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QUENCHING TIME OF SOME NONLINEAR WAVE EQUATIONS

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Abstract. In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_{tt}(x,t) = \varepsilon Lu(x,t) + f\big(u(x,t)\big) \quad \text{in} \quad \Omega \times (0,T)\,, \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T)\,, \\ u(x,0) = 0 \quad \text{in} \quad \Omega\,, \\ u_t(x,0) = 0 \quad \text{in} \quad \Omega\,, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, L is an elliptic operator, ε is a positive parameter, $f(s)$ is a positive, increasing, convex function for $s \in (-\infty, b)$, $\lim_{s\to b} f(s) = \infty$ and $\int_0^b \frac{ds}{f(s)} < \infty$ with $b = \text{const} > 0$. Under some assumptions, we show that the solution of the above problem quenches in a finite time and its quenching time goes to that of the solution of the following differential equation

$$\begin{cases} \alpha''(t) = f(\alpha(t))\,, & t > 0\,, \\ \alpha(0) = 0\,, \quad \alpha'(0) = 0\,, \end{cases}$$

as ε goes to zero. We also show that the above result remains valid if the domain Ω is large enough and its size is taken as parameter. Finally, we give some numerical results to illustrate our analysis.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_{tt}(x,t) = \varepsilon Lu(x,t) + f\big(u(x,t)\big) \quad \text{in} \quad \Omega \times (0,T)\,, \tag{1}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T)\,, \tag{2}$$

$$u(x,0) = 0 \quad \text{in} \quad \Omega\,, \tag{3}$$

$$u_t(x,0) = 0 \quad \text{in} \quad \Omega\,, \tag{4}$$

where ε is a positive parameter, $f(s)$ is a positive, increasing and convex function for $s \in (-\infty, b)$, $\lim_{s\to b} f(s) = \infty$, $\int_0^b \frac{ds}{f(s)} < +\infty$ with $b = \text{const} > 0$. The operator

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L is defined as follows

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big),$$

where $a_{ij}\colon \overline{\Omega} \to \mathbb{R}$, $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$, $1 \le i, j \le N$ and there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge C\|\xi\|^2 \quad \forall x \in \overline{\Omega}, \quad \forall \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N,$$

where $\|\cdot\|$ stands for the Euclidean norm of $\mathbb{R}^N$. Here $(0,T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t\to T} \|u(\cdot,t)\|_\infty = b$$

where $\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega} |u(x,t)|$. In this last case, we say that the solution u quenches in a finite time and the time T is called the quenching time of the solution u. Introduce the function $F(s) = \int_0^s f(\sigma)d\sigma$. Throughout this paper, we suppose that $\int_0^b \frac{ds}{\sqrt{F(s)}} < +\infty$.

Solutions of nonlinear wave equations which quench in a finite time have been the subject of investigation of many authors (see [4], [13], [11], [15], [18], and the references cited therein).

By standard methods, local existence, uniqueness, quenching and global existence have been treated (see for instance [18]). In this paper, we are interested in the asymptotic behavior of the quenching time when ε approaches zero. Our work was motivated by the paper of Friedman and Lacey in [5], where they have considered the following initial-boundary value problem

$$\begin{cases} u_t(x,t) = \varepsilon\Delta u(x,t) + f(u(x,t)) \quad \text{in} \quad \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

where Δ is the Laplacian, $f(s)$ is a positive, increasing, convex function for the nonnegative values of s, $\int_0^\infty \frac{ds}{f(s)} < \infty$, $u_0(x)$ is a continuous function in Ω. Under some additional conditions on the initial data, they have shown that the solution of the above problem blows up in a finite time and its blow-up time tends to that of the solution $\lambda(t)$ of the following differential equation

$$\lambda'(t) = f(\lambda(t)), \quad \lambda(0) = M, \tag{5}$$

as ε goes to zero where $M = \sup_{x\in\Omega} u_0(x)$ (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time).

The proof developed in [5] is based on the construction of upper and lower solutions and it is difficult to extend the method in [5] to the problem described in (1)–(4). In the present paper, we prove a similar result. More precisely, we show that when ε is small enough, the solution u of (1)–(4) quenches in a finite time and its quenching time tends to that of the solution $\alpha(t)$ of the following differential equation

$$\alpha^{''}(t) = f\big(\alpha(t)\big)\,, \quad \alpha(0) = 0\,, \quad \alpha^{'}(0) = 0\,, \tag{6}$$

as ε goes to zero. We also prove that the above result remains valid if Ω is large enough and its size is taken as parameter. Our paper is written in the following manner. In the next section, under some assumptions, we show that the solution u of (1)–(4) quenches in a finite time and its quenching time goes to that of the solution $\alpha(t)$ of the differential equation defined in (6) when the parameter ε goes to zero. We also extend this result taking the size of the domain Ω as parameter. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Quenching times

In this section, under some assumptions, we show that the solution u of (1)–(4) quenches in a finite time and its quenching time goes to that of the solution of the differential equation defined in (6) when ε tends to zero.

