ARCHIVUM MATHEMATICUM (BRNO)
Tomus 45 (2009), 19–35

VISCOSITY SUBSOLUTIONS AND SUPERSOLUTIONS FOR NON-UNIFORMLY AND DEGENERATE ELLIPTIC EQUATIONS

Aris S. Tersenov

Abstract. In the present paper we study the Dirichlet boundary value problem for quasilinear elliptic equations including non-uniformly and degenerate ones. In particular, we consider mean curvature equation and pseudo p-Laplace equation as well. It is well-known that the proof of the existence of continuous viscosity solutions is based on Ishii's implementation of Perron's method. In order to use this method one has to produce suitable subsolution and supersolution. Here we introduce new methods to construct subsolutions and supersolutions for the above mentioned problems. Using these subsolutions and supersolutions one may prove the existence of unique continuous viscosity solution for a wide class of degenerate and non-uniformly elliptic equations.

0. Introduction and main results

Consider the following problem

$$-\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},\nabla u)u_{x_i x_j} + f(\mathbf{x},u,\nabla u) = 0 \quad \text{in} \quad \Omega \subset \mathbf{R^n}\,, \tag{0.1}$$

$$u(\mathbf{x}) = 0 \quad \text{on} \quad \partial\Omega\,, \tag{0.2}$$

where for $\mathbf{x} \in \Omega$, $\mathbf{p} \in \mathbf{R}^n$ and $\xi = (\xi_1,\dots,\xi_n) \in \mathbf{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},\mathbf{p})\xi_i\xi_j \geq 0\,. \tag{0.3}$$

In this paper we focus our attention on the problem of construction of viscosity subsolutions and supersolutions for equation (0.1) that satisfy boundary condition (0.2). Put

$$\Phi(\mathbf{x},u,\nabla u,\nabla^2 u) = -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},\nabla u)u_{x_i x_j} + f(\mathbf{x},u,\nabla u)\,,$$

where we suppose that $\Phi(\mathbf{x},u,\nabla u,\nabla^2 u)$ is a continuous function in all its variables. Let us recall the definition of continuous viscosity sub- and supersolutions of (0.1). According to [6], u is a continuous viscosity subsolution of $\Phi = 0$ if u is upper

2000 *Mathematics Subject Classification*: primary 35J60; secondary 35D05.

Key words and phrases: viscosity subsolution, viscosity supersolution, mean curvature equation, pseudo p-Laplace equation.

Received February 12, 2007, revised November 2008. Editor O. Došlý.

semicontinuous on Ω and for every $\phi \in C^2(\Omega)$ and local maximum point $\hat{x} \in \Omega$ of $u - \phi$, one has

$$\Phi(\hat{\mathbf{x}}, u(\hat{\mathbf{x}}), \nabla\phi(\hat{\mathbf{x}}), \nabla^2\phi(\hat{\mathbf{x}})) \leq 0\,.$$

The notion of a continuous viscosity supersolution of $\Phi = 0$ appears by replacing "upper semicontinuous" by "lower semicontinuous", "max" by "min" and reversing the inequality to

$$\Phi(\hat{\mathbf{x}}, u(\hat{\mathbf{x}}), \nabla\phi(\hat{\mathbf{x}}), \nabla^2\phi(\hat{\mathbf{x}})) \geq 0\,.$$

A continuous viscosity solution of $\Phi = 0$ is a function which is simultaneously a continuous viscosity subsolution and a continuous viscosity supersolution.

The notion of viscosity solution was firstly introduced for Hamilton-Jacobi equation by M. G. Crandall and P. L. Lions [5] and extended to the case of second order elliptic equations with continuous coefficients by Ishii and Lions [6]. It is well known that the proof of the existence of a continuous viscosity solution is based on Ishii-Perron method. In order to use this method in the case of Dirichlet boundary value problem one has to provide subsolution and supersolution that vanish on the boundary (see [4, Theorem 4.1]).

The above mentioned theorem leaves open the question of when such subsolution and supersolution can be found. In [3] viscosity sub- and supersolutions were constructed for the Dirichlet boundary value problem as well as for the initial problem in the case of uniformly elliptic and parabolic fully nonlinear equations. In [8] these solutions were constructed in the case of the Dirichlet problem for the p-Laplace equation.

In this paper we present a new method of constructing subsolutions and supersolutions for equation (0.1) that satisfy boundary condition (0.2) in an orthogonal parallelepiped under general assumptions on the structure of nonlinearities. The subsolution (respectively the supersolution) is the maximum (respectively the minimum) of the set of solutions of certain ordinary differential equations. Also using the idea of [8], we construct subsolution and supersolution for the Dirichlet boundary value problem for mean curvature as well as for pseudo p-Laplace equations in convex domains. These subsolution and supersolution are solutions of ordinary differential equations, where the role of independent variable is played by the function which describes the boundary of Ω. Note that all these subsolutions and supersolutions are given explicitly and can be used to provide modulus of continuity estimates on solutions of (0.1), (0.2) as well as regularity results (see Remarks 1.1, 2.1). To the best of our knowledge there are no such results concerning problem (0.1), (0.2), including particular cases, i.e. mean cuvature equation and pseudo p-Laplace equation. Having these sub- and supersolutions one may invoke Ishii-Perron method in order to prove the existence of continuous viscosity solution of (0.1), (0.2). Concerning applications of our results see also Corollaries 1.1, 1.2, 2.1, 3.1, 4.1.

We say that the strong comparison result holds if from the fact that $u_* \leq u^*$ on $\partial\Omega$, where u_* is a subsolution and u^* is a supersolution of

$$\Phi(\mathbf{x}, u, \nabla u, \nabla^2 u) \equiv -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, \nabla u) u_{x_i x_j} + f(\mathbf{x}, u, \nabla u) = 0 \quad \text{in} \quad \Omega\,,$$

it follows that $u_* \le u^*$ in Ω. To assert that comparison result holds via the theorems of [4] (see conditions (3.13), (3.14) in [4]) one has to impose structure conditions on Φ. According to [4], if there exists a $\nu > 0$ such that

$$\nu(r-s) \le f(\mathbf{x}, r, \mathbf{p}) - f(\mathbf{x}, s, \mathbf{p}), \quad s \le r, \quad (\mathbf{x}, \mathbf{p}) \in \overline{\Omega} \times R^n \tag{0.4}$$

and there is a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0+) = 0$ such that

$$\Phi(\mathbf{y}, r, \beta(\mathbf{x}-\mathbf{y}), Y) - \Phi(\mathbf{x}, r, \beta(\mathbf{x}-\mathbf{y}), X) \le \omega(\beta|\mathbf{x}-\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|)\,, \tag{0.5}$$

whenever $\mathbf{x}, \mathbf{y} \in \Omega$, $r \in R$, $X, Y \in S(n)$ satisfying the matrix inequality for $\beta > 0$ and any $\varepsilon > 0$,

$$-\left(\frac{1}{\varepsilon} + 2\beta\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \beta\,(1 + 2\varepsilon\beta)\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then comparison holds (see Lemma 3.1, Theorems 3.2, 3.3 in [4]). Here $S(n)$ is the set of symmetric matrices and I stands for identity matrix.

This paper is organized as follows. In Section 1 we consider problem (0.1), (0.2) in $\Omega = \{\mathbf{x} : |x_i| < l_i,\ i = 1, \dots, n\}$. Put $\mathbf{p}_i = (0, \dots, 0, p_i, 0, \dots, 0)$. We assume that

$$|f(\mathbf{x}, 0, \mathbf{p}_i)| \le a_{ii}(\mathbf{x}, 0, \mathbf{p}_i)\psi_i(|\mathbf{p}_i|)\,, \quad i = 1, \dots, n\,, \tag{0.6}$$

where ψ_i, $i = 1, \dots, n$ are positive continuous functions such that

$$\int_0^{+\infty} \frac{d\rho}{\psi_i(\rho)} > l_i\,, \quad i = 1, \dots, n\,. \tag{0.7}$$

The main result of this section (Theorem 1.1) is a construction of subsolution and supersolution that satisfy (0.2), carried out using only the monotonicity condition (0.4) with $\nu = 0$. If conditions (0.4), (0.5) hold, then one may apply Ishii-Perron method to prove the existence of unique continuous viscosity solution of (0.1), (0.2) (Corollary 1.1). By direct calculations one can easily see that if, for example $a_{ij} = a_{ij}(\mathbf{p})$, $f = f_1(r, \mathbf{p}) + f_2(\mathbf{x})$ with f_2 uniformly continuous function, then (0.5) is fulfilled. Concerning the Lipschitz regularity of solutions of (0.1), (0.2) see Corollary 1.2.

