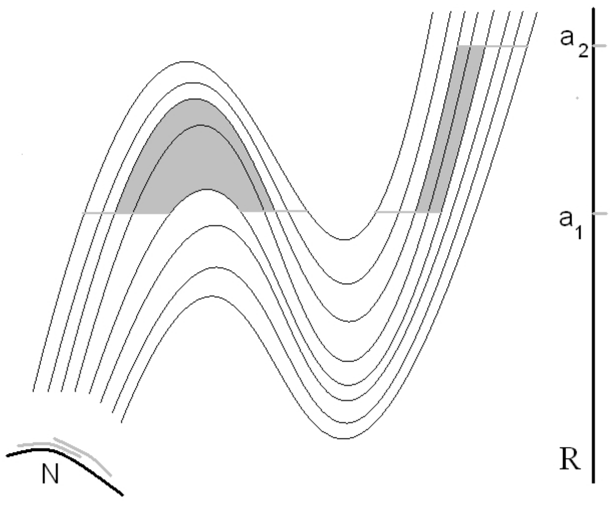
$$p\colon M \to N\ ,\ f\colon M \to \mathbb{R}$$

be the projections, using the last coordinate of point in $\mathbb{R}^k$ for f. Forget the embedding $M \to N \times \mathbb{R}^k$ for a while, but keep p and f. Write $M_y = p^{-1}(y)$. The restriction $f|M_y$ is proper for every $y \in N$. The map $(p, f)\colon M \to N \times \mathbb{R}$ is proper, but we are not allowed to assume that it is a submersion. But: we can choose a locally finite open covering $\mathcal{U} = (U_i)_{i \in J}$ of N and, for every $i \in J$, a real number a_i which is a regular value of the function $f|M_y$, for every $y \in U_i$. (Proceed as follows: first choose, for every $y \in N$, a regular value a_y of $f|M_y$. The same a_y will also be a regular value of $f|M_z$ for every z in a small neighborhood U_y of y. Then $(U_y)_{y \in N}$ is an open covering of N. Refine by a locally finite open covering $\mathcal{U}$.) Now let's see. I claim that we see a $\mathcal{C}^0_k$-cocycle $(\mathcal{U}, \varphi)$ on N.

– On objects of $N_{\mathcal{U}}$ define φ by
$$(S, y) \mapsto M_y \cap f^{-1}(a_S)$$
where $a_S = \max\{a_i \mid i \in S\}$. Shift downwards by an amount a_S.

– On morphisms define φ by
$$\big((S, y) \le (T, y)\big) \mapsto M_y \cap f^{-1}\big([a_S, a_T]\big) = \ .$$
Shift donwards by an amount a_S.

(One has to show that φ is a "continuous" functor, and this uses Ehresmann's fibration theorem.) The picture below illustrates the construction in the case $d = 1$. This establishes one direction of the proposition. The other direction is similar (and less interesting). □



For a generalization of this result, let's fix N as before and let's look at smooth manifolds $M \subset N \times \mathbb{R}^k$ such that the projection $p : M \to N$ is a smooth submersion with d-dimensional oriented fibers, and the projection $M \to N \times \mathbb{R}^i \times \mathbb{R}$ (using the first i coordinates and the last coordinate of points in $\mathbb{R}^k \cong \mathbb{R}^i \times \mathbb{R}^{k-i-1} \times \mathbb{R}$) is proper.

Theorem 3.8. *Concordance classes of such $M \subset N \times \mathbb{R}^k$ are in a natural bijection with homotopy classes of maps $N \to B\mathcal{C}_k^i$.*

Proof. Like the proof of the special case $i = 0$. □

Corollary 3.9. *$B\mathcal{C}_k^i$ is homotopy equivalent to* $\mathrm{mor}(\emptyset, \emptyset)$ *in $\mathcal{C}_k^{i+1}$, provided $0 \leq i < k-1$.*

