ARCHIVUM MATHEMATICUM (BRNO)
Tomus 45 (2009), 37–46

A SUBCLASS OF HARMONIC FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

G. Murugusundaramoorthy[1], K. Vijaya[1], and R. K. Raina[2]

Abstract. Making use of the Dziok-Srivastava operator, we introduce a new class of complex valued harmonic functions which are orientation preserving and univalent in the open unit disc and are related to uniformly convex functions. We investigate the coefficient bounds, distortion inequalities and extreme points for this generalized class of functions.

1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, we may express

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \overline{\sum_{m=1}^{\infty} b_m z^m}\,, \quad |b_1| < 1\,. \tag{2}$$

where the analytic functions h and g are of the forms

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m\,, \quad g(z) = b_1 z + \sum_{m=2}^{\infty} b_m z^m \quad (0 \le b_1 < 1)\,.$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero, that is $g \equiv 0$.

2000 *Mathematics Subject Classification*: primary 30C45; secondary 30C50.
Key words and phrases: harmonic univalent starlike functions, Dziok-Srivastava operator, distortion bounds, extreme points, uniformly convex functions.
Received January 3, 2008, revised December 2008. Editor O. Došlý.

Let the Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{m=2}^{\infty} \phi_m z^m$$

and

$$\psi(z) = z + \sum_{m=2}^{\infty} \psi_m z^m$$

in $\mathcal{S}$ be defined by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{m=2}^{\infty} \phi_m \psi_m z^m .$$

For complex parameters $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ ($\beta_j \neq 0, -1, -2 \ldots$; $j = 1, \ldots, q$) the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$(3) \qquad {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) := \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{z^m}{m!}$$

$$(p \leq q+1;\ p, q \in N_0 := N \cup \{0\};\ z \in \mathcal{U}),$$

where N denotes the set of all positive integers, and $(a)_m$ is the Pochhammer symbol defined by

$$(4) \qquad (a)_m = \begin{cases} 1, & m = 0 \\ a(a+1)(a+2)\ldots(a+m-1), & m \in N . \end{cases}$$

For real values of $\alpha_i > 0$ ($i = 1, \ldots, p$), $\beta_j > 0$ ($j = 1, \ldots, q$); $p \leq q+1$; $p, q \in N_0 = N \cup \{0\}$, let

$$H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q) \colon \mathcal{S} \to \mathcal{S}$$

be a linear operator defined by

$$\begin{aligned} \big[\big(H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)\big)(\phi)\big](z) &:= z\ {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) * \phi(z) \\ &= z + \sum_{m=2}^{\infty} \Gamma(\alpha_1, m)\ a_m\ z^m , \end{aligned} \qquad (5)$$

where

$$(6) \qquad \Gamma(\alpha_1, m) = \frac{(\alpha_1)_{m-1} \ldots (\alpha_p)_{m-1}}{(\beta_1)_{m-1} \ldots (\beta_q)_{m-1}} \frac{1}{(m-1)!} .$$

For notational simplicity, we use the notation $H_q^p[\alpha_1, \beta_1]$ for $H(\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q)$ in the sequel. It follows from (5) that

$$H_0^1[1]\phi(z) = \phi(z), \quad H_0^1[2]\phi(z) = z\phi'(z) .$$

The linear operator $H_q^p[\alpha_1, \beta_1]$ is the Dziok-Srivastava operator (see [4]), which contains such well known operators as the Hohlov linear operator, Saitho generalized linear operator, the Carlson-Shaffer linear operator [2], the Ruscheweyh derivative operator [10] as well as its generalized versions, the Bernardi-Libera-Livingston operator, and the Srivastave-Owa fractional derivative operator. One may refer to [2], [4] and [12] for more details concerning these operators.