In Sections 2, 3 we consider the Dirichlet boundary value problem for the mean curvature equation

$$-(1 + |\nabla u|^2)\Delta u + \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} - nH(\mathbf{x}, u)(1 + |\nabla u|^2)^{\frac{3}{2}} = 0 \tag{0.8}$$

in $\Omega \subset \mathbf{R}^n$, where we suppose that Ω satisfies the following condition
(A) Ω *is a smooth convex domain that lies in the parallelepiped*

$$\Omega \subset \{\mathbf{x} = (x_1, \dots, x_n) : -l_i \le x_i \le l_i,\ i = 1, \dots, n\}\,.$$

The parts of $\partial\Omega \in \mathbf{C^2}$ *lying in the half-spaces* $x_1 \le 0$ *and* $x_1 \ge 0$ *can be expressed as*

$$x_1 = F(x_2, x_3, \dots, x_n)\,, \quad x_1 = G(x_2, x_3, \dots, x_n)$$

respectively.

We consider two cases. In the first case (Section 2) we suppose that

$$|\nabla G| < \infty, \quad |\nabla F| < \infty \quad \text{in} \quad \overline{\Omega} \tag{0.9}$$

and

$$2nlH_0\sqrt{\gamma} < 1\,, \tag{0.10}$$

where

$$H_0 = \max_{\mathbf{x}\in\overline{\Omega}} |H(\mathbf{x},0)|\,, \quad \gamma = 1 + \max_{\mathbf{x}\in\overline{\Omega}} |\nabla G|^2 + \max_{\mathbf{x}\in\overline{\Omega}} |\nabla F|^2$$

and $2l$ is the length of the interval where x_1 varies.

In the second case (Section 3) we suppose that there exist positive constants K_1 and K_2 such that

$$\kappa_1 \equiv \frac{\triangle G}{(1+|\nabla G|^2)^{\frac{3}{2}}} \le -nH_0\frac{\sqrt{1+K_1^2}}{K_1}\,, \tag{0.11}$$

$$\kappa_2 \equiv \frac{\triangle F}{(1+|\nabla F|^2)^{\frac{3}{2}}} \ge nH_0\frac{\sqrt{1+K_2^2}}{K_2}\,. \tag{0.12}$$

The main results of these sections concern the construction of subsolution and supersolution that satisfy (0.2) and are formulated in Theorems 2.1, 3.1. Note that if (0.5) holds for the mean curvature operator and $H(x,u)$ is a strictly decreasing function with respect to u, then one may prove the existence of unique continuous viscosity solution of (0.8), (0.2) using Ishii-Perron method (Corollaries 2.1, 3.1). Concerning the Lipschitz regularity of solutions see Remark 2.1. We mention here that subsolution and supersolution were constructed in [2] for generalized mean curvature flow equations in the case of initial problem.

In Section 4 we treat the Dirichlet boundary value problem for the pseudo p-Laplace equation

$$-\sum_{i=1}^{n} \mu_i(|u_{x_i}|^{p_i}u_{x_i})_{x_i} + c(\mathbf{x})g(u) + f(\mathbf{x}) = 0 \quad \text{in} \quad \Omega \subset \mathbf{R}^{\mathbf{n}}\,. \tag{0.13}$$

Here $\mu_i > 0$ and $p_i \ge 0$ are constants, Ω satisfies condition (A). The main result of Section 4 is formulated in Theorem 4.1 and concerns the construction of subsolution and supersolution that satisfy (0.2). If now (0.5) holds for the pseudo p-Laplace operator and $c(\mathbf{x})g(u)$ is a strictly increasing function with respect to u, then we obtain the existence of unique continuous viscosity solution of (0.13), (0.2) (Corollary 4.1).

1. Construction of sub- and supersolutions in orthogonal parallelepiped

Consider problem (0.1), (0.2) in $\Omega = \{\mathbf{x} : |x_i| < l_i, i = 1,\dots,n\}$. Introduce functions $h_i(x_i)$ as solutions of the following problems

$$h_i'' + \psi_i(|h_i'|) = 0\,, \quad h_i(-l_i) = 0\,, \quad h_i(0) = D_i\,, \quad i = 1,\dots,n\,, \tag{1.1}$$

where positive constants D_i, $i = 1, \dots, n$ will be defined below. Put $h_i'(x_i) = q_i$, $i = 1, \dots, n$. Hence from (1.1) it follows that

$$dx_i = -\frac{dq_i}{\psi_i(|q_i|)}\,, \quad dh_i = q_i dx_i = -\frac{q_i dq_i}{\psi_i(|q_i|)}\,, \quad i = 1, \dots, n\,.$$

Represent the solution of problem (1.1) in parametric form

$$h_i(q_i) = \int_{q_i}^{b_i} \frac{\rho d\rho}{\psi_i(\rho)}\,, \quad x_i(q_i) = \int_{q_i}^{b_i} \frac{d\rho}{\psi_i(\rho)} - l_i\,, \quad i = 1, \dots, n\,,$$

where $q_i \in [a_i, b_i]$ and a_i, b_i are chosen such that $0 < a_i < b_i < +\infty$ and

$$\int_{a_i}^{b_i} \frac{d\rho}{\psi_i(\rho)} = l_i\,, \quad i = 1, \dots, n\,.$$

This is possible due to (0.7). Put $D_i = \int_{a_i}^{b_i} \frac{\rho d\rho}{\psi_i(\rho)}$, $i = 1, \dots, n$. Obviously $x_i(a_i) = 0$, $h_i(a_i) = D_i$, $x_i(b_i) = -l_i$, $h_i(b_i) = 0$, $i = 1, \dots, n$.

Define functions $\tilde{h}_i(x_i)$, $i = 1, \dots, n$ by the following

$$\tilde{h}_i(x_i) = \begin{cases} h_i(x_i)\,, & \text{for } \ x_i \in [-l_i, 0]\,, \\ h_i(-x_i)\,, & \text{for } \ x_i \in [0, l_i]\,. \end{cases} \tag{1.2}$$

Put

$$h^*(x_1, x_2, \dots, x_n) = \min\{\tilde{h}_1(x_1), \tilde{h}_2(x_2), \dots, \tilde{h}_n(x_n)\}\,. \tag{1.3}$$

Theorem 1.1. *Let $\Omega = \{\mathbf{x} : |x_i| < l_i, i = 1, \dots, n\}$. Suppose that $a_{ij}(\mathbf{x}, \mathbf{p}) \in \mathbf{C}(\Omega \times \mathbf{R}^n)$, $i, j = 1, \dots, n$, $f(\mathbf{x}, u, \mathbf{p}) \in \mathbf{C}(\Omega \times \mathbf{R} \times \mathbf{R}^n)$. If conditions* (0.3), (0.4) *with $\nu = 0$,* (0.6), (0.7) *hold, then h^* and $h_* = -h^*$ defined by* (1.3) *are viscosity supersolution and subsolution of* (0.1) *respectively and $h^*(\mathbf{x}) = h_*(\mathbf{x}) = 0$ on $\partial\Omega$.*

Proof. Define operator L by

$$L[u] \equiv -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, \nabla u) u_{x_i x_j} + f(\mathbf{x}, u, \nabla u)\,.$$

