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SOME RESULTS ON THE GROWTH PROPERTIES OF WRONSKIANS

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Abstract. The aim of this paper paper is to study the comparative growth properties of the composition of entire and meromorphic functions and wronskians generated by them improving some earlier results.

1. Introduction, definitions and notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromorphic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of meromorphic functions which are available in [3]. In the sequel we use the following notation:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \ \mathit{for}\ i = 1, 2, 3, \dots \quad \mathit{and} \quad \log^{[0]} x = x .$$

We recall the following definitions:

Definition 1. The order $\rho(f)$ and lower order $\mu(f)$ of a meromorphic function f are defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r} .$$

If f is entire, one can easily verify that,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \mu(f) = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} .$$

Definition 2. The hyper order $\rho_2(f)$ and hyper lower order $\mu_2(f)$ of a meromorphic function f are defined as follows:

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} \quad \text{and} \quad \mu_2(f) = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r} .$$

If f is entire, then

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} \quad \text{and} \quad \mu_2(f) = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r} .$$

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Definition 3. The type $\tau(f)$ of a meromorphic function f is defined as :

$$\tau(f) = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho(f)}}, \quad 0 < \rho(f) < \infty .$$

When f is entire, then

$$\tau(f) = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho(f)}}, \quad 0 < \rho(f) < \infty .$$

Definition 4. A function $\mu_f(r)$ is called a lower proximate order of a meromorphic function f of finite lower order $\mu(f)$ if

(i) $\mu_f(r)$ is non negative and continuous for $r \geq r_0$, say,

(ii) $\mu_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\mu'_f(r+0)$ and $\mu'_f(r-0)$ exists,

(iii) $\lim_{r\to\infty} \mu_f(r) = \mu(f)$,

(iv) $\lim_{r\to\infty} r\mu'_f(r)\log r = 0$ and

(v) $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\mu_f(r)}} = 1$.

Definition 5. Let a be a complex number, finite or infinite. The Nevanlinna deficiency and Valiron deficiency of 'a' with respect to a meromorphic function f is defined as

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)} .$$

From the second fundamental theorem it follows that the set of values of $a \in C \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) \leq 2$ (cf. [3, p. 43]). If in particular $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximal deficiency sum.

Definition 6. A meromorphic function $a = a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.

Definition 7. Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the Wronskian determinant of $a_1, a_2, \ldots, a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \ldots & a_k & f \\ a'_1 & a'_2 & \ldots & a'_k & f' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{(k)} & a_2^{(k)} & \ldots & a_k^{(k)} & f^{(k)} \end{vmatrix} .$$

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials *viz.,* the Wronskians. In the paper we prove some new results depending on the comparative growth properties of composite entire or meromorphic functions and Wronskians generated by one of the factors which improve some earlier theorems.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1]). *If f is meromorphic and g is entire then for all sufficiently large values of r*

$$T\left(f\circ g\right)\leq\left\{1+o\left(1\right)\right\}\frac{T\left(r,g\right)}{\log M\left(r,g\right)}T\left(M\left(r,g\right),f\right)\,.$$

Lemma 2 ([2]). *Let f be meromorphic and g be entire and suppose that $0<\mu\leq\rho_g\leq\infty$. Then for a sequence of values of r tending to infinity,*

$$T\left(r,f\circ g\right)\geq T\left(\exp\left(r^{\mu}\right),f\right)\,.$$

Lemma 3 ([5]). *Let f be a transcendental meromorphic function having maximal deficiency sum. Then*

$$\lim_{r\to\infty}\frac{T\left(r,L\left(f\right)\right)}{T\left(r,f\right)}=1+k-k\delta\left(\infty;f\right)\,.$$

Lemma 4. *If f be a transcendental meromorphic function with the maximal deficiency sum, then*

$$\rho\left(L\left(f\right)\right)=\rho\left(f\right)$$

and

$$\mu\left(L\left(f\right)\right)=\mu\left(f\right)\,.$$

Also

$$\tau\left(L\left(f\right)\right)=\left\{1+k-k\delta\left(\infty;f\right)\right\}\tau\left(f\right)\,.$$