Applying the Dziok-Srivastava operator to the harmonic function $f = h + \overline{g}$ given by (1), we readily get

$$H_q^p[\alpha_1, \beta_1]f(z) = H_q^p[\alpha_1, \beta_1]h(z) + \overline{H_q^p[\alpha_1, \beta_1]\, g(z)}\,. \tag{7}$$

Goodman [5] introduced two interesting subclasses of S, namely, uniformly convex functions (UCV) and uniformly starlike functions (UST), and Ronning [8] introduced a subclass of starlike functions S_p corresponding to the class (UCV).

Motivated by the earlier works of [6] to [7], [9] and [13] on the subject of harmonic functions, we introduce here a new subclass $G_{\mathcal{H}}([\alpha_1], \gamma)$ of $\mathcal{H}$.

For $0 \leq \gamma < 1$, let $G_{\mathcal{H}}([\alpha_1], \gamma)$ denote the subfamily of starlike harmonic functions $f \in \mathcal{H}$ of the form (1) such that

$$\mathrm{Re}\Big\{(1 + e^{i\psi})\frac{z(H_q^p[\alpha_1, \beta_1]f(z))'}{z'(H_q^p[\alpha_1, \beta_1]f(z))} - e^{i\psi}\Big\} \geq \gamma \tag{8}$$

for $z \in \mathcal{U}$ where $(H_q^p[\alpha_1, \beta_1]f(z))' = \frac{\partial}{\partial\theta}(H_q^p[\alpha_1, \beta_1]f(re^{i\theta})$, $z' = \frac{\partial}{\partial\theta}(z = re^{i\theta})$ and $H_q^p[\alpha_1, \beta_1]f(z)$ is defined by (7).

We also let $V_{\mathcal{H}}([\alpha_1], \gamma) = G_{\mathcal{H}}([\alpha_1], \gamma) \bigcap V_{\mathcal{H}}$ where $V_{\mathcal{H}}$ is the class of harmonic functions with varying arguments introduced by Jahangiri and Silverman [6] consisting of functions f of the form (1) in $\mathcal{H}$ for which there exists a real number ϕ such that

$$\eta_m + (m-1)\phi \equiv \pi (\mathrm{mod}\, 2\pi)\,, \quad \delta_m + (m-1)\phi \equiv 0 \ (m \geq 2)\,, \tag{9}$$

where $\eta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$.

In this paper we obtain a sufficient coefficient condition for functions f given by (2) to be in the class $G_{\mathcal{H}}([\alpha_1], \gamma)$. It is shown that this coefficient condition is necessary also for functions belonging to the class $V_{\mathcal{H}}([\alpha_1], \gamma)$. Further, distortion results and extreme points for functions in $V_{\mathcal{H}}([\alpha_1], \gamma)$ are also obtained.

2. The class $G_{\mathcal{H}}([\alpha_1], \gamma)$

We begin deriving a sufficient coefficient condition for the functions belonging to the class $G_{\mathcal{H}}([\alpha_1], \gamma)$. This result is contained in the following.

Theorem 1. *Let $f = h + \overline{g}$ be given by (2). If*

$$\sum_{m=2}^{\infty}\Big(\frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m|\Big)\Gamma(\alpha_1, m) \leq 1 - \frac{3+\gamma}{3-\gamma}b_1 \tag{10}$$

where $0 \leq \gamma < 1$, then $f \in G_{\mathcal{H}}([\alpha_1], \gamma)$.

Proof. We first show that if the inequality (10) holds for the coefficients of $f = h+\overline{g}$, then the required condition (8) is satisfied. Using (7) and (8), we can write

$$\mathrm{Re}\,\Big\{(1 + e^{i\psi})\Big[\frac{z(H_q^p[\alpha_1, \beta_1]h(z))' - \overline{z(H_q^p[\alpha_1, \beta_1]g(z))'}}{(H_q^p[\alpha_1, \beta_1]h(z)) + \overline{(H_q^p[\alpha_1, \beta_1]g(z))}} - e^{i\psi}\Big]\Big\} = \mathrm{Re}\ \frac{A(z)}{B(z)}\,,$$

where

$$A(z) = (1+e^{i\psi})\big[z\big(H_q^p[\alpha_1,\beta_1]h(z)\big)' - \overline{z\big(H_q^p[\alpha_1,\beta_1]g(z)\big)'}\big]$$
$$- e^{i\psi}\big[\big(H_q^p[\alpha_1,\beta_1]h(z)\big) + \overline{\big(H_q^p[\alpha_1,\beta_1]g(z)\big)}\big]$$

and

$$B(z) = \big(H_q^p[\alpha_1,\beta_1]h(z)\big) + \overline{\big(H_q^p[\alpha_1,\beta_1]g(z)\big)}\,.$$