We also obtain an analogous result in the case where the domain Ω is large enough and its size plays the role of parameter. Before starting, let us recall a well known result. Consider the following eigenvalue problem

$$-L\varphi = \lambda\varphi \quad \text{in} \quad \Omega\,, \tag{7}$$

$$\varphi = 0 \quad \text{on} \quad \partial\Omega\,, \tag{8}$$

$$\varphi > 0 \quad \text{in} \quad \Omega\,. \tag{9}$$

The above problem admits a solution (φ, λ) with $\lambda > 0$. We can normalize φ so that $\int_\Omega \varphi dx = 1$.

Our first result on the quenching time concerns the case where the domain Ω is fixed and ε is small enough. It is stated in the following theorem.

Theorem 2.1. *Let $A = \lambda \int_0^b \frac{ds}{f(s)}$. If $\varepsilon < A$ then the solution u of* (1)–(4) *quenches in a finite time and its quenching time T satisfies the following estimates*

$$T_e \le T \le (1 + \varepsilon A/2)T_e + o(\varepsilon) \tag{10}$$

where $T_e = \frac{1}{\sqrt{2}} \int_0^b \frac{ds}{\sqrt{F(s)}}$ is the quenching time of the solution $\alpha(t)$ of the differential equation defined in (6).

Proof. Since $(0, T)$ is the maximal time interval on which the solution u exists, our aim is to show that T is finite and satisfies the above estimates. Introduce the function $v(t)$ defined as follows

$$v(t) = \int_\Omega \varphi(x)u(x,t)\,dx \quad \text{for} \quad t \in (0,T)\,.$$

Take the derivative of v in t and use (1) to obtain

$$v''(t) = \varepsilon \int_\Omega \varphi L u\,dx + \int_\Omega f(u)\varphi\,dx\,.$$

Applying Green's formula, we arrive at

$$v''(t) = \varepsilon \int_\Omega uL\varphi \, dx + \int_\Omega f(u)\varphi \, dx \, .$$

Using (7) and Jensen's inequality, we find that

$$v''(t) \geq -\varepsilon\lambda v(t) + f\big(v(t)\big) \, ,$$

which implies that

$$v''(t) \geq f\big(v(t)\big)\Big(1 - \frac{\varepsilon\lambda v(t)}{f(v(t))}\Big) \, .$$

We observe that

$$\int_0^b \frac{ds}{f(s)} \geq \sup_{0\leq t\leq b} \int_0^t \frac{ds}{f(s)} \geq \sup_{0\leq t\leq b} \frac{t}{f(t)}$$

because $f(s)$ is an increasing function for the nonnegative values of s. We deduce that $v''(t) \geq (1-\varepsilon A)f(v(t))$ which implies that

$$v'(t) \geq (1-\varepsilon A)\int_0^t f\big(v(s)\big)\, ds \, , \quad t \in (0,T) \, , \tag{11}$$

$$v(0) = 0 \, . \tag{12}$$

Let $\gamma(t)$ be the solution of the following differential equation

$$\gamma'(t) = (1-\varepsilon A)\int_0^t f\big(\gamma(s)\big)\, ds \, , \quad t \in (0,T_0) \, , \tag{13}$$

$$\gamma(0) = 0 \, , \tag{14}$$

where $(0,T_0)$ is the maximal time interval of existence of $\gamma(t)$. It is not hard to see that

$$\gamma''(t) = (1-\varepsilon A)f\big(\gamma(t)\big) \, .$$

Multiply both sides of the above equality by $\gamma'(t)$ to obtain

$$\Big(\frac{(\gamma'(t))^2}{2}\Big)' = (1-\varepsilon A)\big(F(\gamma(t))\big)_t \, . \tag{15}$$

Integrating the equality in (15) over $(0,t)$, we find that

$$\frac{(\gamma'(t))^2}{2} = (1-\varepsilon A)\big(F(\gamma(t))\big) \, , \tag{16}$$

which implies that

$$\gamma'(t) = \sqrt{2(1-\varepsilon A)F\big(\gamma(t)\big)} \, .$$

This equality may be rewritten as follows

$$\frac{d\gamma}{\sqrt{F(\gamma)}} = \sqrt{2(1-\varepsilon A)}\, dt \, .$$

After integration over $(0, T_0)$, we discover that

$$T_0 = \frac{1}{\sqrt{2(1-\varepsilon A)}} \int_0^b \frac{ds}{\sqrt{F(s)}}\,.$$

Since the above integral converges, we see that $\gamma(t)$ quenches at the time T_0. On the other hand, the maximum principle implies that

$$v(t) \geq \gamma(t) \quad \text{for} \quad t \in (0, T_*)\,, \tag{17}$$

where $T_* = \min\{T_0, T\}$. We deduce that $T \leq T_0$. Indeed, suppose that $T > T_0$. From (17), it is not difficult to see that $v(T_0) = b$ which implies that u quenches at the times T_0. But this contradicts the fact that $(0, T)$ is the maximal time interval of existence of the solution u. Hence, we have

$$T \leq T_0 = \frac{1}{\sqrt{2(1-\varepsilon A)}} \int_0^b \frac{ds}{\sqrt{F(s)}}\,. \tag{18}$$

