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SEVERAL q-SERIES IDENTITIES FROM THE EULER EXPANSIONS OF $(a;q)_\infty$ AND $\frac{1}{(a;q)_\infty}$

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Abstract. In this paper, we first give several operator identities which extend the results of Chen and Liu, then make use of them to two q-series identities obtained by the Euler expansions of $(a;q)_\infty$ and $\frac{1}{(a;q)_\infty}$. Several q-series identities are obtained involving a q-series identity in Ramanujan's Lost Notebook.

1. Introduction

Throughout this paper, assume that $0<q<1$ and adopt the customary notation in [6] for q-series. Let

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)\,. \tag{1}$$

For any integer n, the q-shifted factorial $(a;q)_n$ is given by

$$(a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}\,. \tag{2}$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}\,. \tag{3}$$

We will also use frequently the following equations:

$$(q/a;q)_n=(-a)^{-n}q^{\binom{n+1}{2}}\frac{(q^{-n}a;q)_\infty}{(a;q)_\infty} \tag{4}$$

or

$$(q^{-n}a;q)_\infty=(-a)^nq^{-\binom{n+1}{2}}(q/a;q)_n(a;q)_\infty\,. \tag{5}$$

For convenience, we employ the following notation for multiple q-shifted factorial:

$$(a_1,a_2,\dots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\dots(a_m;q)_n\,,$$

where n is an integer or ∞.

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The q-differential operator D_q and the q-shifted operator are defined by

$$D_q\{f(a)\} = \frac{1}{a}(f(a) - f(aq))$$

and

$$\eta\{f(a)\} = f(aq)\,,$$

respectively. These two operators have been introduced in [9, 10, 11] due to Rogers, who applied them for proving Roger-Ramanujan identities. Built on the D_q and η, the operator appeared in the work of Roman [12] and will be denoted by

$$\theta = \eta^{-1} D_q\,.$$

Two operators introduced in the papers [4] and [5] by Chen and Liu are the exponential operators constructed from D_q and θ:

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}$$

and

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{\binom{n}{2}}}{(q;q)_n}\,.$$

Then the following operator identities were obtained.

Theorem 1.1 (Chen and Liu, [4] and [5])**.** *Let $T(bD_q)$ and $E(b\theta)$ be respectively defined as above. Then*

$$T(bD_q)\Big\{\frac{1}{(at;q)_\infty}\Big\} = \frac{1}{(at,bt;q)_\infty}\,, \tag{6}$$

$$T(bD_q)\Big\{\frac{1}{(as,at;q)_\infty}\Big\} = \frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}\,, \tag{7}$$

$$E(b\theta)\big\{(at;q)_\infty\big\} = (at,bt;q)_\infty\,, \tag{8}$$

$$E(b\theta)\big\{(as,at;q)_\infty\big\} = \frac{(as,at,bs,bt;q)_\infty}{(abst/q;q)_\infty}\,. \tag{9}$$

By these operator identities, in [4, 5, 7, 8, 13], a lot of q-series identities can be derived. In [14], Theorem 1.1 was extended. Applying generalized operator identities to some terminating basic hypergeometric series and q-integrals, we obtain some summation formulas involving ${}_3\phi_2$ summation and some q-integrals involving ${}_3\phi_2$ summation which extend some famous q-integrals, such as the Ismail-Stanton-Viennot integral, Gasper integral formula.

In this paper, we first give several operator identities which extend the results of Chen and Liu. Then making use of them to two q-series identities obtained by the Euler expansions of $(a;q)_\infty$ and $\frac{1}{(a;q)_\infty}$, several q-series identities are obtained involving a q-series identity in Ramanujan's Lost Notebook.

Throughout this paper, n and k are two nonnegative integers.

2. Several operator identities

First of all, by induction we give a result which is to be used later.