From (0.4), (1.1) and the fact that $h_1(x_1) \geq 0$ it follows that for $x_1 \in (-l_1, 0)$

$$\begin{aligned} L[h_1(x_1)] &= -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, \nabla h_1) h_{1x_i x_j} + f(\mathbf{x}, h_1, \nabla h_1) \\ &\geq -\sum_{i,j=1}^{n} a_{ij}(\mathbf{x}, \nabla h_1) h_{1x_i x_j} + f(\mathbf{x}, 0, \nabla h_1) \\ &= -a_{11}(\mathbf{x}, h_1', 0, \dots, 0) h_1'' + f(\mathbf{x}, 0, h_1', 0, \dots, 0) \\ &= a_{11}(\mathbf{x}, h_1, h_1', 0, \dots, 0) \psi_1(|h_1'|) + f(\mathbf{x}, 0, h_1', 0, \dots, 0)\,. \end{aligned}$$

Hence due to (0.6) we have

$$L[h_1(x_1)] \geq 0 \quad \text{for} \quad x_1 \in (-l_1, 0)\,.$$

Similarly $L[h_i(x_i)] \geq 0$ for $x_i \in (-l_i, 0)$, $i = 2, \ldots, n$. By direct calculations one may easily derive that $L[h_i(-x_i)] \geq 0$ for $x_i \in (0, l_i)$, $i = 1, \ldots, n$.

Thus we proved that $h_i(x_i)$ for $x_i \in (-l_i, 0)$ as well as $h_i(-x_i)$ for $x_i \in (0, l_i)$, $i = 1, \ldots, n$ are classical supersolutions of (0.1). Consider now functions $\tilde{h}_i(x_i)$, $i = 1, \ldots, n$ defined by (1.2). Obviously these functions are classical supersolutions of (0.1) in domains $\{\mathbf{x} : \mathbf{x} \in \Omega, -l_i < x_i < 0\}$ and $\{\mathbf{x} : \mathbf{x} \in \Omega, 0 < x_i < l_i\}$, $i = 1, \ldots, n$. On the line $x_i = 0$, $i = 1, \ldots, n$, $\tilde{h}_i(x_i)$ are only Lipschitz continuous. From the definition of a viscosity supersolution it follows that $\tilde{h}_i(x_i)$, $i = 1, \ldots, n$ are viscosity supersolutions of (0.1), (0.2) in Ω. In fact, this follows easily from $h_i'(x_i) > 0$ which implies that there does not exist any $\mathbf{C^2}$ function whose graph touches the graph of $\tilde{h}_i$ from below at a point belonging to the set $\{x \in \Omega : x_i = 0\}$.

Consider now the function h^* defined by (1.3). One may easily see that the Lipschitz continuous function $h^*(\mathbf{x})$ is nonnegative and satisfies $h^*(\mathbf{x}) = 0$ on $\partial\Omega$. Moreover, $h^*(\mathbf{x})$ is a viscosity supersolution of (0.1) being the minimum of the set of viscosity supersolutions. Obviously the function $h_*(\mathbf{x}) = -h^*(\mathbf{x})$,

$$h_*(x_1, x_2, \ldots, x_n) = \max\{-\tilde{h}_1(x_1), -\tilde{h}_2(x_2), \ldots, -\tilde{h}_n(x_n)\},$$

is a subsolution of the same problem and $h_*(\mathbf{x}) = 0$ on $\partial\Omega$. □

Using the Ishii-Perron method and the solutions constructed in Theorem 1.1, one may prove the following

Corollary 1.1. *Suppose that conditions of Theorem 1.1 are fulfilled, where* (0.4) *holds with* $\nu > 0$. *If additionally* (0.5) *holds, then there exists a unique continuous viscosity solution of problem* (0.1), (0.2).

Remark 1.1. Consider the case where the operator in (0.1) is locally strictly elliptic and independent of $\mathbf{x}$, i.e.

$$\lambda_K \mathrm{trace}(X - Y) \leq \Phi(r, p, Y) - \Phi(r, p, X) \tag{1.4}$$

for $X \geq Y$ and $|r|$, $|p|$, $\|X\|$, $\|Y\| \leq K$ and for some positive constant λ_K depending on K. We assume that $\Phi(r, p, X)$ is continuous in all its variables and locally Lipschitz continuous in p, i.e.

$$\Phi(r, p, X) - \Phi(r, q, X)| \leq C_K |p - q| \tag{1.5}$$

for $|r|$, $|p|$, $|q|$, $\|X\| \leq K$ and for some positive constant C_K depending on K. In [7] it was shown that under assumptions (0.4) with $\nu = 0$, (1.4), (1.5), the strong comparison principle for semicontinuous viscosity sub- and supersolutions holds. Furthermore it was shown that if additionally a continuous viscosity solution is Lipschitz continuous on the boundary, then the solution is Lipschitz continuous in the whole domain. It is well known that a sufficient condition for the Lipschitz continuity on the boundary is the existence of Lipschitz sub- and supersolutions in $\overline{\Omega}$ that satisfy (0.2). By direct calculations one may show that the functions $h_*(\mathbf{x})$, $h^*(\mathbf{x})$ are $C^{0,1}(\overline{\Omega})$ viscosity sub- and supersolutions of (0.1), satisfying (0.2).

As a consequence the following corollary takes place

Corollary 1.2. *Suppose that* $a_{ij}(\mathbf{p}) \in \mathbf{C}(\mathbf{R}^n)$, $i,j = 1,\dots,n$, $f(u,\mathbf{p}) \in \mathbf{C}(\mathbf{R}\times\mathbf{R}^n)$ *and conditions* (0.4) *with* $\nu = 0$, (1.4), (1.5) *hold. Then there exists a unique Lipschitz continuous viscosity solution of problem* (0.1), (0.2).

Note here that if we additionally suppose that $f(0,p) = 0$, then Corollary 1.2 implies that the problem

$$-\sum_{i,j=1}^{n} a_{ij}(\nabla u)u_{x_i x_j} + f(u,\nabla u) = C \quad \text{in} \quad \Omega = \{\mathbf{x} : |x_i| < l_i, i = 1,\dots,n\},$$

$$u(\mathbf{x}) = 0 \quad \text{on} \quad \partial\Omega,$$

where C is a constant, has a unique Lipschitz continuous viscosity solution without any restrictions on the nonlinearities of f with respect to ∇u and the size of the domain Ω.

Concerning the higher regularity of Lipschitz continuous solutions of (0.1), (0.2) with a_{ij}, $i = 1,\dots,n$ and f independent of $\mathbf{x}$, for example interior $C^{1+\alpha}$ regularity under additional structural assumptions on the elliptic operator, see [9]. For the Lipschitz a priori estimates of viscosity solutions of (0.1), (0.2) see also [1].

2. Construction of subsolutions and supersolutions for the mean curvature equation. Case I

Consider the problem (0.8), (0.2). Let $\varphi(\tau)$ be the solution of the problem

$$\varphi'' + nH_0(1+\gamma\varphi'^2)^{\frac{3}{2}} = 0\,, \quad \varphi(0) = 0 \tag{2.1}$$

such that $\varphi'(\tau) > 0$ for $\tau \in [0, 2l]$, where

$$H_0 = \max_{\mathbf{x}\in\overline{\Omega}} |H(\mathbf{x},0)|\,, \quad \gamma = 1 + \max_{\mathbf{x}\in\overline{\Omega}} |\nabla G|^2 + \max_{\mathbf{x}\in\overline{\Omega}} |\nabla F|^2$$

and $2l$ is the length of the interval where x_1 varies. Put $\varphi'(\tau) = q$. Represent the solution of (2.1) in parametric form as

$$\varphi(q) = \frac{1}{nH_0}\int_q^{q_1} \frac{\rho d\rho}{(1+\gamma\rho^2)^{\frac{3}{2}}}\,, \quad \tau(q) = \frac{1}{nH_0}\int_q^{q_1} \frac{d\rho}{(1+\gamma\rho^2)^{\frac{3}{2}}}\,,$$

where the parameter q is lying in the interval $[q_0, q_1]$. Select $0 < q_0 < q_1 < +\infty$ such that $\tau(q_0) = 2l$. The latter is possible due to assumption (0.10). In fact,

$$\int \frac{d\rho}{(1+\gamma\rho^2)^{\frac{3}{2}}} = \frac{\rho}{(1+\gamma\rho^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{1}{nH_0}\int_0^{+\infty} \frac{d\rho}{(1+\gamma\rho^2)^{\frac{3}{2}}} = \frac{1}{n\sqrt{\gamma}H_0}\,.$$

Thus we can select q_0 and q_1 such that $\tau(q_0) = 2l$ if $2l < \left(n\sqrt{\gamma}H_0\right)^{-1}$. One may easily see that condition (0.10) is a restriction on the size of the domain Ω, but only in one direction.