Proof. By Lemma 3, $\lim\limits_{r\to\infty}\frac{\log T(r,L(f))}{\log T(r,f)}$ exists and equal to 1. So

$$\begin{aligned}\rho\left(L\left(f\right)\right)&=\limsup_{r\to\infty}\frac{\log T\left(r,f\right)}{\log r}\cdot\lim_{r\to\infty}\frac{\log T\left(r,L\left(f\right)\right)}{\log T\left(r,f\right)}\\&=\rho\left(f\right)\cdot 1=\rho\left(f\right)\,.\end{aligned}$$

In a similar manner, $\mu\left(L\left(f\right)\right)=\mu\left(f\right)$.
Again

$$\begin{aligned}\tau\left(L\left(f\right)\right)&=\limsup_{r\to\infty}\frac{T\left(r,L\left(f\right)\right)}{r^{\rho(L(f))}}\\&=\limsup_{r\to\infty}\frac{T\left(r,f\right)}{r^{\rho(f)}}\cdot\lim_{r\to\infty}\frac{T\left(r,L\left(f\right)\right)}{T\left(r,f\right)}\\&=\left\{1+k-k\delta\left(\infty;f\right)\right\}\tau\left(f\right)\,.\end{aligned}$$

This proves the lemma. □

Lemma 5. *Let f be a transcendental meromorphic function with the maximal deficiency sum. Then the hyper order and (hyper lower order) of $L(f)$ and f are equal.*

The proof of Lemma 5 is omitted because it can be carried out in the line of Lemma 4.

Lemma 6. *For a meromorphic function f of finite lower order, lower proximate order exists.*

The lemma can be proved in the line of Theorem 1 [4] and so the proof is omitted. For further reference see also [8] and [9].

Lemma 7. *Let f be a meromorphic function of finite lower order $\mu(f)$. Then for $\delta(>0)$ the function $r^{\mu(f)+\delta-\mu_f(r)}$ is ultimately an increasing function of r.*

Proof. Since

$$\frac{d}{dr}r^{\mu(f)+\delta-\mu_f(r)} = \left\{\mu(f)+\delta-\mu_f(r)-r\mu_f'(r)\log r\right\}\frac{r^{\mu(f)+\delta-\mu_f(r)}}{r} > 0$$

for all sufficiently large values of r, the lemma follows. □

Lemma 8 ([6])**.** *Let f be an entire function of finite lower order. If there exists entire functions $a_i\,(i=1,2,\ldots,n; n\leq\infty)$ satisfying $T(r,a_i)=o\{T(r,f)\}$ and $\sum\limits_{i=1}^{n}\delta(a_i;f)=1$, then $\lim\limits_{r\to\infty}\frac{T(r,f)}{\log M(r,f)}=\frac{1}{\pi}$.*

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f be a meromorphic function and g be a transcendental entire function with maximal deficiency sum. If $\mu(f)$ and $\mu(g)$ are both finite, then*

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,L(g))} \leq \frac{3\cdot\rho(f)\cdot 2^{\mu(g)}}{1+k-k\delta(\infty;g)}\,.$$

Proof. If $\rho(f)=\infty$, the theorem is obvious. So we suppose that $\rho(f)<\infty$. Since $T(r,g)\leq\log^{+}M(r,g)$, in view of Lemma 1, we get for all sufficiently large values of r,

$$\begin{aligned} T(f\circ g) &\leq \{1+o(1)\}\,T(M(r,g),f)\ , \\ \text{i.e.,}\quad \log T(r,f\circ g) &\leq \log\{1+o(1)\}+\log T(M(r,g),f) \\ &\leq o(1)+(\rho(f)+\varepsilon)\log M(r,g)\ , \end{aligned}$$

$$\text{i.e.,}\quad \liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)} \leq (\rho(f)+\varepsilon)\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}\,.$$