Proof. The idea is that two spaces Y, Z are (weakly) homotopy equivalent if there is a natural bijection $[N, Y] \to [N, Z]$ for every smooth manifold N without boundary. It is important to allow noncompact N.
Using the theorem just above, we have a description of $[N, B\mathcal{C}_k^i]$ for every smooth N in terms of concordance classes of $M \subset N \times \mathbb{R}^k$ subject to certain conditions. It is not a serious restriction to assume that $M \subset N \times \mathbb{R}^i \times [0,1]^{k-i-1} \times \mathbb{R}$. Then, looking at the situation fiberwise over N, we see a map from N to the space of morphisms $\mathrm{mor}(\emptyset, \emptyset)$ in the topological category $\mathcal{C}_k^1$. □

Corollary 3.10. *$B\mathcal{C}_k^i \simeq \Omega B\mathcal{C}_k^{i+1}$ for $0 \leq i < k-1$.*

Proof. The idea of this proof is that the category $\mathcal{C}_k^{i+1}$ is a lot like a topological groupoid. *Essentially*, all morphisms in the category are invertible. The base point component of the classifying space will then be the classifying space of the endomorphism "group" $\mathrm{mor}(\emptyset, \emptyset)$ of the object $\emptyset$ corresponding to the base point. Therefore we ought to expect

$$\Omega B\mathcal{C}_k^{i+1} \simeq \mathrm{mor}(\emptyset, \emptyset) \text{ in } \mathcal{C}_k^{i+1}$$

by analogy with $\Omega BG \simeq G$ for an honest topological group G. Then we use the result about $\mathrm{mor}(\emptyset, \emptyset)$ from the previous corollary.
Why are these claims about invertibility of morphisms in the topological category $\mathcal{C}_k^{i+1}$ *essentially* true ? The thing to show is that the morphisms are always invertible up to homotopy. That is, if $M \in \mathrm{mor}(L, L')$, then there exists $M' \in \mathrm{mor}(L', L)$ such that $M' \circ M$ and $M \circ M'$ are in the path component(s) of the respective identity morphisms. To show this, recall first that L and L' are certain $(d-1)$-manifolds in $\mathbb{R}^{k-1}$ for which the projections $L, L' \to \mathbb{R}^{i+1}$ are proper. Without loss of generality, these projections are also transverse to the origin in $\mathbb{R}^{i+1}$. Let $L_0 \subset L$ and $L'_0 \subset L'$ be the preimages of the origin in $\mathbb{R}^{i+1}$. Now a morphism $M \in \mathrm{mor}(L, L')$ is an oriented d-manifold M in $\mathbb{R}^{k-1} \times [0, a] \subset \mathbb{R}^k$ with boundary $\partial M \cong L \cup L'$ etc., such that the projection $M \to \mathbb{R}^{i+1}$ is proper. Without loss of generality that projection is again transverse to the origin. The preimage of the origin is then a $(d-i-1)$-dimensional oriented cobordism M_0 embedded in $\mathbb{R}^{k-i-1}$. Using some shrinking and stretching, one can verify that the connected component of M in $\mathrm{mor}(L, L')$ is completely determined by the embedded oriented cobordism M_0 from L_0 to L'_0. Conversely, if the embedded oriented cobordism is prescribed, a morphism M in $\mathrm{mor}(L, L')$ can be found which extends it. With that, it is easy to establish invertibility of morphisms up to homotopy. □

3.4. **Zooming.** We fix $d > 0$ as in the previous (sub)section. By theorem 3.8, specialized to the case $i = k-1$, homotopy classes of maps from a smooth manifold N to $B\mathcal{C}_k^{k-1}$ are in a natural bijection with concordance classes of smooth submanifolds

$$M \subset N \times \mathbb{R}^k \ ,$$

without boundary, closed as subsets of $N \times \mathbb{R}^k$, and such that the projection $p : M \to N$ is a submersion with d-dimensional oriented fibers. We can formulate this in a more illuminating way. Let Y be the "space" of smooth oriented d-dimensional submanifolds $F \subset \mathbb{R}^k$, without boundary, closed as subsets of $\mathbb{R}^k$. The message of theorem 3.8 in the special case $i = k-1$ is