Lemma 2.1. *We have*

$$\theta^n\{a^k\} = \begin{cases} 0\,, & \text{if} \quad n > k\,, \\ (-1)^n q^n (q^{-k};q)_n a^{k-n}\,, & \text{if} \quad n \le k\,, \end{cases} \tag{10}$$

$$D_q^n\{a^k\} = \begin{cases} 0\,, & \text{if} \quad n > k\,, \\ (q^{k-n+1};q)_n a^{k-n}\,, & \text{if} \quad n \le k\,, \end{cases} \tag{11}$$

$$\theta^n\{a^{-k}\} = (-q)^n (q^k;q)_n a^{-(k+n)}\,, \tag{12}$$

$$D_q^n\{a^{-k}\} = q^{-\binom{n}{2}-kn}(q^k;q)_n a^{-(k+n)}\,. \tag{13}$$

In [13], we obtained the following operator identity.

$$\begin{aligned} &E(d\theta)\big\{a^k(at,as;q)_\infty\big\} \\ &\quad = a^k \frac{(as,at,ds,dt;q)_\infty}{(adst/q)_\infty}\, {}_3\phi_2\left(\begin{matrix} q^{-k}, & q/as, & q/at \\ & 0, & q^2/adst \end{matrix}; q, q\right)\,. \end{aligned} \tag{14}$$

This identity can be rewritten by

$$\begin{aligned} E(d\theta)\big\{a^k(at,as;q)_\infty\big\} &= a^k \frac{(at,as,dt,ds;q)_\infty}{(adst/q;q)_\infty} \\ &\times \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} \frac{(q/at,q/as;q)_j}{(q^2/adst;q)_j} q^{\binom{j+1}{2}-kj}\,. \end{aligned} \tag{15}$$

Next we give three operator identities which are extensions of Theorem 1.1. They specialize to Theorem 1.1 by setting $k=0$.

Theorem 2.2. *We have*

$$T(dD_q)\Big\{\frac{a^k}{(at,as;q)_\infty}\Big\} = a^k \frac{(adst;q)_\infty}{(at,as,dt,ds;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(at,as;q)_j}{(adst;q)_j} \Big(\frac{d}{a}\Big)^j\,. \tag{16}$$

$$\begin{aligned} &E(d\theta)\big\{a^{-k}(at,as;q)_\infty\big\} \\ &\quad = a^{-k} \frac{(at,as,dtq^k,dsq^k;q)_\infty}{(adstq^{k-1};q)_\infty} \sum_{n=0}^{\infty} \frac{(q^k,adstq^{k-1};q)_n}{(q,dtq^k,dsq^k;q)_n} q^{\binom{n+1}{2}} \Big(-\frac{d}{a}\Big)^n \\ &\quad = a^{-k} \frac{(at,as,dt,ds;q)_\infty}{(adst/q;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^k;q)_n}{(q;q)_n} \frac{(adst/q;q)_{n+k}}{(dt,ds;q)_{n+k}} q^{\binom{n+1}{2}} \Big(-\frac{d}{a}\Big)^n \end{aligned} \tag{17}$$

where $|d/a| < 1$, *and*

$$T(dD_q)\Big\{\frac{a^{-k}}{(at,as;q)_\infty}\Big\}$$

$$= a^{-k}\frac{(adtsq^{-k};q)_\infty}{(at,as,dtq^{-k},dsq^{-k};q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{(q^k,q^{1+k}/adst;q)_n}{(q,q^{1+k}/dt,q^{1+k}/ds;q)_n}q^n$$

$$= \Big(-\frac{1}{d}\Big)^k q^{\binom{k+1}{2}}\frac{(adts;q)_\infty}{(at,as,dt,ds;q)_\infty}\sum_{n=0}^{\infty}\frac{(q^k;q)_n}{(q;q)_n}\frac{(q/adst;q)_{n+k}}{(q/dt,q/ds;q)_{n+k}}(-q)^n\,. \tag{18}$$