Since $\varepsilon\ (>0)$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log T(r,f\circ g)}{T(r,g)} \leq \rho(f)\cdot\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}\,. \tag{1}$$

As $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\mu_f(r)}} = 1$, so for given ε $(0 < \varepsilon < 1)$ we get for a sequence of values of r tending to infinity,

$$T\left(r,g\right) \leq \left(1+\varepsilon\right) r^{\mu_g(r)} \tag{2}$$

and for all sufficiently large values of r,

$$T\left(r,g\right) \geq \left(1-\varepsilon\right) r^{\mu_g(r)} \; . \tag{3}$$

Since $\log M\left(r,g\right) \leq 3T\left(2r,g\right)$, we have by (2), for a sequence of values of r tending to infinity,

$$\log M\left(r,g\right) \leq 3T\left(2r,g\right) \leq 3\left(1+\varepsilon\right)\left(2r\right)^{\mu_g(2r)} \; . \tag{4}$$

Combining (3) and (4) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log M\left(r,g\right)}{T\left(r,g\right)} \leq \frac{3\left(1+\varepsilon\right)}{\left(1-\varepsilon\right)} \cdot \frac{\left(2r\right)^{\mu_g(2r)}}{r^{\mu_g(r)}} \; .$$

Now for any δ (> 0), for a sequence of values of r tending to infinity,

$$\frac{\log M\left(r,g\right)}{T\left(r,g\right)} \leq \frac{3\left(1+\varepsilon\right)}{\left(1-\varepsilon\right)} \cdot \frac{\left(2r\right)^{\mu(g)+\delta}}{\left(2r\right)^{\mu(g)+\delta-\mu_g(2r)}} \cdot \frac{1}{r^{\mu_g(r)}}$$

$$\text{i.e.,}\ \frac{\log M\left(r,g\right)}{T\left(r,g\right)} \leq \frac{3\left(1+\varepsilon\right)}{\left(1-\varepsilon\right)} \cdot 2^{\mu(g)+\delta} \tag{5}$$

because $r^{\mu(g)+\delta-\mu_g(2r)}$ is ultimately an increasing function of r by Lemma 7. Since ε (> 0) and δ (> 0) are arbitrary, it follows from (5) that

$$\liminf_{r\to\infty} \frac{\log M\left(r,g\right)}{T\left(r,g\right)} \leq 3 \cdot 2^{\mu(g)} \; . \tag{6}$$

Thus from (1) and (6) we obtain that

$$\liminf_{r\to\infty} \frac{\log T\left(r, f\circ g\right)}{T\left(r,g\right)} \leq 3 \cdot \rho\left(f\right) \cdot 2^{\mu(g)} \; . \tag{7}$$

Now in view of Lemma 3 and (7) we get

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log T\left(r, f\circ g\right)}{T\left(r,L\left(g\right)\right)} &= \liminf_{r\to\infty} \frac{\log T\left(r, f\circ g\right)}{T\left(r,g\right)} \cdot \lim_{r\to\infty} \frac{T\left(r,g\right)}{T\left(r,L\left(g\right)\right)} \\ &\leq \frac{3\cdot\rho\left(f\right)\cdot 2^{\mu(g)}}{1+k-k\delta\left(\infty;g\right)} \; .\end{aligned}$$

This proves the theorem. □

Theorem 2. *Let f be a meromorphic function of finite order and g be transcendental entire of finite lower order with maximal deficiency sum. Then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T\left(r, f\circ g\right)}{\log T\left(r,L\left(g\right)\right)} \leq 1 \; .$$

Proof. Since $T(r,g) \le \log^+ M(r,g)$, in view of Lemma 1, we get for all sufficiently large values of r,

$$\begin{aligned} \log T(r, f\circ g) &\le \log\{1+o(1)\} + \log T(M(r,g), f) \\ &\le o(1) + (\rho(f)+\varepsilon)\log M(r,g) \end{aligned}$$

$$\text{i.e.,}\quad \log^{[2]} T(r, f\circ g) \le \log^{[2]} M(r,g) + O(1)\,. \tag{8}$$