$$\boxed{B\mathcal{C}_k^{k-1} \simeq Y} \, .$$

Lemma 3.11. *$Y \simeq \mathrm{Th}\,(V(d,k))$.*

Proof. Let $Y' \subset Y$ consists of those (surfaces) $F \subset \mathbb{R}^k$ in Y which avoid the origin $0 \in \mathbb{R}^k$. Let $Y'' \subset Y$ consists of those surfaces $F \subset \mathbb{R}^k$ in Y which either pass through 0 or come very close to it, so that the norm function $x \mapsto \|x\|$ on F has a unique nondegenerate minimum. Then $Y = Y' \cup Y''$, and we will assume in the following that Y' and Y'' are both open in Y. (This must remain a little vague since I did not fully explain how the topology in Y should be defined.) Then Y is homotopy equivalent to the double mapping cylinder of the inclusion maps

$$Y' \leftarrow Y' \cap Y'' \rightarrow Y'' \, .$$

As this is a homotopy invariant construction, we can now look at the homotopy types of Y', $Y' \cap Y''$ and Y'' separately and draw our conclusions. A "zooming in at the origin" argument shows that Y' is contractible. More precisely, we have a homotopy $(h_t : Y' \to Y')$ where $t \in [1, \infty]$ and $h_t(F) = tF$ for a d-manifold $F \in Y'$ (that is, h_t magnifies by a factor t). In particular h_1 is the identity and h_∞ is the constant map which takes every $F \in Y'$ to the point $\emptyset \in Y'$. A similar zooming argument shows that Y'' is homotopy equivalent to the Grassmannian $G_d(\mathbb{R}^k)$ of oriented d-planes in $\mathbb{R}^k$. (Here it is better to make the zooming procedure dependent on $F \in Y''$, and to magnify only those directions which are tangent to F at x, where $\|x\|$ is that point on F which is closest to the origin.) The same argument also shows that $Y' \cap Y''$ is homotopy equivalent to the unit sphere bundle $S(V)$ associated with the tautological $(k-d)$-dimensional vector bundle V on the Grassmannian $G_d(\mathbb{R}^k)$. Therefore Y is homotopy equivalent to the double mapping cylinder of the maps

$$\text{point} \leftarrow S(V) \rightarrow G_d(\mathbb{R}^k) \, .$$

But that is exactly $\mathrm{Th}\,(V(d,k))$. □

Corollary 3.12. *$B\mathcal{C}_k^{k-1} \simeq \mathrm{Th}\,(V(d,k))$.*

Remark 3.13. Tools from optics like "zooming" and also "scanning" were introduced long ago in a somewhat similar setting by Graeme Segal [22].

4. Back to mapping class groups

4.1. **Cobordism categories and mapping class groups.** The main achievement of the last chapter (specialized to $d = 2$) was the construction of a homotopy equivalence

$$B\mathcal{C}_k^0 \longrightarrow \Omega^{k-1}\mathrm{Th}\,(V(2,k))$$

where $\mathcal{C}_k^0$ is the surface cobordism category of definition 3.3. It is not very hard to let k tend to infinity in this. Then we have a homotopy equivalence

$$B\mathcal{C}_\infty^0 \longrightarrow \varinjlim_k \Omega^{k-1}\mathrm{Th}\,(V(2,k))$$

and consequently another homotopy equivalence

$$\Omega B\mathcal{C}_\infty^0 \longrightarrow \varinjlim_k \Omega^{k}\mathrm{Th}\,(V(2,k))\,.$$

The cohomological properties of the target have already been discussed. In particular, every connected component of $\varinjlim_k \Omega^k\mathrm{Th}\,(V(2,k))$ has rational cohomology isomorphic to

$$\mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \dots]$$

where κ_i lives in degree $2i$. Now it remains to say how $\Omega B\mathcal{C}_\infty^0$ is related to $B\Gamma_\infty$. The relationship that we will get is similar to the homotopy equivalence

$$\Omega B\mathcal{C}_k^i \simeq \mathrm{mor}(\emptyset, \emptyset) \text{ in } \mathcal{C}_k^i$$

which we found *for* $i > 0$ in the previous chapter (proof of corollary 3.10). If this were true for $i = 0$ and $k = \infty$, it would say that $\Omega B\mathcal{C}_\infty^0$ is homotopy equivalent to $\mathrm{mor}(\emptyset, \emptyset)$ in $\mathcal{C}_\infty^0$. That is the *space of all compact oriented surfaces without boundary*, and it carries a tautological surface bundle which is universal among such bundles (with oriented fibers, compact without boundary). However, that is not a correct description of $\Omega B\mathcal{C}_\infty^0$.