Proof. We note that the rule of Leibniz for D_q^n (see [4])

$$D_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)}\begin{bmatrix}n\\k\end{bmatrix}D_q^k\{f(a)\}D_q^{n-k}\{g(aq^k)\}\,. \tag{19}$$

Therefore, we have

$$\begin{aligned}
T(dD_q)\Big\{\frac{a^k}{(at,as;q)_\infty}\Big\} &= \sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}D_q^n\Big\{\frac{a^k}{(at,as;q)_\infty}\Big\}\\
&= \sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}\sum_{j=0}^{n}q^{j(j-n)}\begin{bmatrix}n\\j\end{bmatrix}D_q^j\{a^k\}D_q^{n-j}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}\\
&= \sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}\sum_{j=0}^{n}q^{j(j-n)}\begin{bmatrix}n\\j\end{bmatrix}\frac{(q;q)_k}{(q;q)_{k-j}}a^{k-j}D_q^{n-j}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}\\
&= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\frac{d^n}{(q;q)_n}q^{j(j-n)}\begin{bmatrix}n\\j\end{bmatrix}\frac{(q;q)_k}{(q;q)_{k-j}}a^{k-j}D_q^{n-j}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}\\
&= \sum_{j=0}^{\infty}\frac{d^j(q;q)_k}{(q;q)_j(q;q)_{k-j}}a^{k-j}\sum_{n=0}^{\infty}\frac{d^n}{(q;q)_n}q^{-nj}D_q^n\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}\\
&= a^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\Big(\frac{d}{a}\Big)^j\sum_{n=0}^{\infty}\frac{(dq^{-j}D_q)^n}{(q;q)_n}\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}\\
&= a^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\Big(\frac{d}{a}\Big)^j\cdot T(dq^{-j}D_q\Big\{\frac{1}{(atq^j,asq^j;q)_\infty}\Big\}\\
&= a^k\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\Big(\frac{d}{a}\Big)^j\frac{(adstq^j;q)_\infty}{(atq^j,asq^j,dt,ds;q)_\infty}\\
&= a^k\frac{(adst;q)_\infty}{(at,as,dt,ds;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(at,as;q)_j}{(adst;q)_j}\Big(\frac{d}{a}\Big)^j\,.
\end{aligned}$$

We obtain (16). (17) and (18) can be proved similarly. The proof is completed. □

Taking $s = 0$ in Theorem 2.2, we have

Corollary 2.3.

$$T(dD_q)\Big\{\frac{a^k}{(at;q)_\infty}\Big\} = a^k \frac{1}{(at,dt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(at;q)_j\Big(\frac{d}{a}\Big)^j . \tag{20}$$

$$\begin{aligned} E(d\theta)\{a^{-k}(at;q)_\infty\} &= a^{-k}(at,dtq^k;q)_\infty \\ &\quad\times\sum_{n=0}^{\infty}\frac{(q^k;q)_n}{(q;q)_n}\frac{1}{(dt;q)_{n+k}}q^{\binom{n+1}{2}}\Big(-\frac{d}{a}\Big)^n \end{aligned}\tag{21}$$

where $|d/a| < 1$, *and*

$$\begin{aligned} T(dD_q)\Big\{\frac{a^{-k}}{(at;q)_\infty}\Big\} &= \Big(-\frac{1}{adt}\Big)^k q^{\binom{k+1}{2}}\frac{1}{(at,dt;q)_\infty} \\ &\quad\times\sum_{n=0}^{\infty}\frac{(q^k;q)_n}{(q;q)_n}\frac{1}{(q/dt;q)_{n+k}}\Big(-\frac{q}{at}\Big)^n , \end{aligned}\tag{22}$$

where $|q/at| < 1$.

Further, setting $k = 1$ or $t = 0$ in (21), we get the following results of Chen and Liu.