It is well known that for any entire function g, $\log M(r,g) \le 3T(2r,g)$ (cf. [3]). Then for $0<\varepsilon<1$ and $\delta\ (>0)$, for a sequence of values of r tending to infinity it follows from (5) that

$$\log^{[2]} M(r,g) \le \log T(r,g) + O(1)\,. \tag{9}$$

Now combining (8) and (9) we obtain for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, f\circ g) \le \log T(r,g) + O(1)$$

$$\text{i.e.,}\quad \frac{\log^{[2]} T(r, f\circ g)}{\log T(r,g)} \le 1\,. \tag{10}$$

As by Lemma 3, $\lim\limits_{r\to\infty} \frac{\log T(r,g)}{\log T(r,L(g))}$ exists and is equal to 1, from (10) we obtain that

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log T(r,L(g))} &= \liminf_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log T(r,g)} \cdot \lim_{r\to\infty} \frac{\log T(r,g)}{\log T(r,L(g))} \\ &\le 1\cdot 1 = 1\,. \end{aligned}$$

Thus the theorem is established. □

Theorem 3. *Let f and g be two transcendental entire functions each having maximal deficiency sum such that $\mu(f)>0$, $\rho(g)<\mu(f)\le\rho(f)<\infty$. Also let there exist entire functions a_i $(i=1,2,\cdots,n;\, n\le\infty)$ with*

(i) *$T(r,a_i) = o\{T(r,g)\}$ as $r\to\infty$ for $i=1,2,\ldots,n$ and*

(ii) *$\sum\limits_{i=1}^{n} \delta(a_i;g) = 1$. Then*

$$\lim_{r\to\infty} \frac{\{\log T(r, f\circ g)\}^2}{T(r,L(f))\,T(r,L(g))} = 0\,.$$

Proof. In view of the inequality $T(r,g)\le \log^+ M(r,g)$ and Lemma 1 we obtain for all sufficiently large values of r,

$$\begin{aligned} T(f\circ g) &\le \{1+o(1)\}\,T(M(r,g),f) \\ \text{i.e.,}\quad \log T(r,f\circ g) &\le \log\{1+o(1)\} + \log T(M(r,g),f) \\ &\le o(1) + (\rho(f)+\varepsilon)\log M(r,g) \\ &\le o(1) + (\rho(f)+\varepsilon)\, r^{\rho(g)+\varepsilon}\,. \end{aligned} \tag{11}$$

Again in view of Lemma 4, we get for all sufficiently large values of r,

$$\log T(r, L(f)) > (\mu(L(f)) - \varepsilon) \log r$$
$$\text{i.e.,} \quad \log T(r, L(f)) > (\mu(f) - \varepsilon) \log r$$
$$\text{i.e.,} \quad T(r, L(f)) > r^{\mu(f)-\varepsilon}. \tag{12}$$

Now combining (11) and (12) it follows for all sufficiently large values of r,

$$\frac{\log T(r, f \circ g)}{T(r, L(f))} \le \frac{o(1) + (\rho(f) + \varepsilon) r^{\rho(g)+\varepsilon}}{r^{\mu(f)-\varepsilon}}. \tag{13}$$