In view of the simple assertion that Re $(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0\,. \tag{11}$$

Substituting for $A(z)$ and $B(z)$ the appropriate expressions in (11), we get

$$\begin{aligned}
&|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)|\\
&\quad\geq (2-\gamma)|z| - \sum_{m=2}^{\infty}(2m-\gamma)\Gamma(\alpha_1,m)|a_m|\,|z|^m - \sum_{m=1}^{\infty}(2m+\gamma)\Gamma(\alpha_1,m)|b_m|\,|z|^m\\
&\qquad - \gamma|z| - \sum_{m=2}^{\infty}(2m-2-\gamma)\Gamma(\alpha_1,m)|a_m|\,|z|^m - \sum_{m=1}^{\infty}(2m+2+\gamma)\Gamma(\alpha_1,m)|b_m|\,|z|^m\\
&\quad\geq 2(1-\gamma)|z|\Big\{1 - \frac{3+\gamma}{1-\gamma}b_1 - \Big(\sum_{m=2}^{\infty}\Big[\frac{2m-1-\gamma}{1-\gamma}\Gamma(\alpha_1,m)|a_m|\\
&\qquad + \frac{2m+1+\gamma}{1-\gamma}\Gamma(\alpha_1,m)|b_m|\Big]\Big)\Big\} \geq 0
\end{aligned}$$

by virtue of the inequality (10). This implies that $f \in G_{\mathcal{H}}\big([\alpha_1],\gamma\big)$. $\square$

Now we obtain the necessary and sufficient condition for function $f = h + \overline{g}$ be given with condition (9).

Theorem 2. *Let $f = h + \overline{g}$ be given by (2). Then $f \in V_{\mathcal{H}}\big([\alpha_1],\gamma\big)$ if and only if*

$$\sum_{m=2}^{\infty}\Big\{\frac{2m-1-\gamma}{1-\gamma}|a_m| + \frac{2m+1+\gamma}{1-\gamma}|b_m|\Big\}\Gamma(\alpha_1,m) \leq 1 - \frac{3+\gamma}{1-\gamma}b_1 \tag{12}$$

where $0 \leq \gamma < 1$.

Proof. Since $V_{\mathcal{H}}\big([\alpha_1],\gamma\big) \subset G_{\mathcal{H}}\big([\alpha_1],\gamma\big)$, we only need to prove the necessary part of the theorem. Assume that $f \in V_{\mathcal{H}}\big([\alpha_1],\gamma\big)$, then by virtue of (7) to (8), we obtain

$$\text{Re}\,\Big\{(1+e^{i\psi})\Big[\frac{z(H_q^p[\alpha_1,\beta_1]h(z))' - \overline{z(H_q^p[\alpha_1,\beta_1]g(z))'}}{(H_q^p[\alpha_1,\beta_1]h(z)) + \overline{(H_q^p[\alpha_1,\beta_1]g(z))}} - (e^{i\psi}+\gamma)\Big]\Big\} \geq 0\,.$$

