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FUNDAMENTAL GROUP OF Symp(M,ω) WITH NO CIRCLE ACTION

Jarek Kędra

Abstract. We show that $\pi_1(\mathrm{Symp}(M,\omega))$ can be nontrivial for M that does not admit any symplectic circle action.

1. Introduction

Let (M,ω) be a closed symplectic manifold and let $\mathrm{Symp}(M,\omega)$ denote the group of symplectic diffeomorphisms of (M,ω). This group is equipped with the C^∞-topology. We are interested in the relation between the fundamental group $\pi_1(\mathrm{Symp}(M,\omega),\mathrm{Id})$ and symplectic circle actions on (M,ω). A symplectic circle action is a homomorphism $\alpha\colon S^1 \to \mathrm{Symp}(M,\omega)$ and it defines an element of the fundamental group of the group of symplectic diffeomorphisms.

Question 1.1. Suppose that $\pi_1(\mathrm{Symp}(M,\omega))$ is nontrivial. Is it true that some nonzero element is represented by a symplectic circle action?

If G is a Lie group then every element of $\pi_1(G)$ is represented by a loop that is a homomorphism. Examples of elements in $\pi_1(\mathrm{Symp}(M,\omega))$ which are not represented by a circle action were described by Anjos and McDuff [2, 8]. In the present paper, we provide a family of symplectic four manifolds (M,ω) such that $\pi_1(\mathrm{Symp}(M,\omega))$ is non-trivial and (M,ω) admits no circle action. More precisely we prove the following result.

Theorem 1.2. *Let (K,ω_K) be a simply connected symplectic four manifolds that is neither $\mathbb{CP}^2$ nor a ruled surface up to a blow-up. Let (M,ω) be a symplectic blow-up (K,ω_K) in a small ball. Then (M,ω) admits no symplectic circle action and the fundamental group $\pi_1(\mathrm{Symp}(M,\omega))$ is nontrivial.*

Recall that the blow-up of a symplectic manifold is defined as follows. Let $B\subset(M,\omega)$ be an open symplectic ball. This means that the restriction of the symplectic form ω to B is the standard symplectic form $\sum dx^i\wedge dy^i$. Such balls always exist due to the Darboux theorem. The boundary of $M-B$ is diffeomorphic to an odd dimensional sphere S^{2n-1}. Taking the quotient of this sphere as in the

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Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$ we obtain a closed symplectic manifold called the blow-up of (M,ω) in a ball B (see Section 7.1 in [10] for details). The blow-up contains $\mathbb{CP}^{n-1}$ as a symplectic submanifold. It is called the exceptional divisor.

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2. Proof of Theorem 1.2

There are very few manifolds admitting a circle action. On the other hand, the topology of groups of symplectic diffeomorphisms is rather complicated [6]. Hence one can expect nontrivial fundamental groups. The argument consists of several steps:

Step 1: Take a closed simply connected symplectic manifold (K,ω_K). Choose a point $p \in M$ and consider the evaluation fibration

$$\operatorname{Symp}(K,p) \to \operatorname{Symp}_0(K) \xrightarrow{ev} K,$$

defined by $ev(f) := f(p)$. Here $\operatorname{Symp}(K,p) \subset \operatorname{Symp}_0(K)$ denote the isotropy subgroup and $\operatorname{Symp}_0(K)$ denotes the identity component of the group of symplectic diffeomorphisms. We claim that

the rank of $\pi_1(\operatorname{Symp}(K,p))$ *is positive.*

Observe that $ev_*\colon \pi_2(\operatorname{Symp}(K)) \to \pi_2(K)$ is trivial up to torsion. Indeed, if $ev_*(\sigma)$ were nontorsion then the corresponding map on rational cohomology would be nonzero, say $ev^*(\alpha) \neq 0$ for $\alpha \in H^2(K,\mathbb{Q})$ such that $\langle \alpha, \sigma\rangle \neq 0$. Then we would have that $0 = ev^*(\alpha^{n+1}) = ev^*(\alpha)^{n+1}$, where $\dim K = 2n$. But $\operatorname{Symp}(K)$ is a topological group so its rational cohomology is free graded algebra. Thus if $ev^*(\alpha)^{n+1} = 0$ then $ev^*(\alpha)$ has to be a sum of products of degree one cohomology classes. Hence it has to vanish on spheres. On the other hand, $\langle ev^*(\alpha), \sigma\rangle = \langle \alpha, ev_*(\sigma)\rangle \neq 0$ which is a contradiction.