Since $\rho(g) < \mu(f)$, we can choose ε (> 0) in such a way that

$$\rho(g) + \varepsilon < \mu(f) - \varepsilon. \tag{14}$$

So in view of (13) and (14) it follows that

$$\lim_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} = 0. \tag{15}$$

Again from Lemma 3 and Lemma 8 we get for all sufficiently large values of r,

$$\frac{\log T(r, f \circ g)}{T(r, L(g))} \le \frac{o(1) + (\rho(f) + \varepsilon) \log M(r, g)}{T(r, L(g))}$$
$$\text{i.e.,} \quad \limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \le (\rho(f) + \varepsilon) \limsup_{r\to\infty} \frac{\log M(r, g)}{T(r, L(g))}$$
$$\text{i.e.,} \quad \limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \le (\rho(f) + \varepsilon) \limsup_{r\to\infty} \frac{\log M(r, g)}{T(r, g)} \cdot \lim_{r\to\infty} \frac{T(r, g)}{T(r, L(g))}$$
$$\text{i.e.,} \quad \limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \le (\rho(f) + \varepsilon) \cdot \pi \cdot \frac{1}{1 + k - k\delta(\infty; g)}.$$

Since ε (> 0) is arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \le \rho(f) \cdot \pi \cdot \frac{1}{1 + k - k\delta(\infty; g)}. \tag{16}$$

In view of (15) i.e., $\lim\limits_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r, L(f))} = 0$ and using (16) we obtain that

$$\begin{aligned} \limsup_{r\to\infty} \frac{\{\log T(r, f \circ g)\}^2}{T(r, L(f))\, T(r, L(g))} &= \lim_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} \cdot \limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \\ &\le 0 \cdot \frac{\pi \rho(f)}{1 + k - k\delta(\infty; g)} = 0, \end{aligned}$$

$$\text{i.e.,} \quad \lim_{r\to\infty} \frac{\{\log T(r, f \circ g)\}^2}{T(r, L(f))\, T(r, L(g))} = 0.$$

This proves the theorem. □

Theorem 4. *If f and g be two entire functions with f to be transcendental having maximal deficiency sum satisfying the following conditions:*

(i) $\mu(f) > 0$, (ii) $\rho_2(f) < \infty$ *and* (iii) $0 < \mu(g) \leq \rho(g)$. *Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \max\left\{\frac{\mu(g)}{\mu_2(f)}, \frac{\rho(g)}{\rho_2(f)}\right\}.$$

Proof. We know that for $r > 0$ (cf. [7]) and for all sufficiently large values of r,

$$T(r, f\circ g) \geq \frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + o(1), f\right\}. \tag{17}$$

Since $\mu(f)$ and $\mu(g)$ are the lower orders of f and g respectively then for given ε (> 0) and for large values of r we obtain that

$$\log M(r, f) > r^{\mu(f)-\varepsilon} \quad \text{and} \quad \log M(r, g) > r^{\mu(g)-\varepsilon}$$

where $0 < \varepsilon < \min\{\mu(f), \mu(g)\}$. So from (17) we have for all sufficiently large values of r,

$$T(r, f\circ g) \geq \frac{1}{3}\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + o(1)\right\}^{\mu(f)-\varepsilon}$$

$$\text{i.e.,}\quad T(r, f\circ g) \geq \frac{1}{3}\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right)\right\}^{\mu(f)-\varepsilon}$$

$$\text{i.e.,}\quad \log T(r, f\circ g) \geq O(1) + (\mu(f) - \varepsilon)\log M\left(\frac{r}{4}, g\right)$$

$$\text{i.e.,}\quad \log T(r, f\circ g) \geq O(1) + (\mu(f) - \varepsilon)\left(\frac{r}{4}\right)^{\mu(g)-\varepsilon}$$

$$\text{i.e.,}\quad \log^{[2]} T(r, f\circ g) \geq O(1) + (\mu(g) - \varepsilon)\log r. \tag{18}$$

Again in view of Lemma 5, we get for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, L(f)) \leq (\mu_2(L(f)) + \varepsilon)\log r$$

$$\text{i.e.,}\quad \log^{[2]} T(r, L(f)) \leq (\mu_2(f) + \varepsilon)\log r. \tag{19}$$

Combining (18) and (19) it follows for a sequence of values of r tending to infinity

$$\frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \frac{O(1) + (\mu(g) - \varepsilon)\log r}{(\mu_2(f) + \varepsilon)\log r}.$$

Since ε (> 0) is arbitrary, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \frac{\mu(g)}{\mu_2(f)}. \tag{20}$$