The above inequality is equivalent to

$$\begin{aligned}
&\operatorname{Re}\Big\{\Big(z+\Big(\sum_{m=2}^{\infty}[m(1+e^{i\psi})-\gamma-e^{i\psi}]\Gamma(\alpha_1,m)|a_m|z^m \\
&\qquad -\sum_{m=2}^{\infty}[m(1+e^{i\psi})+\gamma+e^{i\psi}]\Gamma(\alpha_1,m)|b_m|\overline{z}^m\Big)\Big) \\
&\qquad \times\Big(z+\sum_{m=2}^{\infty}\Gamma(\alpha_1,m)|a_m|z^m+\sum_{m=2}^{\infty}\Gamma(\alpha_1,m)|b_m|\overline{z}^m\Big)^{-1}\Big\} \\
&\quad =\operatorname{Re}\Big\{\Big((1-\gamma)+\sum_{m=2}^{\infty}[m(1+e^{i\psi})-e^{i\psi}-\gamma]\Gamma(\alpha_1,m)|a_m|z^{m-1} \\
&\qquad -\frac{\overline{z}}{z}\sum_{m=2}^{\infty}[m(1+e^{i\psi})+e^{i\psi}+\gamma]\Gamma(\alpha_1,m)|b_m|\overline{z}^{m-1}\Big) \\
&\qquad \times\Big(1+\sum_{m=2}^{\infty}\Gamma(\alpha_1,m)|a_m|z^{m-1}+\frac{\overline{z}}{z}\sum_{m=2}^{\infty}\Gamma(\alpha_1,m)|b_m|\overline{z}^{m-1}\Big)^{-1}\Big\}\geq 0\,.
\end{aligned}$$

This condition must hold for all values of z, such that $|z|=r<1$. Upon choosing ϕ according to (9) and noting that $\operatorname{Re}(-e^{i\psi})\geq -|e^{i\psi}|=-1$, the above inequality reduces to

$$\begin{aligned}
&\Big((1-\gamma)-(1-b_1)-\Big[\sum_{m=2}^{\infty}(2m-1-\gamma)\Gamma(\alpha_1,m)|a_m|r^{m-1} \\
&\qquad +(2m+1+\gamma)\Gamma(\alpha_1,m)|b_m|r^{m-1}\Big]\Big) \\
&\qquad \times\Big(1-\sum_{m=2}^{\infty}\Gamma(\alpha_1,m)|a_m|r^{m-1}+\sum_{m=1}^{\infty}\Gamma(\alpha_1,m)|b_m|r^{m-1}\Big)^{-1}\geq 0\,.
\end{aligned}\tag{13}$$

If (12) does not hold, then the numerator in (13) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0=r_0$ in (0,1) for which the quotient in (13) is negative. This contradicts our assumption that $f\in V_{\mathcal{H}}([\alpha_1],\gamma)$. We thus conclude that it is both necessary and sufficient that the coefficient bound inequality (12) holds true when $f\in V_{\mathcal{H}}([\alpha_1],\gamma)$. This completes the proof of Theorem 2. □

If we put $\phi=2\pi/k$ in (9), then Theorem 2 gives the following corollary.

Corollary 1. *A necessary and sufficient condition for $f=h+\overline{g}$ satisfying* (12) *to be starlike is that*

$$\arg(a_m)=\pi-2(m-1)\pi/k\,,$$

and

$$\arg(b_m)=2\pi-2(m-1)\pi/k\quad (k=1,2,3,\dots)\,.$$

3. Distortion and extreme points

In this section we obtain the distortion bounds for the functions $f\in V_{\mathcal{H}}([\alpha_1],\gamma)$ that lead to a covering result for the family $V_{\mathcal{H}}([\alpha_1],\gamma)$.

Theorem 3. *If* $f \in V_{\mathcal{H}}\big([\alpha_1], \gamma\big)$ *then*

$$|f(z)| \le (1+|b_1|)r + \frac{\beta_1}{\alpha_1}\Big(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\Big)r^2$$

and

$$|f(z)| \ge (1-|b_1|)r - \frac{\beta_1}{\alpha_1}\Big(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\Big)r^2 .$$

Proof. We will only prove the right-hand inequality of the above theorem. The arguments for the left-hand inequality are similar and so we omit it. Let $f \in V_{\mathcal{H}}\big([\alpha_1], \gamma\big)$ taking the absolute value of f, we obtain

$$\begin{aligned}|f(z)| &\le (1+|b_1|)r + \sum_{m=2}^{\infty}(|a_m|+|b_m|)r^m \\ &\le (1+b_1)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m|) .\end{aligned}$$