Lemma 2.1. *Suppose that* Ω *satisfies condition* (A) *and* $H(\mathbf{x},u) \in \mathbf{C}\left(\overline{\Omega}\times\mathbf{R}\right)$ *is a non-increasing function with respect to* u. *If conditions* (0.9), (0.10) *hold, then* $\varphi(G - x_1)$ *defined by* (2.1) *is a classical supersolution of* (0.8) *and* $\varphi \geq 0$ *on* $\partial\Omega$, $\varphi = 0$ *for* $x_1 = G(x_2, x_3, \dots, x_n)$.

Proof. Define L by the following

$$Lu \equiv -(1+|\nabla u|^2)\Delta u + \sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} - nH(\mathbf{x},u)(1+|\nabla u|^2)^{\frac{3}{2}}\,.$$

Put $\xi \equiv G(\tilde{x}) - x_1$, $\tilde{x} = (x_2,\dots,x_n)$ and $\varphi' = \varphi_\xi$, $\varphi'' = \varphi_{\xi\xi}$. Hence

$$\nabla\varphi(\xi) \equiv (\varphi_{x_1},\varphi_{x_2},\dots,\varphi_{x_n}) = (-\varphi',\varphi' G_{x_2}(\tilde{x}),\dots,\varphi' G_{x_n}(\tilde{x}))\,,$$
$$|\nabla\varphi(\xi)|^2 = \varphi'^2\left(1+|\nabla G(\tilde{x})|^2\right)\,,$$
$$\varphi_{x_1x_1} = \varphi'', \quad \varphi_{x_1x_j} = -\varphi'' G_{x_j}\,, \quad \varphi_{x_ix_j} = \varphi'' G_{x_i}G_{x_j} + \varphi' G_{x_ix_j}, \quad i,j = 2,\dots,n\,.$$

By direct calculations we obtain

$$\begin{aligned}
L\varphi(G(\tilde{x})-x_1) &= -(1+|\nabla\varphi|^2)\Delta\varphi + \sum_{i,j=1}^{n}\varphi_{x_i}\varphi_{x_j}\varphi_{x_ix_j} - nH(\mathbf{x},\varphi)(1+|\nabla\varphi|^2)^{\frac{3}{2}}\\
&= -\left(1+\varphi'^2\left(1+|\nabla G|^2\right)\right)\left(\varphi''+\varphi''|\nabla G|^2+\varphi'\Delta G\right)+\varphi'^2\varphi''\\
&\quad+\sum_{i=2}^{n}\varphi'^2G_{x_i}^2(\varphi''G_{x_i}^2+\varphi'G_{x_ix_i})+2\varphi'^2\varphi''|\nabla G|^2\\
&\quad+\sum_{i,j=2,i\neq j}^{n}\varphi'^2G_{x_i}G_{x_j}(\varphi''G_{x_i}G_{x_j}+\varphi'G_{x_ix_j})- nH(\mathbf{x},\varphi)(1+|\nabla\varphi|^2)^{\frac{3}{2}}\\
&= -\left(1+\varphi'^2\left(1+|\nabla G|^2\right)\right)\varphi'' - \left(1+\varphi'^2\left(1+|\nabla G|^2\right)\right)\varphi''|\nabla G|^2\\
&\quad-\left(1+\varphi'^2\left(1+|\nabla G|^2\right)\right)\varphi'\Delta G+\varphi'^2\varphi''+\varphi'^2\varphi''\sum_{i=2}^{n}G_{x_i}^4\\
&\quad+\varphi'^3\sum_{i=2}^{n}G_{x_i}^2G_{x_ix_i}+2\varphi'^2\varphi''|\nabla G|^2+\varphi'^2\varphi''\sum_{i,j=2,i\neq j}^{n}G_{x_i}^2G_{x_j}^2\\
&\quad+\varphi'^3\sum_{i,j=2,i\neq j}^{n}G_{x_i}G_{x_j}G_{x_ix_j}-nH(\mathbf{x},\varphi)(1+|\nabla\varphi|^2)^{\frac{3}{2}}\\
&= -\varphi''-\varphi'^2\varphi''\left(1+|\nabla G|^2\right)-\varphi''|\nabla G|^2-\varphi'^2\varphi''\left(1+|\nabla G|^2\right)|\nabla G|^2\\
&\quad-\varphi'\Delta G-\varphi'^3\left(1+|\nabla G|^2\right)\Delta G+\varphi'^2\varphi''+\varphi'^2\varphi''\sum_{i=2}^{n}G_{x_i}^4\\
&\quad+\varphi'^3\sum_{i=2}^{n}G_{x_i}^2G_{x_ix_i}+2\varphi'^2\varphi''|\nabla G|^2+\varphi'^2\varphi''\sum_{i,j=2,i\neq j}^{n}G_{x_i}^2G_{x_j}^2\\
&\quad+\varphi'^3\sum_{i,j=2,i\neq j}^{n}G_{x_i}G_{x_j}G_{x_ix_j}-nH(\mathbf{x},\varphi)(1+|\nabla\varphi|^2)^{\frac{3}{2}}\\
&= -\varphi''\left(1+|\nabla G|^2\right)-\varphi'\Delta G
\end{aligned}$$

$$
\begin{aligned}
&-\varphi'^2\varphi''\Big(\left(1+|\nabla G|^2\right)^2 - \sum_{i=2}^{n} G_{x_i}^4 - 2|\nabla G|^2 - \sum_{i,j=2,i\neq j}^{n} G_{x_i}^2 G_{x_j}^2 - 1\Big)\\
&-\varphi'^3\Big(\left(1+|\nabla G|^2\right)\Delta G - \sum_{i=2}^{n} G_{x_i}^2 G_{x_i x_i} - \sum_{i,j=2 i\neq j}^{n} G_{x_i} G_{x_j} G_{x_i x_j}\Big)\\
&-nH(\mathbf{x},\varphi)(1+|\nabla\varphi|^2)^{\frac{3}{2}}\,.
\end{aligned}
$$

Due to the fact that

$$
\left(1+|\nabla G|^2\right)^2 - \sum_{i=2}^{n} G_{x_i}^4 - 2|\nabla G|^2 - \sum_{i,j=2,i\neq j}^{n} G_{x_i}^2 G_{x_j}^2 - 1 = 0\,, \tag{2.2}
$$

we conclude that

$$
\begin{aligned}
L\varphi(G(\tilde{x})-x_1) = &-\varphi''\left(1+|\nabla G|^2\right) - \varphi'\Delta G\\
&-\varphi'^3\Big(\left(1+|\nabla G|^2\right)\Delta G - \sum_{i=2}^{n} G_{x_i}^2 G_{x_i x_i} - \sum_{i,j=2,i\neq j}^{n} G_{x_i} G_{x_j} G_{x_i x_j}\Big)\\
&-nH(\mathbf{x},\varphi)(1+|\nabla\varphi|^2)^{\frac{3}{2}}\,.
\end{aligned}
$$