Corollary 2.4 (Chen and Liu, [5])**.**

$$E(b\theta)\{a^{-1}\} = a^{-1}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}} b^n a^{-n} , \tag{23}$$

$$E(b\theta)\{a^{-1}(-a;q)_\infty\} = a^{-1}(-a;q)_\infty\sum_{m=0}^{\infty}(-b/a)^m q^{\binom{m+1}{2}}(-bq^{m+1};q)_\infty . \tag{24}$$

3. Several q-series identities from expansion of $(a;q)_\infty$

From the Euler expansion of $(a;q)_\infty$ (see [3]):

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} a^n}{(q;q)_n} = (a;q)_\infty ,$$

the following identity can be verified.

$$\sum_{n=0}^{\infty}(-aq^{n+1};q)_\infty q^n = -a^{-1} + a^{-1}(-a;q)_\infty . \tag{25}$$

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq,-dq;q)_n}q^n\sum_{j=0}^{k+1}\frac{(q^{-(k+1)},-q^{-n}/a,q/at;q)_j}{(q,-q^{1-n}/adt;q)_j}q^j$$
$$=-\,a^{-1}\frac{(-adt;q)_\infty}{(-aq,-dq;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},q/at;q)_j}{(q;q)_j}(dt)^j$$
$$+\,a^{-1}\frac{(1+a)(1+d)}{1+adt/q}\sum_{j=0}^{k}\frac{(q^{-k},-q/a,q/at;q)_j}{(q,-q^2/adt;q)_j}q^j\,. \tag{26}$$

Proof. Multiply both sides in equation (25) by $a^{k+1}(at;q)_\infty$. Then

$$\sum_{n=0}^{\infty}a^{k+1}(-aq^{n+1},at;q)_\infty q^n=-a^k(at;q)_\infty+a^k(-a,at;q)_\infty\,. \tag{27}$$

Apply the operator $E(d\theta)$ to its both sides with respect to the variable a. Then

$$\sum_{n=0}^{\infty}q^nE(d\theta)\big\{a^{k+1}(-aq^{n+1},at;q)_\infty\big\}=-E(d\theta)\big\{a^k(at;q)_\infty\big\}$$
$$+\,E(d\theta)\big\{a^k(-a,at;q)_\infty\big\}\,. \tag{28}$$

Applying Theorem 2.2 and Corollary 2.3, we have

$$E(d\theta)\big\{a^{k+1}(-aq^{n+1},at;q)_\infty\big\}=a^{k+1}\frac{(-aq^{n+1},at,-dq^{n+1},dt;q)_\infty}{(-adtq^n;q)_\infty}$$
$$\times\sum_{j=0}^{k+1}\frac{(q^{-(k+1)},-q^{-n}/a,q/at;q)_j}{(q,-q^{1-n}/adt;q)_j}q^j\,,$$

$$E(d\theta)\big\{a^k(at;q)_\infty\big\}=a^k(at,dt;q)_\infty\sum_{j=0}^{k}\frac{(q^{-k},q/at;q)_j}{(q;q)_j}(dt)^j\,,$$

$$E(d\theta)\big\{a^k(-a,at;q)_\infty\big\}=a^k\frac{(-a,at,-d,dt;q)_\infty}{(-adt/q;q)_\infty}\sum_{j=0}^{k}\frac{(q^{-k},-q/a,q/at;q)_j}{(q,-q^2/adt;q)_j}q^j\,.$$

Substituting these three identities into (27) and then using (4), we obtain (3.1) after simplifying. This completes the proof. □

Taking $k=0$ in Theorem 3.1, we have

Corollary 3.2.

$$a\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq,-dq;q)_n}q^n+d(1-at/q)\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq;q)_n(-dq;q)_{n+1}}q^{n+1}$$
$$=-\,\frac{(-adt;q)_\infty}{(-aq,-dq;q)_\infty}+1+a\,. \tag{29}$$

Taking $t=0$ in Corollary 3.2 we have

Corollary 3.3.