Again from (17) we get for a sequence of values of r tending to infinity,

$$\log T(r, f\circ g) \geq O(1) + (\mu(f) - \varepsilon)\left(\frac{r}{4}\right)^{\rho(g)-\varepsilon}$$

$$\text{i.e.,}\quad \log^{[2]} T(r, f\circ g) \geq O(1) + (\rho(g) - \varepsilon)\log r. \tag{21}$$

Also in view of Lemma 5, for all sufficiently large values of r, we have

$$\log^{[2]} T(r, L(f)) \leq (\rho_2(L(f)) + \varepsilon)\log r = (\rho_2(f) + \varepsilon)\log r. \tag{22}$$

Now from (21) and (22) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \frac{O(1) + (\rho(g) - \varepsilon)\log r}{(\rho_2(f) + \varepsilon)\log r}\,.$$

As ε $(0 < \varepsilon < \rho(g))$ is arbitrary, we obtain from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \frac{\rho(g)}{\rho_2(f)}\,. \tag{23}$$

Therefore from (20) and (23) we get that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f\circ g)}{\log^{[2]} T(r, L(f))} \geq \max\left\{\frac{\mu(g)}{\mu_2(f)}, \frac{\rho(g)}{\rho_2(f)}\right\}.$$

Thus the theorem is established. □

Theorem 5. *Let f be transcendental meromorphic with maximal deficiency sum and g be entire such that* (i) $0 < \mu_2(f) < \rho_2(f)$, (ii) $\rho(g) < \infty$. *If h be transcendental meromorphic function with finite order then*

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, h\circ g)}{\log^{[2]} T(r, L(f))} \leq \min\left\{\frac{\mu(g)}{\mu_2(f)}, \frac{\rho(g)}{\rho_2(f)}\right\}.$$

Proof. In view of Lemma 1 and the inequality $T(r, g) \leq \log^{+} M(r, g)$, we obtain for all sufficiently large values of r

$$\log T(r, h\circ g) \leq o(1) + (\rho(h) + \varepsilon)\log M(r, g)\,. \tag{24}$$

Also for a sequence of values of r tending to infinity,

$$\log M(r, g) \leq r^{\mu(g)+\varepsilon}\,. \tag{25}$$

Combining (24) and (25) it follows for a sequence of values of r tending to infinity,

$$\log T(r, h\circ g) \leq o(1) + (\rho(h) + \varepsilon)\, r^{\mu(g)+\varepsilon}$$

$$\text{i.e.,}\quad \log T(r, h\circ g) \leq \{(\rho(h) + \varepsilon) + o(1)\}\, r^{\mu(g)+\varepsilon}$$

$$\text{i.e.,}\quad \log^{[2]} T(r, h\circ g) \leq O(1) + (\mu(g) + \varepsilon)\log r\,. \tag{26}$$

Again in view of Lemma 5, we have for all sufficiently large values of r,

$$\log^{[2]} T(r, L(f)) > (\mu_2(L(f)) - \varepsilon)\log r = (\mu_2(f) - \varepsilon)\log r\,. \tag{27}$$

Now from (26) and (27) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, h\circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{O(1) + (\mu(g) + \varepsilon)\log r}{(\mu_2(f) - \varepsilon)\log r}\,.$$

As ε (> 0) is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, h\circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{\mu(g)}{\mu_2(f)}\,. \tag{28}$$

In view of Lemma 1 we get for all sufficiently large values of r,

$$\log^{[2]} T(r, h\circ g) \leq O(1) + (\rho(g) + \varepsilon)\log r\,. \tag{29}$$

Also for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, L(f)) > (\rho_2(L(f)) - \varepsilon) \log r = (\rho_2(f) - \varepsilon) \log r \,. \tag{30}$$

Combining (29) and (30) we have for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, h \circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{O(1) + (\rho(g) + \varepsilon) \log r}{(\rho_2(f) - \varepsilon) \log r} \,.$$

Since ε (> 0) is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, h \circ g)}{\log^{[2]} T(r, L(f))} \leq \frac{\rho(g)}{\rho_2(f)} \,. \tag{31}$$

Now from (28) and (29) we get that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, h \circ g)}{\log^{[2]} T(r, L(f))} \leq \min \left\{ \frac{\mu(g)}{\mu_2(f)}, \frac{\rho(g)}{\rho_2(f)} \right\} .$$

This proves the theorem. □

The following theorem is a natural consequence of Theorem 4 and Theorem 5.