Since G is concave, it follows that $\Delta G \leq 0$. Using the fact that $\varphi' > 0$, we conclude that

$$
\begin{aligned}
L\varphi(G(\tilde{x})-x_1) \geq &-\varphi''\left(1+|\nabla G|^2\right)\\
&-\varphi'^3\Big(|\nabla G|^2\Delta G - \sum_{i=2}^{n} G_{x_i}^2 G_{x_i x_i} - \sum_{i,j=2,i\neq j}^{n} G_{x_i} G_{x_j} G_{x_i x_j}\Big)\\
&-nH(\mathbf{x},\varphi)(1+|\nabla\varphi|^2)^{\frac{3}{2}}\,.
\end{aligned}
$$

Let us prove now that

$$
|\nabla G|^2\Delta G - \sum_{i=2}^{n} G_{x_i}^2 G_{x_i x_i} - \sum_{i,j=2,i\neq j}^{n} G_{x_i} G_{x_j} G_{x_i x_j} \leq 0\,. \tag{2.3}
$$

Rewrite the last expression as

$$
\begin{aligned}
|\nabla G|^2\Delta G &- \sum_{i=2}^{n} G_{x_i}^2 G_{x_i x_i} - \sum_{i,j=2,i\neq j}^{n} G_{x_i} G_{x_j} G_{x_i x_j}\\
&= \sum_{i,j=2,i\neq j}^{n} G_{x_i}^2 G_{x_j x_j} - \sum_{i,j=2,i\neq j}^{n} G_{x_i} G_{x_j} G_{x_i x_j}\\
&= \frac{1}{2}\sum_{i,j=2,i\neq j}^{n}\left[G_{x_i}^2 G_{x_j x_j} + G_{x_j}^2 G_{x_i x_i} - 2G_{x_i} G_{x_j} G_{x_i x_j}\right].
\end{aligned}
$$

Consider the common term of the last sum. Every such term can be represented in the following way

$$(G_{x_j}, -G_{x_i}) \begin{pmatrix} G_{x_ix_i} & G_{x_ix_j} \\ G_{x_jx_i} & G_{x_jx_j} \end{pmatrix} (G_{x_j}, -G_{x_i})^T ,$$

where $(G_{x_j}, -G_{x_i})^T$ means a vector column. Since G is concave and the matrix of the second derivatives is negatively defined, we have that for every i, j, $i \neq j$ the matrices

$$\begin{pmatrix} G_{x_ix_i} & G_{x_ix_j} \\ G_{x_jx_i} & G_{x_jx_j} \end{pmatrix}$$

are negatively defined. Hence (2.3) holds. Taking into account (2.3), the fact that $\varphi \geq 0$ and the monotonicity of $H(x,u)$ with respect to u, we obtain that

$$\begin{aligned} L\varphi(G(\tilde{x}) - x_1) &\geq -\varphi'' \left(1 + |\nabla G|^2\right) - nH(\mathbf{x}, 0)(1 + |\nabla\varphi|^2)^{\frac{3}{2}} \\ &\geq -\varphi'' - nH_0(1 + \gamma\varphi'^2)^{\frac{3}{2}} = 0 , \end{aligned} \tag{2.4}$$

where equality is the consequence of (2.1). Thus we proved that $\varphi(\xi)$ is a classical supersolution of (0.8). Obviously $\varphi(\xi) \geq 0$ on $\partial\Omega$ and $\varphi(\xi) = 0$ for $x_1 = G$. □

Lemma 2.2. *Suppose that Ω satisfies condition (A) and $H(\mathbf{x}, u) \in \mathbf{C}\left(\overline{\Omega} \times \mathbf{R}\right)$ is a non-increasing function with respect to u. If conditions* (0.9), (0.10) *hold, then $\varphi(x_1 - F)$ defined by* (2.1) *is a classical supersolution of* (0.8) *and $\varphi \geq 0$ on $\partial\Omega$, $\varphi = 0$ on $x_1 = F(x_2, x_3, \ldots, x_n)$.*

Proof. The proof of Lemma 2.2 is similar to the proof of Lemma 2.1. Put $\zeta = x_1 - F(\tilde{x})$, $\varphi' = \varphi_\zeta$, $\varphi'' = \varphi_{\zeta\zeta}$. Hence

$$\nabla\varphi(\zeta) \equiv (\varphi_{x_1}(\zeta), \varphi_{x_2}(\zeta), \ldots, \varphi_{x_n}(\zeta)) = (\varphi'(\zeta), -\varphi'(\zeta)F_{x_2}(\tilde{x}), \ldots, -\varphi'(\zeta)F_{x_n}(\tilde{x})) ,$$

$$|\nabla\varphi(\zeta)|^2 = \varphi'^2(\zeta)\left(1 + |\nabla F(\tilde{x})|^2\right) ,$$

$$\varphi_{x_1x_1} = \varphi'', \quad \varphi_{x_1x_j} = -\varphi'' F_{x_j}, \quad \varphi_{x_ix_j} = \varphi'' F_{x_i}F_{x_j} - \varphi' F_{x_ix_j}, \quad i, j = 2, \ldots, n .$$

Note that equality (2.2) still holds if we substitute G by F in it. Using this fact and the convexity of F we arrive to

$$\begin{aligned} L\varphi(x_1 - F(\tilde{x})) \geq &- \varphi'' \left(1 + |\nabla F(\tilde{x})|^2\right) \\ &+ \varphi'^3 \Big(|\nabla F(\tilde{x})|^2 \Delta F - \sum_{i=2}^{n} F_{x_i}^2 F_{x_ix_i} - \sum_{i,j=2, i\neq j}^{n} F_{x_i}F_{x_j}F_{x_ix_j} \Big) \\ &- nH(\mathbf{x}, \varphi)(1 + |\nabla\varphi|^2)^{\frac{3}{2}} . \end{aligned}$$

Similarly to (2.3) we obtain that

$$|\nabla F(\tilde{x})|^2 \Delta F - \sum_{i=2}^{n} F_{x_i}^2 F_{x_ix_i} - \sum_{i,j=2, i\neq j}^{n} F_{x_i}F_{x_j}F_{x_ix_j} \geq 0 . \tag{2.5}$$

Hence

$$\begin{aligned} L\varphi(x_1 - F(\tilde{x})) &\geq -\varphi'' \left(1 + |\nabla F(\tilde{x})|^2\right) - nH(\mathbf{x}, 0)(1 + |\nabla\varphi|^2)^{\frac{3}{2}} \\ &\geq -\varphi'' - nH_0(1 + \gamma\varphi'^2)^{\frac{3}{2}} = 0 , \end{aligned}$$

Thus we proved that $\varphi(\zeta)$ is a classical supersolution of (0.1). Obviously $\varphi(\zeta) \geq 0$ on $\partial\Omega$ and $\varphi(\zeta) = 0$ for $x_1 = F$. □

Define

$$\varphi^*(\mathbf{x}) = \begin{cases} \varphi(G(\tilde{x}) - x_1)\,, & \text{for} \quad x_1 \geq \frac{F(\tilde{x})+G(\tilde{x})}{2}\,, \\ \varphi(x_1 - F(\tilde{x}))\,, & \text{for} \quad x_1 < \frac{F(\tilde{x})+G(\tilde{x})}{2}\,. \end{cases} \tag{2.6}$$

Theorem 2.1. *Suppose that Ω satisfies condition (A) and $H(\mathbf{x}, u) \in \mathbf{C}\left(\overline{\Omega} \times \mathbf{R}\right)$ is a non-increasing function with respect to u. If conditions* (0.9), (0.10) *hold, then $\varphi^*(\mathbf{x})$ and $\varphi_*(\mathbf{x}) = -\varphi^*(\mathbf{x})$ defined by* (2.6) *are viscosity supersolution and subsolution of* (0.8) *respectively and $\varphi^*(\mathbf{x}) = \varphi_*(\mathbf{x}) = 0$ on $\partial\Omega$.*