$$
\begin{aligned}
&a\sum_{n=0}^{\infty}\frac{1}{(-aq,-dq;q)_n}q^n+d\sum_{n=0}^{\infty}\frac{1}{(-aq;q)_n(-dq;q)_{n+1}}q^{n+1}\\
&\qquad=-\frac{1}{(-aq,-dq;q)_\infty}+1+a\,. \qquad (30)
\end{aligned}
$$

Taking $k=1$ in Theorem 3.1, we have

Corollary 3.4.

$$
\begin{aligned}
&a^2q\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq,-dq;q)_n}q^n-ad(1+q)(at-q)\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq;q)_n(-dq;q)_{n+1}}q^n\\
&\quad+d^2(at-q)(at-q^2)\sum_{n=0}^{\infty}\frac{(-adt;q)_n}{(-aq;q)_n(-dq;q)_{n+2}}q^n\\
&=\frac{(1+a)(aq+dq+adq^2-adt)}{1+dq}-(aq+dq-adt)\frac{(-adt;q)_\infty}{(-aq,-dq;q)_\infty}\,. \qquad (31)
\end{aligned}
$$

Taking $t=0$ in Corollary 3.4, we have

Corollary 3.5.

$$
\begin{aligned}
&a^2\sum_{n=0}^{\infty}\frac{1}{(-aq,-dq;q)_n}q^n+ad(1+q)\sum_{n=0}^{\infty}\frac{1}{(-aq;q)_n(-dq;q)_{n+1}}q^n\\
&\quad+d^2q^2\sum_{n=0}^{\infty}\frac{1}{(-aq;q)_n(-dq;q)_{n+2}}q^n\\
&=\frac{(1+a)(a+d+adq)}{1+dq}-(a+d)\frac{1}{(-aq,-dq;q)_\infty}\,. \qquad (32)
\end{aligned}
$$

Theorem 3.6. *We have*

$$
\begin{aligned}
&(1-dtq^k)\sum_{n=0}^{\infty}\frac{(-adtq^k;q)_n}{(-aq,-dq^{k+1};q)_n}q^n\sum_{i=0}^{\infty}\frac{(q^k,-adtq^{n+k};q)_i}{(q,-dq^{n+k+1},dtq^k;q)_i}q^{\binom{i+1}{2}}\Big(-\frac{d}{a}\Big)^i\\
&\quad=-a^{-1}\frac{(-adtq^k;q)_\infty}{(-aq,-dq^{k+1};q)_\infty}\sum_{i=0}^{\infty}\frac{(q^{k+1};q)_i}{(q,dtq^{k+1};q)_i}q^{\binom{i+1}{2}}\Big(-\frac{d}{a}\Big)^i\\
&\qquad+(1+a^{-1})\sum_{i=0}^{\infty}\frac{(q^{k+1},-adtq^k;q)_i}{(q,-dq^{k+1},dtq^{k+1};q)_i}q^{\binom{i+1}{2}}\Big(-\frac{d}{a}\Big)^i, \qquad (33)
\end{aligned}
$$

where $|d/a|<1$.

Proof. (25) can be rewritten by

$$
\sum_{n=0}^{\infty}q^na^{-k}(-aq^{n+1},at;q)_\infty=-a^{-(k+1)}(at;q)_\infty+a^{-(k+1)}(-a,at;q)_\infty \qquad (34)
$$

Apply the operator $E(d\theta)$ to its both sides with respect to the variable a. Then

$$\sum_{n=0}^{\infty} q^n E(d\theta)\{a^{-k}(-aq^{n+1}, at; q)_\infty\} = -E(d\theta)\{a^{-(k+1)}(at; q)_\infty\} \\ + E(d\theta)\{a^{-(k+1)}(-a, at; q)_\infty\}.$$