4.2. **Tillmann's theorem.** As a step towards a correct description or calculation, we introduce an important subcategory of $\mathcal{C}_\infty^0$. This is a very clever idea due to Tillmann [23] which, in 1996, started off the whole recent development which led to the proof of the Mumford conjecture and even beyond.

Definition 4.1. The category $\mathcal{C}_{\infty,T}^0$ is a subcategory of $\mathcal{C}_\infty^0$. It has the same objects as $\mathcal{C}_\infty^0$. A morphism $M \in \mathrm{mor}(L, L')$ in $\mathcal{C}_\infty^0$ belongs to $\mathcal{C}_{\infty,T}^0$ if the inclusion of the outgoing boundary L' in the surface M induces a surjection on π_0.

Theorem 4.2 ([23]). *There is a map $\mathbb{Z} \times B\Gamma_\infty \longrightarrow \Omega B\mathcal{C}_{\infty,T}^0$ which induces an isomorphism in integer (co)homology.*

The proof is easy for homotopy theorists, but for most other people there are some unfamiliar concepts and facts in it. The most important of the concepts is that of a *homotopy colimit.*

Let $\mathcal{A}$ be a category (small and discrete, to keep it simple) and let F be a contravariant functor from $\mathcal{A}$ to spaces. Then we can make a new category $F \wr \mathcal{A}$ whose objects are pairs (x, a) where a is an object of $\mathcal{A}$ and $x \in F(a)$. A morphism from (x,a) to (y,b) is a morphism $m : a \to b$ in $\mathcal{A}$ such that $F(m) : F(b) \to F(a)$ takes y to x. The category $F \wr \mathcal{A}$ has (inevitably) a topology on its object set, because the x in objects (x,a) can vary continuously while a is fixed. This forces us to put a topology on the total morphism set as well (details omitted). In any case we let

$$\operatorname*{hocolim}_{\mathcal{A}} F := B(F \wr \mathcal{A}) .$$

There is a forgetful functor $F \wr \mathcal{A} \to \mathcal{A}$ which induces a map from $B(F \wr \mathcal{A}) = \operatorname{hocolim}_{\mathcal{A}} F$ to $B\mathcal{A}$. That map will be very important in the following.

Now we need some "facts" about homotopy colimits. I believe [12] is a good reference.
For the first of these, we say that F is representable if it is isomorphic to a functor of the form $F(a) = \operatorname{mor}(a,c)$ for a fixed object c in $\mathcal{C}$. We met these previously in definition 2.4. If $\mathcal{C}$ is discrete, then such a functor has obviously sets (discrete spaces) as values, but the lemma which follows works also for topological categories.

Lemma 4.3. *If F is a representable functor, then* $\operatorname{hocolim}_{\mathcal{A}} F$ *is (weakly) contractible.*

More facts: for these we need some variations on the idea of a fibration. Let $f : X \to Y$ be any map of spaces. There is a standard way (Serre's construction) to factorise it as $X \to X' \to Y$ such that $X' \to Y$ is a fibration and $X \to X'$ is a homotopy equivalence. If, for each $p \in Y$, the inclusion of fibers $X_p \to X'_p$ is a homotopy equivalence, then we say that $f : X \to Y$ is a *quasifibration.* If, for each $p \in Y$, the inclusion of fibers $X_p \to X'_p$ induces an isomorphism in integer homology, we say that $f : X \to Y$ is a *homology fibration.*

Lemma 4.4. *Suppose that F takes every morphism in $\mathcal{A}$ to a homotopy equivalence. Then the projection*

$$\operatorname*{hocolim}_{\mathcal{A}} F \;=\; B(F \wr \mathcal{A}) \longrightarrow B\mathcal{A}$$

is a quasi-fibration.

Lemma 4.5. *Suppose that F takes every morphism in $\mathcal{A}$ to a homology equivalence (a map inducing an isomorphism in integer homology). Then the projection*

$$\operatorname*{hocolim}_{\mathcal{A}} F \;=\; B(F \wr \mathcal{A}) \longrightarrow B\mathcal{A}$$

is a homology fibration.