Proof. Obviously $\varphi^*(\mathbf{x}) = 0$ on $\partial\Omega$. Note that functions $\varphi(G(\tilde{x}) - x_1)$ and $\varphi(x_1 - F(\tilde{x}))$ are classical supersolutions of (0.8), therefore the function $\varphi^*(\mathbf{x})$ is also a classical supersolution of (0.8) in domains $\left\{\mathbf{x} \colon \mathbf{x} \in \Omega, x_1 > \frac{F(\tilde{x})+G(\tilde{x})}{2}\right\}$ and $\left\{\mathbf{x} \colon \mathbf{x} \in \Omega, x_1 < \frac{F(\tilde{x})+G(\tilde{x})}{2}\right\}$. On the line $x_1 = \frac{F(\tilde{x})+G(\tilde{x})}{2}$ the function $\varphi^*(\mathbf{x})$ is only Lipschitz continuous. The fact that $\varphi^*(\mathbf{x})$ is a viscosity supersolution of (0.8) in Ω follows easily from $\varphi' > 0$, which implies that there does not exist any $\mathbf{C^2}$ function whose graph touches the graph of φ^* from below at a point belonging to the set $\left\{x \in \Omega \colon x_1 = \frac{F(\tilde{x})+G(\tilde{x})}{2}\right\}$.

We show now that $\varphi_* = -\varphi^*$ is a viscosity subsolution of the same problem. Obviously $\varphi_* = 0$ on $\partial\Omega$. Consider $\varphi_1(\xi) = -\varphi(G(\tilde{x}) - x_1)$. By direct computations we obtain

$$\begin{aligned} L(\varphi_1(\xi)) &= L(-\varphi(G(\tilde{x}) - x_1)) = -(1 + |\nabla(-\varphi)|^2)\Delta(-\varphi) \\ &\quad + \sum_{i,j=1}^{n} (-\varphi)_{x_i}(-\varphi)_{x_j}(-\varphi)_{x_i x_j} - nH(\mathbf{x}, -\varphi)(1 + |\nabla(-\varphi)|^2)^{\frac{3}{2}} \\ &= (1 + |\nabla\varphi|^2)\Delta\varphi - \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j} - nH(\mathbf{x}, -\varphi)(1 + |\nabla\varphi|^2)^{\frac{3}{2}} \\ &\leq (1 + |\nabla\varphi|^2)\Delta\varphi - \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j} + nH_0(1 + \gamma\varphi'^2)^{\frac{3}{2}}. \end{aligned} \tag{2.7}$$

From (2.4) it follows that

$$-(1 + |\nabla\varphi|^2)\Delta\varphi + \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j} \geq -\varphi''\left(1 + |\nabla G|^2\right) \geq -\varphi'',$$

so we arrive to

$$\begin{aligned} L(\varphi_1(\xi)) &= L(-\varphi(G(\tilde{x}) - x_1)) \\ &\leq (1 + |\nabla\varphi|^2)\Delta\varphi - \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j} + nH_0(1 + \gamma\varphi'^2)^{\frac{3}{2}} \\ &\leq \varphi'' + nH_0(1 + \gamma\varphi'^2)^{\frac{3}{2}} = 0\,. \end{aligned}$$

Similarly one may obtain that $L(\varphi_1(\zeta)) = L(-\varphi(x_1 - F(\tilde{x}))) \le 0$. Thus the function

$$\varphi_*(\mathbf{x}) = -\varphi^*(\mathbf{x}) = \begin{cases} -\varphi(G(\tilde{x}) - x_1)\,, & \text{for} \quad x_1 \ge \frac{F(\tilde{x})+G(\tilde{x})}{2}\,, \\ -\varphi(x_1 - F(\tilde{x}))\,, & \text{for} \quad x_1 < \frac{F(\tilde{x})+G(\tilde{x})}{2} \end{cases}$$

is a viscosity subsolution of (0.8) that satisfies (0.2). □

So far we have constructed the supersolution $\varphi^*(\mathbf{x})$ and the subsolution $\varphi_*(\mathbf{x})$ of (0.8) that vanish on the boundary of Ω. Having the comparison holds, one may invoke the Ishii-Perron method to obtain the existence and uniqueness of a continuous viscosity solution of (0.1), (0.2).

Corollary 2.1. *Suppose that Ω satisfies condition (A), $H(\mathbf{x}, u) \in \mathbf{C}\left(\overline{\Omega} \times \mathbf{R}\right)$ is a strictly decreasing function with respect to u and conditions* (0.9), (0.10) *hold. Suppose that condition* (0.5) *holds for the left-hand side of* (0.8)*, then there exists a unique continuous viscosity solution of problem* (0.8), (0.2).

Remark 2.1. One may easily see that φ^* and φ_* are Lipschitz continuous functions in $\overline{\Omega}$. As a consequence we have that a continuous viscosity solution of (0.8), (0.2) is Lipschitz continuous on the boundary of Ω. Suppose now that $H(\mathbf{x}, u)$ is independent of $\mathbf{x}$ and non-increasing in u. From results in [7] it follows that a continuous viscosity solution of (0.8), (0.2) is Lipschitz continuous in $\overline{\Omega}$.

3. Construction of subsolutions and supersolutions for the mean curvature equation. Case II

We turn to the construction of subsolution and supersolution for the problem (0.8), (0.2) in the case when (0.11), (0.12) hold.

Lemma 3.1. *Suppose that Ω satisfies condition (A) and $H(\mathbf{x}, u) \in \mathbf{C}\left(\overline{\Omega} \times \mathbf{R}\right)$ is a non-increasing function with respect to u. Suppose that* (0.11) *holds. Then for any positive constant K_1 the function $K_1\xi = K_1(G(\tilde{x}) - x_1)$ is a classical supersolution of* (0.8).

Proof. Indeed,

$$\begin{aligned} L\left(K_1(G(\tilde{x}) - x_1)\right) \ge &- K_1\left(1 + K_1^2\left(1 + |\nabla G|^2\right)\right)\Delta G + K_1^3 \sum_{i,j=2}^{n} G_{x_i} G_{x_j} G_{x_i x_j} \\ &- nH(\mathbf{x}, 0)\left(1 + K_1^2\left(1 + |\nabla G|^2\right)\right)^{\frac{3}{2}} \\ = &- K_1 \Delta G - K_1^3 \Big(\left(1 + |\nabla G|^2\right)\Delta G - \sum_{i,j=2}^{n} G_{x_i} G_{x_j} G_{x_i x_j} \Big) \\ &- nH(\mathbf{x}, 0)\left(1 + K_1^2\left(1 + |\nabla G|^2\right)\right)^{\frac{3}{2}} \\ = &- K_1\left(1 + K_1^2\right)\Delta G - K_1^3 \Big(|\nabla G|^2 \Delta G - \sum_{i,j=2}^{n} G_{x_i} G_{x_j} G_{x_i x_j} \Big) \\ &- nH(\mathbf{x}, 0)\left(1 + K_1^2\left(1 + |\nabla G|^2\right)\right)^{\frac{3}{2}}\,. \end{aligned} \tag{3.1}$$