This implies that

$$\begin{aligned}|f(z)| &\le (1+|b_1|)r + \frac{\beta_1}{\alpha_1}\Big(\frac{1-\gamma}{3-\gamma}\Big)\sum_{m=2}^{\infty}\Big[\Big(\frac{3-\gamma}{1-\gamma}\Big)\frac{\alpha_1}{\beta_1}|a_m| + \Big(\frac{3-\gamma}{1-\gamma}\Big)\frac{\alpha_1}{\beta_1}|b_m|\Big]r^2 \\ &\le (1+|b_1|)r + \frac{\beta_1}{\alpha_1}\Big(\frac{1-\gamma}{3-\gamma}\Big)\Big[1 - \frac{3+\gamma}{1-\gamma}|b_1|\Big]r^2 \\ &\le (1+|b_1|)r + \frac{\beta_1}{\alpha_1}\Big(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\Big)r^2 ,\end{aligned}$$

which establishes the desired inequality. □

As a consequence of the above theorem and Corollary 1, we state the following covering lemma.

Corollary 2. *Let* $f = h + \overline{g}$ *and of the form* (2) *be so that* $f \in V_{\mathcal{H}}\big([\alpha_1], \gamma\big)$. *Then*

$$\Big\{w : |w| < \frac{3\alpha_1 - \beta_1 - (\alpha_1-\beta_1)\gamma}{(3-\gamma)\alpha_1}\,(1-b_1)\Big\} \subset f(\mathcal{U}) .$$

For a compact family, the maximum or minimum of the real part of any continuous linear functional occurs at one of the extreme points of the closed convex hull. Unlike many other classes, characterized by necessary and sufficient coefficient conditions, the family $V_{\mathcal{H}}([\alpha_1], \gamma)$ is not a convex family. Nevertheless, we may still apply the coefficient characterization of the $V_{\mathcal{H}}([\alpha_1], \gamma)$ to determine the extreme points.

Theorem 4. *The closed convex hull of* $V_{\mathcal{H}}\big([\alpha_1], \gamma\big)$ (*denoted by* clco $V_{\mathcal{H}}\big([\alpha_1], \gamma\big)$) *is*

$$\Big\{f(z) = z + \sum_{m=2}^{\infty}|a_m|z^m + \overline{\sum_{m=1}^{\infty}|b_m|z^m}\ : \sum_{m=2}^{\infty} m\big[|a_m|+|b_m|\big] < 1 - b_1\Big\} .$$

By setting $\lambda_m = \frac{(1-\gamma)}{(2m-1-\gamma)\Gamma(\alpha_1,m)}$ *and* $\mu_m = \frac{(1+\gamma)}{(2m+1+\gamma)\Gamma(\alpha_1,m)}$, *then for* b_1 *fixed, the extreme points for* clco $V_{\mathcal{H}}\big([\alpha_1],\gamma\big)$ *are*

$$\big\{z+\lambda_m x z^m + \overline{b_1 z}\big\} \cup \big\{z+\overline{b_1 z+\mu_m x z^m}\big\} \tag{14}$$

where $m \geq 2$ *and* $|x| = 1-|b_1|$.

Proof. Any function f in clco$V_{\mathcal{H}}\big([\alpha_1],\gamma\big)$ may be expressed as

$$f(z) = z + \sum_{m=2}^{\infty} |a_m| e^{i\eta_m} z^m + \overline{b_1 z} + \overline{\sum_{m=2}^{\infty} |b_m| e^{i\delta_m} z^m}\,,$$

where the coefficients satisfy the inequality (10). Set

$$h_1(z) = z\,, \quad g_1(z) = b_1 z\,, \quad h_m(z) = z + \lambda_m e^{i\eta_m} z^m\,, \quad g_m(z) = b_1 z + \mu_m e^{i\delta_m} z^m$$