Finally, we get that the rank of $\pi_1(\operatorname{Symp}(K,p))$ is not smaller than the rank of $\pi_2(K)$. The latter is nonzero because because K is symplectic and simply connected. More precisely, since K is simply connected $\pi_2(K) \cong H_2(M;\mathbb{Z})$. The cohomology class of the symplectic form $[\omega] \in H^2(M;\mathbb{R}) = \operatorname{Hom}(H_2(M;\mathbb{Z}),\mathbb{R})$ is nonzero which proves that the rank of $H_2(M;\mathbb{Z})$ is nonzero which implies that the rank of $\pi_1(\operatorname{Symp}(K,p))$ is positive as claimed.

Step 2: The isotropy subgroup $\operatorname{Symp}(K,p)$ *should* be weakly homotopy equivalent to the group of symplectomorphisms of a one point blow-up of (K,ω_K) in a very small ball. This is proved for a range of 4-dimensional manifolds by Lalonde and Pinsonnault in [7] It is interesting to what extent it is true. Some progress has been made recently by McDuff [9].

More precisely, Lalonde and Pinsonnault proved (Lemma 2.3 and 2.4 in [7]) that, *if for any almost complex structure J compatible with ω there exists unique*

J-holomorphic sphere that is embedded then $\mathrm{Symp}(M, \omega)$ *is weakly homotopy equivalent to* $\mathrm{Symp}^U(K, B_\varepsilon)$. The latter group is a subgroup of $\mathrm{Symp}(K, \omega_K)$ which fixes a ball $B_\varepsilon \subset K$ and acts on it by unitary maps.

Suppose that ω_K is integral and ε is small enough. Then the exceptional divisor has unique J-holomorphic representative for any compatible J. It is easy to prove (Lemma 4.3 in [6]) that $\mathrm{Symp}^U(K, B_\varepsilon)$ is weakly homotopy equivalent to $\mathrm{Symp}(K, p)$.

Step 3: The final step is to find a simply connected symplectic closed manifold that its blow-up does not admit any symplectic circle action. There is a classification, due to Audin [3] and Ahara-Hattori [1], of symplectic manifolds admitting a Hamiltonian circle action (see also Karshon [5]). In the simply connected case the symplectic action is Hamiltonian. According to this classification, a simply connected symplectic manifold admitting an effective circle action is a blow-up of the complex projective plane or a blow-up of a rational ruled surface. These are excluded by our hypothesis. This finishes the proof.

□

3. Remarks and examples

3.1. Let (M, ω) be as in the theorem and assume moreover that $b_2^+ > 1$. Due to a result of Baldridge [4], a simply connected 4-dimensional symplectic manifold with $b_2^+ > 1$ does not admit any smooth circle action. On the other hand, McDuff showed (Corollary 1.4 in [9]) that the fundamental group of $\mathrm{Diff}(M)$ is non-trivial. Combining this two results with our proof we obtain examples of manifolds with nontrivial $\pi_1(\mathrm{Diff}(M))$ and admitting no smooth circle actions.

3.2. Let $K \subset \mathbb{CP}^3$ be a hypersurface of degree d. It is simply connected according to the Lefschetz hyperplane theorem. Moreover, it is not difficult to calculate that

$$b_2^+(K) = 1 + \frac{1}{3}(d-1)(d-2)(d-3)\,.$$

Hence every hypersurface of degree at least 4 satisfies the assumption of Theorem 1.2 and the above smooth analog. For $d = 4$ we obtain K3 surfaces.

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Mathematical Sciences, University of Aberdeen
Meston Building, King's College
Aberdeen AB24 3UE, Scotland, UK
E-mail: kedra@maths.abdn.ac.uk

Institute of Mathematics, University of Szczecin
Wielkopolska 15, 70-451 Szczecin, Poland