Using (2.3) we arrive to

$$
\begin{aligned}
L\left(K_1(G(\tilde{x})-x_1)\right) &\geq -K_1(1+K_1^2)\Delta G - nH(\mathbf{x},0)\left(1+K_1^2\left(1+|\nabla G|^2\right)\right)^{\frac{3}{2}} \\
&\geq -K_1(1+K_1^2)\Delta G - nH_0\left(1+K_1^2\left(1+|\nabla G|^2\right)\right)^{\frac{3}{2}} \\
&= -K_1\left(1+K_1^2\right)\left(1+|\nabla G|^2\right)^{\frac{3}{2}} \\
&\quad \times \left(\frac{\Delta G}{(1+|\nabla G|^2)^{\frac{3}{2}}} + \frac{nH_0}{K_1\,(1+K_1^2)}\left(\frac{1}{(1+|\nabla G|^2)}+K_1^2\right)^{\frac{3}{2}}\right).
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
&\frac{\Delta G}{(1+|\nabla G|^2)^{\frac{3}{2}}} + \frac{nH_0}{K_1\,(1+K_1^2)}\left(\frac{1}{(1+|\nabla G|^2)}+K_1^2\right)^{\frac{3}{2}} \\
&\qquad \leq \frac{\Delta G}{(1+|\nabla G|^2)^{\frac{3}{2}}} + \frac{nH_0}{K_1\,(1+K_1^2)}\left(1+K_1^2\right)^{\frac{3}{2}} \\
&\qquad = \frac{\Delta G}{(1+|\nabla G|^2)^{\frac{3}{2}}} + \frac{nH_0}{K_1}\sqrt{1+K_1^2} \leq 0
\end{aligned}
$$

due to (0.11), and as a consequence we obtain $L\left(K_1(G(\tilde{x})-x_1)\right) \geq 0$. □

Lemma 3.2. *Suppose that $H(\mathbf{x},u)$ and Ω satisfy all conditions of Lemma 3.1, except condition* (0.11)*. Suppose that* (0.12) *holds then for any positive K_2 the function $K_2\zeta = K_2(x_1 - F(\tilde{x}))$ is a classical supersolution of* (0.8)*.*

Proof. The proof of Lemma 3.2 is similar to the proof of the previous one. The only difference is the usage of inequalities (0.12), (2.5) instead of (0.11), (2.3). □

If $K_1 \leq K_2$ put

$$
\tilde{\varphi}(\mathbf{x}) = \begin{cases} K_2(G(\tilde{x})-x_1), & \text{for } \; x_1 \geq \frac{F(\tilde{x})+G(\tilde{x})}{2}, \\ K_2(x_1-F(\tilde{x})), & \text{for } \; x_1 < \frac{F(\tilde{x})+G(\tilde{x})}{2}. \end{cases} \tag{3.2}
$$

If $K_1 > K_2$ put

$$
\tilde{\varphi}(\mathbf{x}) = \begin{cases} K_1(G(\tilde{x})-x_1), & \text{for } \; x_1 \geq \frac{F(\tilde{x})+G(\tilde{x})}{2}, \\ K_1(x_1-F(\tilde{x})), & \text{for } \; x_1 < \frac{F(\tilde{x})+G(\tilde{x})}{2}. \end{cases} \tag{3.3}
$$

Theorem 3.1. *Suppose that Ω satisfies condition (A) and $H(\mathbf{x},u) \in \mathbf{C}(\overline{\Omega}\times\mathbf{R})$ is a strictly decreasing function with respect to u. Suppose that* (0.11)*,* (0.12) *hold, then $\tilde{\varphi}(\mathbf{x})$ and $-\tilde{\varphi}(\mathbf{x})$ defined by* (3.2) *or* (3.3) *are viscosity supersolution and subsolution of* (0.8) *respectively and $\tilde{\varphi}(\mathbf{x}) = 0$ on $\partial\Omega$.*

Proof. The proof is exactly the same as that of Theorem 2.1. □

Using the solutions constructed in Theorem 3.1 one may prove the following

Corollary 3.1. *Suppose that Ω satisfies condition (A) and $H(\mathbf{x},u) \in \mathbf{C}\left(\overline{\Omega}\times\mathbf{R}\right)$ is a strictly decreasing function with respect to u. Suppose that conditions* (0.11)*,* (0.12) *as well as condition* (0.5) *for the left-hand side of* (0.8) *hold. Then there exists a unique continuous viscosity solution of problem* (0.8)*,* (0.2)*.*

4. Construction of subsolutions and supersolutions for the pseudo p-Laplace equation

In this section we consider the Dirichlet problem for (0.13). Recall that the equation has the following form

$$-\sum_{i=1}^{n}\mu_i(|u_{x_i}|^{p_i}u_{x_i})_{x_i}+c(\mathbf{x})g(u)+f(\mathbf{x})=0 \quad \text{in} \quad \Omega\subset\mathbf{R}^{\mathbf{n}}\,,$$

where μ_i, p_i, $i=1,\dots,n$ are constants, $\mu_i>0$, $p_i\ge 0$. Suppose that $c(x)g(u)$ is a nondecreasing function with respect to u. Without loss of generality we assume that

$$c(x)\ge 0\,,\quad g(0)=0\,,\quad g \text{ is a nondecreasing function.} \tag{4.1}$$

Let M be a positive constant such that

$$f_0=\mu_1(p_1+1)M^{p+1}\,,\quad f_0=\max_{\bar\Omega}|f(\mathbf{x})|\,. \tag{4.2}$$

Define the function $v(\tau)$ by the following

$$v(\tau)=-M\frac{\tau^2}{2}+[M(1+2l_1)+\epsilon]\tau\,,\quad \tau\in[0,2l_1]\,. \tag{4.3}$$

Obviously

$$v\ge 0\,,\quad v'\ge M+\epsilon>M\,,\quad v''=-M\,.$$

Define the following set of regularized operators

$$L_\varepsilon u(\mathbf{x})\equiv-\sum_{i=1}^{n}\mu_i((u_{x_i}^{\alpha}+\varepsilon)^{p_i/\alpha}u_{x_i})_{x_i}+c(\mathbf{x})g(u)+f(\mathbf{x})=0\,. \tag{4.4}$$

Here the constant $\alpha\in(0,1)$ is such that $(u_{x_i}^{\alpha})^{p_i/\alpha}=|u_{x_i}|^{p_i}$.

Lemma 4.1. *Suppose that Ω satisfies condition (A) and $c(\mathbf{x}),f(\mathbf{x})\in\mathbf{C}\left(\overline{\Omega}\right)$, $g(u)\in\mathbf{C}(\mathbf{R})$. If condition (4.1) holds, then $v(G-x_1)$ defined by (4.3) is a classical supersolution of (4.4) and $v\ge 0$ on $\partial\Omega$, $v=0$ for $x_1=G(x_2,x_3,\dots,x_n)$.*

Proof. Rewrite (4.4) in non-divergent form as

$$L_\varepsilon u(\mathbf{x})\equiv-\sum_{i=1}^{n}a_{i\varepsilon}(u_{x_i})u_{x_ix_i}+c(\mathbf{x})g(u)+f(\mathbf{x})\,, \tag{4.5}$$

where

$$a_{i\varepsilon}(z)=\mu_i(z^\alpha+\varepsilon)^{\frac{p_i}{\alpha}-1}((p_i+1)z^\alpha+\varepsilon)\,.$$

One may easily see that if $\alpha\in(0,1)$, then for any $p\ge 0$ the expression $E(\varepsilon)=\mu(z^\alpha+\varepsilon)^{\frac{p}{\alpha}-1}\big((p+1)z^\alpha+\varepsilon\big)$ is a nondecreasing function with respect to ε.