By Theorem 2.2 and Corollary 2.3, we have

$$E(d\theta)\{a^{-k}(-aq^{n+1}, at; q)_\infty\} = a^{-k}\frac{(-aq^{n+1}, at, -dq^{n+k+1}, dtq^k; q)_\infty}{(-adtq^{n+k}; q)_\infty} \\ \times \sum_{i=0}^{\infty} \frac{(q^k, -adtq^{n+k}; q)_i}{(q, -dq^{n+k+1}, dtq^k; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i,$$

$$E(d\theta)\{a^{-(k+1)}(at; q)_\infty\} = a^{-(k+1)}(at, dtq^{k+1}; q)_\infty \\ \times \sum_{i=0}^{\infty} \frac{(q^{k+1}; q)_i}{(q, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i,$$

$$E(d\theta)\{a^{-(k+1)}(-a, at; q)_\infty\} = a^{-(k+1)}\frac{(-a, at, -dq^{k+1}, dtq^{k+1}; q)_\infty}{(-adtq^k; q)_\infty} \\ \times \sum_{i=0}^{k} \frac{(q^{k+1}, -adtq^k; q)_i}{(q, -dq^{k+1}, dtq^{k+1}; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i.$$

Substituting these three identities into (34) and then using (4), we obtain the theorem. □

Taking $k = 0$ in Theorem 3.6 we have

Corollary 3.7.

$$(1 - dt)\sum_{n=0}^{\infty} \frac{(-adt; q)_n}{(-aq, -dq; q)_n} q^n = -a^{-1}\frac{(-adt; q)_\infty}{(-aq, -dq; q)_\infty} \sum_{i=0}^{\infty} \frac{1}{(dtq; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i \\ + (1 + a^{-1})\sum_{i=0}^{\infty} \frac{(-adt; q)_i}{(-dq, dtq; q)_i} q^{\binom{i+1}{2}} \left(-\frac{d}{a}\right)^i, \tag{35}$$

where $|d/a| < 1$.

Taking $t = 0$ in Corollary 3.7 we obtain the following Ramanujan's identity.

Corollary 3.8.

$$\sum_{n=0}^{\infty} \frac{1}{(-aq, -dq; q)_n} q^n = -a^{-1}\frac{1}{(-aq, -dq; q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} (d/a)^n \\ + (1 + a^{-1})\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+1}{2}} (d/a)^n}{(-dq; q)_n}, \tag{36}$$

where $|d/a| < 1$.

Note. This identity is a formula in Ramanujan's Lost Notebook, and its proofs are given by Andrews [1, 2]. In [5], Chen and Liu gave also a simple proof by using the method of the operator identity.

4. Several q-series identities from expansion of $\frac{1}{(a;q)_\infty}$

From the Euler expansion of $\frac{1}{(a;q)_\infty}$ (see [3]):

$$\sum_{n=0}^{\infty} \frac{1}{(q;q)_n} a^n = \frac{1}{(a;q)_\infty},$$

the following identity can be verified.

$$\sum_{n=0}^{\infty} \frac{1}{(aq^n;q)_\infty} q^n = a^{-1} \frac{1}{(a;q)_\infty} - a^{-1}. \tag{37}$$

From this identity, by the operator $T(dD_q$ we can obtain the following results.

Theorem 4.1. *We have*

$$\sum_{n=0}^{\infty} q^n (d;q)_n \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} (at;q)_j \frac{(a;q)_{n+j}}{(adt;q)_{n+j}} \left(\frac{d}{a}\right)^j = a^{-1} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a,at;q)_j}{(adt;q)_j} \left(\frac{d}{a}\right)^j$$
$$- a^{-1} \frac{(a,d;q)_\infty}{(adt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (at;q)_j \left(\frac{d}{a}\right)^j. \tag{38}$$

Proof. By (37), we have

$$\sum_{n=0}^{\infty} \frac{a^{k+1}}{(aq^n, at;q)_\infty} q^n = \frac{a^k}{(a,at;q)_\infty} - \frac{a^k}{(at;q)_\infty}. \tag{39}$$