for $m = 2,3,\ldots$. Writing $X_m = \frac{|a_m|}{\lambda_m}$, $Y_m = \frac{|b_m|}{\mu_m}$, $m = 2,3,\ldots$ and $X_1 = 1-\sum_{m=2}^{\infty} X_m$; $Y_1 = 1-\sum_{m=2}^{\infty} Y_m$, we get

$$f(z) = \sum_{m=1}^{\infty} \big(X_m h_m(z) + Y_m g_m(z)\big)\,.$$

In particular, putting

$$f_1(z) = z + \overline{b_1 z} \quad \text{and} \quad f_m(z) = z + \lambda_m x z^m + \overline{b_1 z} + \overline{\mu_m y z^m}\,,$$
$$(m \geq 2, |x|+|y| = 1-|b_1|)$$

we see that extreme points of clco$V_{\mathcal{H}}\big([\alpha_1],\gamma\big) \subset \big\{f_m(z)\big\}$.

To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z) = \frac{1}{2}\{f_1(z) + \lambda_2(1-|b_1|)z^2\} + \frac{1}{2}\{f_1(z) - \lambda_2(1-|b_1|)z^2\}\,,$$

a convex linear combination of functions in clco $V_{\mathcal{H}}\big([\alpha_1],\gamma\big)$.

To see that f_m is not an extreme point if both $|x| \neq 0$ and $|y| \neq 0$, we will show that it can then also be expressed as a convex linear combinations of functions in clco $V_{\mathcal{H}}\big([\alpha_1],\gamma\big)$. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1+\epsilon$ and $B = 1-\left|\frac{\epsilon x}{y}\right|$. We then see that both

$$t_1(z) = z + \lambda_m A x z^m + \overline{b_1 z + \mu_m y B z^m}$$

and

$$t_2(z) = z + \lambda_m(2-A)x z^m + \overline{b_1 z + \mu_m y(2-B) z^m}$$

are in clco $V_{\mathcal{H}}\big([\alpha_1],\gamma\big)$ and that

$$f_m(z) = \frac{1}{2}\big\{t_1(z) + t_2(z)\big\}\,.$$

The extremal coefficient bounds show that functions of the form (14) are the extreme points for *clco* $V_{\mathcal{H}}([\alpha_1],\gamma)$, and so the proof is complete. ◻

4. Inclusion relation

Following Avici and Zlotkiewicz [1] (see also Ruscheweyh [11]), we refer to the δ-neighborhood of the function $f(z)$ defined by (2) to be the set of functions F for which

$$N_\delta(f) = \Big\{F(z) = z + \sum_{m=2}^{\infty} A_m z^m + \sum_{m=1}^{\infty} \overline{B_m z^m}, \tag{15}$$
$$\sum_{m=2}^{\infty} m\big(|a_m - A_m| + |b_m - B_m| + |b_1 - B_1|\big) \le \delta\Big\}.$$

In our case, let us define the generalized δ-neighborhood of f to be the set

$$N_\delta(f) = \Big\{F : \sum_{m=2}^{\infty} \Gamma(\alpha_1, m)\big[(2m-1-\gamma)(|a_m - A_m| + (2m+1+\gamma))|b_m - B_m|\big]$$
$$+ (1-\gamma)|b_1 - B_1| \le (1-\gamma)\delta\Big\}. \tag{16}$$

Theorem 5. *Let f be given by* (2)*. If f satisfies the conditions*

$$\sum_{m=2}^{\infty} m(2m-1-\gamma)|a_m|\Gamma(\alpha_1, m) + \sum_{m=1}^{\infty} m(2m+1+\gamma)|b_m|\Gamma(\alpha_1, m) \le (1-\gamma)\,, \tag{17}$$

where $0 \le \gamma < 1$ and

$$\delta = \frac{1-\gamma}{3-\gamma}\Big(1 - \frac{3+\gamma}{1-\gamma}|b_1|\Big), \tag{18}$$

then $N(f) \subset G_{\mathcal{H}}\big([\alpha_1], \gamma\big)$.