Put $\xi=G-x_1$. Our goal is to show that $L_\varepsilon v\ge 0$. Applying the operator L_ε to v we obtain that

$$\begin{aligned}L_\varepsilon v(\xi)={}&-a_{1\varepsilon}(v'(\xi))v''(\xi)\\&-\sum_{i=2}^{n}a_{i\varepsilon}(v'(\xi)G_{x_i})(v''(\xi)G_{x_i}^2+v'(\xi)G_{x_ix_i})+c(\mathbf{x})g(v)+f(\mathbf{x})\,.\end{aligned}$$

Taking into account that $G_{x_i x_i} \leq 0$, $v'' < 0$, $v' > 0$ we conclude that

$$v''(\xi)G_{x_i}^2 + v'(\xi)G_{x_i x_i} \leq 0\,, \tag{4.6}$$

hence

$$L_\varepsilon v(\xi) \geq -a_{1\varepsilon}(v'(\xi))v''(\xi) + c(\mathbf{x})g(v) + f(\mathbf{x})\,. \tag{4.7}$$

From (4.1) and (4.7) it follows that

$$L_\varepsilon v(\xi) \geq -a_{1\varepsilon}(v'(\xi))v''(\xi) + f(\mathbf{x})\,. \tag{4.8}$$

By direct calculations one estimates from below the first term on the right side of (4.8) in the following way

$$-\, a_{1\varepsilon}(v'(\xi))v''(\xi) \geq \mu_1(p_1+1)M^{p_1+1}\,. \tag{4.9}$$

Here we use the fact that $v'(\xi) \geq M$, $v''(\xi) = -M$, $\alpha < 1$ and $E(\varepsilon)$ is a nondecreasing function with respect to ε. From (4.8) and (4.9) we conclude that

$$L_\varepsilon v(\xi) \geq \mu_1(p_1+1)M^{p_1+1} - f_0\,, \tag{4.10}$$

and finally from (4.10) and (4.2) we obtain the desired inequality

$$L_\varepsilon v(\xi) \geq 0\,. \tag{4.11}$$

□

Lemma 4.2. *Suppose that Ω satisfies condition (A) and $c(\mathbf{x}), f(\mathbf{x}) \in \mathbf{C}\left(\overline{\Omega}\right)$, $g(u) \in \mathbf{C}\left(\mathbf{R}\right)$. If condition* (4.1) *holds, then $v(x_1 - F)$ defined by* (4.3) *is a classical supersolution of* (4.4) *and $v \geq 0$ on $\partial\Omega$, $v = 0$ on $x_1 = F(x_2, x_3, \ldots, x_n)$.*

Proof. The proof of Lemma 4.2 is similar to the proof of the previous one. In order to obtain inequality analogous to (4.7) one only has to notice that

$$v''(\zeta)F_{x_i}^2 - v'(\zeta)F_{x_i x_i} \leq 0\,.$$

□

Lemma 4.3. *Suppose that Ω satisfies condition (A) and $c(\mathbf{x}), f(\mathbf{x}) \in \mathbf{C}\left(\overline{\Omega}\right)$, $g(u) \in \mathbf{C}\left(\mathbf{R}\right)$. If condition* (4.1) *holds, then $v(G - x_1)$ and $v(x_1 - F)$ defined by* (4.3) *are classical supersolutions of* (0.13) *and $v(G - x_1) \geq 0$ on $\partial\Omega$, $v(G - x_1) = 0$ on $x_1 = G(x_2, x_3, \ldots, x_n)$, $v(x_1 - F) \geq 0$ on $\partial\Omega$, $v(x_1 - F) = 0$ on $x_1 = F(x_2, x_3, \ldots, x_n)$.*

Proof. One only has to notice that the results of Lemmas 4.1, 4.2 hold for any $\varepsilon > 0$ and $\lim_{\varepsilon\to 0} L_\varepsilon v = Lv$. □

Define

$$v^*(\mathbf{x}) = \begin{cases} v(G(\tilde{x}) - x_1)\,, & \text{for} \quad x_1 \geq \frac{F(\tilde{x})+G(\tilde{x})}{2}\,, \\ v(x_1 - F(\tilde{x}))\,, & \text{for} \quad x_1 < \frac{F(\tilde{x})+G(\tilde{x})}{2}\,. \end{cases} \tag{4.12}$$

Theorem 4.1. *Suppose that Ω satisfies condition (A) and $c(\mathbf{x}), f(\mathbf{x}) \in \mathbf{C}\left(\overline{\Omega}\right)$, $g(u) \in \mathbf{C}\left(\mathbf{R}\right)$. If condition* (4.1) *holds, then $v^*(\mathbf{x})$ and $v_*(\mathbf{x}) = -v^*(\mathbf{x})$ defined by* (4.12) *are viscosity supersolution and subsolution of* (0.13) *respectively and $v^*(\mathbf{x}) = v_*(\mathbf{x}) = 0$ on $\partial\Omega$.*

Proof. Using the arguments of Theorem 2.1 concerning $\varphi^*(\mathbf{x})$, one may easily prove that $v^*(\mathbf{x})$ is a viscosity supersolution of (0.13) and $v^*(\mathbf{x}) = 0$ on $\partial\Omega$.

Let us show that $v_* = -v^*$ is a viscosity subsolution of the same problem. Obviously $v_* = 0$ on the boundary of Ω. Consider $v_1(\xi) = -v(G(\tilde{x}) - x_1)$. Substituting $v_1(\xi) = -v(G(\tilde{x}) - x_1)$ into L_ε and using the fact that $a_{i\varepsilon}(z) = a_{i\varepsilon}(-z)$ we obtain

$$\begin{aligned} L_\varepsilon(v_1(\xi)) = L_\varepsilon(-v(G(\tilde{x}) - x_1)) &= a_{1\varepsilon}(v'(\xi))v''(\xi) \\ &+ \sum_{i=2}^{n} a_{i\varepsilon}(v'(\xi)G_{x_i})\left(v''(\xi)G_{x_i}^2 + v'(\xi)G_{x_i x_i}\right) \\ &+ c(\mathbf{x})g(-v(\xi)) + f(\mathbf{x})\,. \end{aligned} \tag{4.13}$$

Using (4.2), (4.6), (4.9) and the fact that $c(\mathbf{x})g(-v(\xi)) \le 0$, from (4.13) it follows

$$L_\varepsilon(v_1(\xi)) \le -\mu_1(p_1 + 1)M^{p_1+1} + f_0 = 0\,. \tag{4.14}$$

Similarly one may obtain $L_\varepsilon(v_1(\zeta)) = L_\varepsilon(-v(x_1 - F(\tilde{x}))) \le 0$. Hence $L(v_1(\xi)) \le 0$, $L(v_1(\zeta)) \le 0$. By using Lemma 4.3 and the arguments of Theorem 2.1 we conclude that $v_* = -v^*$ is a viscosity subsolution of (0.13) which satisfies (0.2). □

The following corollary is an immediate consequence of Theorem 4.1.

Corollary 4.1. *Suppose that Ω satisfies condition (A) and $c(\mathbf{x}), f(\mathbf{x}) \in \mathbf{C}\left(\overline{\Omega}\right)$, $g(u) \in \mathbf{C}(\mathbf{R})$. Suppose that $c(x)g(u)$ is a strictly increasing function with respect to u and condition* (0.5) *holds for the left-hand side of* (0.13)*, then there exists a unique continuous viscosity solution of problem* (0.13), (0.2).

References

[1] Barles, G., *A weak Bernstein method for fully nonlinear elliptic equations*, Differential Integral Equations **4** (2) (1991), 241–262.

[2] Chen, Y. G., Giga, Y., Goto, S., *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), 749–786.

[3] Crandall, M., Kocan, M., Lions, P. L., Swięch, A., *Existence results for uniformly elliptic and parabolic fully nonlinear equations*, Electron. J. Differential Equations **24** (1999), 1–20.

[4] Crandall, M. G., Ishii, H., Lions, P. L., *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. **27** (1992), 1–67.

[5] Crandall, M. G., Lions, P. L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1–42.

[6] Ishii, H., Lions, P. L., *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations*, J. Differential Equations **83** (1990), 26–78.

[7] Kawohl, B., Kutev, N., *Comparison principle and Lipschitz regularity for viscosity solutions of some classes of nonlinear partial differential equations*, Funkcial. Ekvac. **43** (2000), 241–253.

[8] Tersenov, Al., Tersenov, Ar., *Viscosity solutions of p-Laplace equation with nonlinear source*, Arch. Math. (Basel) **88** (3) (2007), 259–268.

[9] Trudinger, N. S., *Holder gradient estimates for fully nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 57–65.

University of Cyprus, Department of Mathematics and Statistics
P.O. Box 20537, 1678 Nicosia, Cyprus
E-mail: aterseno@ucy.ac.cy

Sobolev Institute of Mathematics
Siberian Branch of the Russian Academy of Sciences
4 Acad. Koptyug avenue, 630090 Novosibirsk, Russia