Apply the operator $T(dD_q)$ to its both sides with respect to the variable a. Then

$$\sum_{n=0}^{\infty} q^n \cdot T(dD_q) \Big\{ \frac{a^{k+1}}{(aq^n, at;q)_\infty} \Big\} = T(dD_q) \Big\{ \frac{a^k}{(a,at;q)_\infty} \Big\}$$
$$- T(dD_q) \Big\{ \frac{a^k}{(at;q)_\infty} \Big\}. \tag{40}$$

By Theorem 2.2 and Corollary 2.3, we have

$$T(dD_q) \Big\{ \frac{a^{k+1}}{(aq^n, at;q)_\infty} \Big\} = a^{k+1} \frac{(adtq^n;q)_\infty}{(aq^n, at, dq^n, dt;q)_\infty}$$
$$\times \sum_{j=0}^{k+1} \begin{bmatrix} k+1 \\ j \end{bmatrix} \frac{(aq^n, at;q)_j}{(adtq^n;q)_j} \left(\frac{d}{a}\right)^j, \tag{41}$$

$$T(dD_q) \Big\{ \frac{a^k}{(a,at;q)_\infty} \Big\} = a^k \frac{(adt;q)_\infty}{(a,at,d,dt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a,at;q)_j}{(adt;q)_j} \left(\frac{d}{a}\right)^j \tag{42}$$

and

$$T(dD_q)\Big\{\frac{a^k}{(at;q)_\infty}\Big\} = a^k \frac{1}{(at,dt;q)_\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}(at;q)_j\Big(\frac{d}{a}\Big)^j . \tag{43}$$

Substituting these three identities into (40) and then using (4), we obtain the theorem. □

Taking $k=0$ in Theorem 4.1 we have

Corollary 4.2.

$$a\sum_{n=0}^{\infty}\frac{(a,d;q)_n}{(adt;q)_n}q^n + d(1-at)\sum_{n=0}^{\infty}(d;q)_n\frac{(a;q)_{n+1}}{(adt;q)_{n+1}}q^n = 1-\frac{(a,d;q)_\infty}{(adt;q)_\infty} . \tag{44}$$

Specially, taking $t=0$ we have

Corollary 4.3.

$$a\sum_{n=0}^{\infty}(a,d;q)_nq^n + d\sum_{n=0}^{\infty}(d;q)_n(a;q)_{n+1}q^n = 1-(a,d;q)_\infty . \tag{45}$$

Taking $k=1$ in Theorem 4.1 we have

Corollary 4.4.

$$\begin{aligned}a^2\sum_{n=0}^{\infty}\frac{(a,d;q)_n}{(adt;q)_n}q^n &+ ad(1+q)(1-at)\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}(d;q)_n}{(adt;q)_{n+1}}q^n\\ &+ d^2(1-at)(1-atq)\sum_{n=0}^{\infty}\frac{(a;q)_{n+2}(d;q)_n}{(adt;q)_{n+2}}q^n\\ &= a+d\frac{(1-a)(1-at)}{1-adt}-\frac{(a,d;q)_\infty}{(adt;q)_\infty}(a+d-adt) .\end{aligned} \tag{46}$$

Specially, taking $t=0$ we have

Corollary 4.5.

$$\begin{aligned}a^2\sum_{n=0}^{\infty}(a,d;q)_nq^n + ad(1+q)\sum_{n=0}^{\infty}(a;q)_{n+1}(d;q)_nq^n + d^2\sum_{n=0}^{\infty}(a;q)_{n+2}(d;q)_nq^n\\ = (a+d)(1-(a,d;q)_\infty)-ad .\end{aligned} \tag{47}$$