Proof. Let f satisfy (17) and $F(z)$ be given by

$$F(z) = z + \overline{B_1 z} + \sum_{m=2}^{\infty} \big(A_m z^m + \overline{B_m z^m}\big)$$

which belongs to $N(f)$. We obtain

$$\begin{aligned}
(3+\gamma)|B_1| &+ \sum_{m=2}^{\infty} ((2m-1-\gamma)|A_m| + (2m+1+\gamma)|B_m|)\,\Gamma(\alpha_1, m) \\
&\le (3+\gamma)|B_1 - b_1| + (3+\gamma)|b_1| \\
&\quad + \sum_{m=2}^{\infty} \Gamma(\alpha_1, m)\,[(2m-1-\gamma)|A_m - a_m| + (2m+1+\gamma)|B_m - b_m|] \\
&\quad + \sum_{m=2}^{\infty} \Gamma(\alpha_1, m)\,[(2m-1-\gamma)|a_m| + (2m+1+\gamma)|b_m|]
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\gamma)\delta + (3+\gamma)|b_1| \\
&\quad + \frac{1}{3-\gamma}\sum_{m=2}^{\infty} m\Gamma(\alpha_1, m)\left((2m-1-\gamma)|a_m| + (2m+1+\gamma)|b_m|\right) \\
&\leq (1-\gamma)\delta + (3+\gamma)|b_1| + \frac{1}{3-\gamma}\left[(1-\gamma) - (3+\gamma)|b_1|\right] \leq 1-\gamma\,.
\end{aligned}$$

Hence for $\delta = \frac{1-\gamma}{3-\gamma}\bigl(1 - \frac{3+\gamma}{1-\gamma}|b_1|\bigr)$, we infer that $F(z) \in G_{\mathcal{H}}\bigl([\alpha_1], \gamma\bigr)$ which concludes the proof of Theorem 5. □

Acknowledgement. The authors would like to thank the referee for his suggestions.

References

[1] Avici, Y., Zlotkiewicz, E., *On harmonic univalent mappings*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **44** (1990), 1–7.

[2] Carlson, B. C., Shaffer, B., *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15** (2002), 737 – 745.

[3] Clunie, J., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Math. **9** (1984), 3–25.

[4] Dziok, J., Srivastava, H. M., *Certain subclasses of analytic functions associated with the generalized hypergeometric function*, Integral Transform. Spec. Funct. **14** (2003), 7–18.

[5] Goodman, A. W., *On uniformly convex functions*, Ann. Polon. Math. **1991** (56), 87–92.

[6] Jahangiri, J. M., Silverman, H., *Harmonic univalent functions with varying rrguments*, Int. J. Appl. Math. **8** (3) (2002), 267–275.

[7] Murugusundaramoorthy, G., *A class of Ruscheweyh-Type harmonic univalent functions with varying arguments*, Southwest J. Pure Appl. Math. **2** (2003), 90–95.

[8] Rønning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), 189–196.

[9] Rosy, T., Stephen, B. A., Subramanian, K. G., Jahangiri, J. M., *Goodman-Ronning type harmonic univalent functions*, Kyungpook Math. J. **41** (2001), 45–54.

[10] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[11] Ruscheweyh, S., *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81** (1981), 521–528.

[12] Srivastava, H. M., Owa, S., *Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators and certain subclasses of analytic functions*, Nagoya Math. J. **106** (1998), 1–28.

[13] Vijaya, K., *Studies on certain subclasses of harmonic functions*, Ph.D. thesis, VIT University, 2006.

[1] School of Science and Humanities, VIT University
Vellore - 632014, India
E-mail: gmsmoorthy@yahoo.com kvavit@yahoo.co.in

[2] 10/11, Ganpati Vihar, Opposite Sector 5
Udaipur 313001, Rajasthan, India
E-mail: rkraina_7@hotmail.com