Theorem 6. *Let f and h be transcendental entire functions and g be an entire function such that* (i) $0 < \mu_2(f) \leq \rho_2(f) < \infty$,(ii) $0 < \mu(h) \leq \rho(h) < \infty$, (iii) $0 < \mu(g) \leq \rho(g) < \infty$. *Also let f has a maximal deficiency sum. Then*

$$\begin{aligned} \liminf_{r \to \infty} \frac{\log^{[2]} T(r, h \circ g)}{\log^{[2]} T(r, L(f))} &\leq \min \left\{ \frac{\mu(g)}{\mu_2(f)}, \frac{\rho(g)}{\rho_2(f)} \right\} \leq \max \left\{ \frac{\mu(g)}{\mu_2(f)}, \frac{\rho(g)}{\rho_2(f)} \right\} \\ &\leq \limsup_{r \to \infty} \frac{\log^{[2]} T(r, h \circ g)}{\log^{[2]} T(r, L(f))} \,. \end{aligned}$$

Taking $f = h$ in Theorem 4 the above theorem can be proved in view of Theorem 5.

Theorem 7. *Let f be transcendental meromorphic with maximal deficiency sum and g be entire such that $0 < \mu(f) \leq \rho(f) < \infty$. Then*

$$\limsup_{r \to \infty} \frac{\log^{[2]} T\left(\exp\left(r^{\rho(g)}\right), f \circ g\right)}{\log^{[2]} T(\exp(r^{\mu}), L(f))} = \infty \,.$$

Proof. Let $0 < \mu' < \rho(g)$. Then in view of Lemma 2 we get for a sequence of values of r tending to infinity,

$$\begin{aligned} \log T(r, f \circ g) &\geq \log T(\exp(r^{\mu'}), f) \\ &\geq (\mu(f) - \varepsilon) \log(\exp(r^{\mu'})) \\ &\geq (\mu(f) - \varepsilon) r^{\mu'} \\ \text{i.e.,} \quad \log^{[2]} T(r, f \circ g) &\geq O(1) + \mu' \log r \,. \end{aligned}$$

So for a sequence of values of r tending to infinity,

$$\log^{[2]} T(\exp(r^{\rho(g)}), f \circ g) \geq O(1) + \mu' \log(\exp(r^{\rho(g)}))$$

$$\text{i.e.,} \quad \log^{[2]} T(\exp(r^{\rho(g)}), f \circ g) \geq O(1) + \mu' r^{\rho(g)} . \tag{32}$$

Again in view of Lemma 4, we have for all sufficiently large values of r,

$$\log T(\exp(r^{\mu}), L(f)) \leq (\rho(L(f)) + \varepsilon) \log(\exp(r^{\mu}))$$

$$\text{i.e.,} \quad \log T(\exp(r^{\mu}), L(f)) \leq (\rho(f) + \varepsilon) r^{\mu}$$

$$\text{i.e.,} \quad \log^{[2]} T(\exp(r^{\mu}), L(f)) \leq O(1) + \mu \log r . \tag{33}$$

Now combining (32) and (33) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T\left(\exp\left(r^{\rho(g)}\right), f \circ g\right)}{\log^{[2]} T\left(\exp\left(r^{\mu}\right), L(f)\right)} \geq \frac{O(1) + \mu' r^{\rho(g)}}{O(1) + \mu \log r} ,$$

from which the theorem follows. □

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