Theorem 4.6. *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}q^n\frac{(a,d;q)_n}{(adt;q)_n}\sum_{j=0}^{\infty}\frac{(q^k;q)_j}{(q;q)_j}\frac{(q^{1-n}/adt;q)_{k+j}}{(q/dt,q^{1-n}/d;q)_{k+j}}(-q)^j\\ = -\frac{q^{k+1}}{d}\sum_{j=0}^{\infty}\frac{(q^{k+1};q)_j}{(q;q)_j}\frac{(q/adt;q)_{k+1+j}}{(q/d,q/dt;q)_{k+1+j}}(-q)^j\\ + \frac{1}{d}\frac{(a,d;q)_\infty}{(adt;q)_\infty}\sum_{j=0}^{\infty}\frac{(q^{k+1};q)_j}{(q;q)_j}\frac{(-1)^j}{(q/dt;q)_{k+1+j}}(q/at)^{k+1+j} ,\end{aligned} \tag{48}$$

where $|q/at| < 1$.

Proof. We rewrite (37) into the following form:

$$\sum_{n=0}^{\infty} \frac{a^{-k}}{(aq^n, at; q)_\infty} q^n = \frac{a^{-(k+1)}}{(a, at; q)_\infty} - \frac{a^{-(k+1)}}{(at; q)_\infty}. \tag{49}$$

Apply the operator $T(dD_q)$ to its both sides with respect to the variable a. Then

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \cdot T(dD_q)\Big\{\frac{a^{-k}}{(aq^n, at; q)_\infty}\Big\} &= T(dD_q)\Big\{\frac{a^{-(k+1)}}{(a, at; q)_\infty}\Big\} \\ &- T(dD_q)\Big\{\frac{a^{-(k+1)}}{(at; q)_\infty}\Big\}.\end{aligned} \tag{50}$$

By Theorem 2.2 and Corollary 2.3, we have

$$\begin{aligned}T(dD_q)\Big\{\frac{a^{-k}}{(aq^n, at; q)_\infty}\Big\} &= (-d)^{-k} q^{\binom{k+1}{2}} \frac{(adtq^n; q)_\infty}{(aq^n, at, dq^n, dt; q)_\infty} \\ &\times \sum_{j=0}^{\infty} \frac{(q^k; q)_j}{(q; q)_j} \frac{(q^{1-n}/adt; q)_{k+j}}{(q/dt, q^{1-n}/d; q)_{k+j}} (-q)^j,\end{aligned} \tag{51}$$

$$\begin{aligned}T(dD_q)\Big\{\frac{a^{-(k+1)}}{(a, at; q)_\infty}\Big\} &= (-d)^{-(k+1)} q^{\binom{k+2}{2}} \frac{(adt; q)_\infty}{(a, at, d, dt; q)_\infty} \\ &\times \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{(q/adt; q)_{k+j+1}}{(q/d, q/dt; q)_{k+j+1}} (-q)^j\end{aligned} \tag{52}$$

and

$$\begin{aligned}T(dD_q)\Big\{\frac{a^{-(k+1)}}{(at; q)_\infty}\Big\} &= \Big(-\frac{1}{adt}\Big)^{-(k+1)} q^{\binom{k+2}{2}} \frac{1}{(at, dt; q)_\infty} \\ &\times \sum_{j=0}^{\infty} \frac{(q^{k+1}; q)_j}{(q; q)_j} \frac{1}{(q/dt; q)_{k+j+1}} \Big(-\frac{q}{at}\Big)^j.\end{aligned} \tag{53}$$

Substituting these three identities into (50) and then using (4), we obtain the theorem. □

Taking $k = 0$ in Theorem 4.6 we have

Corollary 4.7.

$$\begin{aligned}\sum_{n=0}^{\infty} q^n \frac{(a, d; q)_n}{(adt; q)_n} &= -\frac{q}{d} \sum_{j=0}^{\infty} \frac{(q/adt; q)_{j+1}}{(q/d, q/dt; q)_{j+1}} (-q)^j \\ &+ \frac{1}{d} \frac{(a, d; q)_\infty}{(adt; q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{(q/dt; q)_{j+1}} \Big(\frac{q}{at}\Big)^{j+1},\end{aligned} \tag{54}$$

where $|q/at| < 1$.

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