Proof of Tillmann's theorem (sketch). We abbreviate

$$\mathcal{C} = \mathcal{C}^0_{\infty,T} .$$

For an object L in $\mathcal{C}$, let $F_u(L) = \operatorname{mor}(L, S^1)$, the *space* of morphisms from L to S^1 in $\mathcal{C}$. (This is homotopy equivalent to a disjoint union

$$\coprod_{g=0}^{\infty} B\Gamma_{g,b+1}$$

where b is the number of connected components of L.) Let $z \in \mathrm{mor}(S^1, S^1)$ be the genus 1 morphism from S^1 to S^1. Composition with z gives a natural map $F_u(L) \to F_u(L)$ and we make this "invertible" by taking the iterated mapping cylinder (telescope) of

$$F_u(L) \xrightarrow{z\cdot} F_u(L) \xrightarrow{z\cdot} F_u(L) \xrightarrow{z\cdot} F_u(L) \xrightarrow{z\cdot} \cdots$$

Call this $F(L)$. Now we only have to check two things:

(i) The functor F takes all morphisms to homology equivalences

(ii) $\mathrm{hocolim}_{\mathcal{C}}\, F$ is contractible.

Then we can apply lemma 4.5, looking specifically at the fiber over the point in $B\mathcal{C}$ which corresponds to the object S^1. That fiber is $\mathbb{Z} \times B\Gamma_\infty$. The lemma says that it maps by a homology equivalence to $\Omega B\mathcal{C}$, and that is the end of it. Claim (i) is a direct consequence of the Harer-Ivanov stability theorems. Claim (ii) is a consequence of the observation that hocolim F can be identified with a telescope made from the diagram

$$\mathrm{hocolim}\ F_u \xrightarrow{z\cdot} \mathrm{hocolim}\ F_u \xrightarrow{z\cdot} \mathrm{hocolim}\ F_u \xrightarrow{z\cdot} \mathrm{hocolim}\ F_u \xrightarrow{z\cdot} \cdots ,$$

and lemma 4.3, which says that hocolim F_u is contractible. □

Remark 4.6. When Tillmann proved this theorem, she was probably not (yet) interested in the Mumford conjecture. It was known that Γ_∞ , the fundamental group of $B\Gamma_\infty$, is a perfect group [19]. It was known that for a connected reasonable space X with perfect fundamental group, there exists a simply connected space X^+ and a map $X \to X^+$ which induces an isomorphism in integer homology. This determines X^+ up to homotopy equivalence. (The construction of X^+ is a special case of *Quillen's plus construction.*) So Tillmann's theorem turns into

$$\mathbb{Z} \times B\Gamma_\infty^+ \ \simeq\ \Omega B\mathcal{C}^0_{\infty,T}\,.$$

It was known that, for a category $\mathcal{C}$ with a "symmetric monoidal" structure[2], the space $\Omega B\mathcal{C}$ is always an *infinite loop space.*[3] In particular, $\mathcal{C}^0_{\infty,T}$ has a disjoint union operation, and so ΩB of it is an infinite loop space. Therefore by Tillmann's theorem, $\mathbb{Z} \times B\Gamma_\infty^+$ is an infinite loop space. This came as a big surprise at the time.

Lemma 4.7. *The inclusion $B\mathcal{C}^0_{\infty,T} \to B\mathcal{C}^0_\infty$ is a homotopy equivalence.*

The proof is not long, but a little technical. Perhaps the best way to appreciate it is to read it in [7, §6].

Therefore we have $\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega B\mathcal{C}^0_{\infty,T} \simeq \Omega B\mathcal{C}^0_\infty \simeq \varinjlim_k \Omega^k \mathrm{Th}\,(V(2,k))$.

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[2] Examples are the disjoint union in the category of finite sets, or the direct sum in the category of finite dimensional vector spaces over $\mathbb{Q}$.

[3] That means: a space which is homotopy equivalent to the loop space of some other space which is again homotopy equivalent to the loop space of another space and so on. These infinite loop spaces are important in algebraic topology because they correspond roughly to generalized cohomology theories. The most famous example is $BU \times \mathbb{Z}$, which is an infinite loop space because it is homotopy equivalent to the 2-fold loop space of itself.

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