Now let us define the function $U(t)$ as follows

$$U(t) = \sup_{x\in\Omega} u(x,t) \quad \text{for} \quad t \in (0, T)\,.$$

Obviously, we have $U(0) = 0$, $U^{'}(0) = 0$ and there exists $x_0 \in \Omega$ such $U(t) = u(x_0, t)$. It is not hard to see that $Lu(x_0, t) \leq 0$. Consequently, we get

$$\begin{cases} U''(t) \leq f(U(t))\,, \quad t \in (0, T)\,, \\ U(0) = 0\,, \quad U'(0) = 0\,, \end{cases}$$

which implies that

$$U'(t) \leq \int_0^t f\big(U(s)\big)\, ds\,, \quad t \in (0, T)\,, \tag{19}$$

$$U(0) = 0\,. \tag{20}$$

Let $\beta(t)$ be the solution of the differential equation below

$$\beta'(t) = \int_0^t f\big(\beta(s)\big)\, ds\,, \quad t \in (0, T_1)\,, \tag{21}$$

$$\beta(0) = 0\,, \tag{22}$$

where $(0, T_1)$ is the maximal time interval of existence of $\beta(t)$. As we have seen for the solution $\gamma(t)$, $\beta(t)$ quenches at the time $T_1 = \frac{1}{\sqrt{2}} \int_0^b \frac{ds}{\sqrt{F(s)}}$. By the maximum principle, we find that

$$U(t) \leq \beta(t) \quad \text{for} \quad t \in (0, T_{**})\,,$$

where $T_{**} = \min\{T, T_1\}$. This implies that $T_{**} = T$. In fact, if $T_1 > T$, we obtain $U(T) \leq \beta(T) < b$ which is a contradiction. Therefore

$$T \geq T_1 = \frac{1}{\sqrt{2}} \int_0^b \frac{ds}{\sqrt{F(s)}}\,. \tag{23}$$

Apply Taylor's expansion to obtain

$$\frac{1}{\sqrt{1-\varepsilon A}} = 1 + \frac{\varepsilon A}{2} + o(\varepsilon)\,.$$

Use (18), (23) and the above relation to complete the rest of the proof. □

Now, let us consider the case where the domain Ω is large enough and ε is fixed. Let $a \in \Omega$ be such that $\delta = \mathrm{dist}(a, \partial\Omega) > 0$. Consider the following eigenvalue problem

$$\begin{aligned} -L\psi(x) &= \lambda_\delta \psi(x) \quad \text{in} \quad B(a,\delta)\,,\\ \psi(x) &= 0 \quad \text{on} \quad \partial B(a,\delta)\,,\\ \psi(x) &> 0 \quad \text{in} \quad B(a,\delta)\,, \end{aligned}$$

where $B(a,\delta) = \{x \in \mathbb{R}^N; \|x-a\| < \delta\} \subset \Omega$. It is well known that the above problem admits a solution (ψ, λ_δ) such that $0 < \lambda_\delta \le \frac{D}{\delta^2}$ where D is a positive constant which depends only on the upper bound of the coefficients of the operator L and the dimension N. We have the following result.

Theorem 2.2. *Let $Q = D\int_0^b \frac{ds}{f(s)}$ and suppose that $\mathrm{dist}(a,\partial\Omega) > \sqrt{\varepsilon Q}$. Then the solution u of (1)–(4) quenches in a finite time and its quenching time T satisfies the following estimates*

$$T_e \le T \le T_e + \frac{\varepsilon Q T_e}{2(\mathrm{dist}(a,\partial\Omega))^2} + o\Big(\frac{1}{(\mathrm{dist}(a,\partial\Omega))^2}\Big)\,,$$

where $T_e = \frac{1}{\sqrt{2}}\int_0^b \frac{ds}{\sqrt{F(s)}}$ is the quenching time of the solution $\alpha(t)$ of the differential equation defined in (6).

Proof. Since $B(a,\delta) \subset \Omega$ then we have $0 < \lambda \le \lambda_\delta$ where λ is the eigenvalue of the eigenvalue problem defined in (7)–(9). Reasoning as in the proof of Theorem 2.1, it is not hard to see that

$$\frac{1}{\sqrt{2}}\int_0^b \frac{d\sigma}{\sqrt{F(\sigma)}} \le T \le \frac{1}{\sqrt{2(1-\varepsilon A)}}\int_0^b \frac{d\sigma}{\sqrt{F(\sigma)}}$$

where $A = \lambda\int_0^b \frac{ds}{f(s)}$. Obviously, we have

$$1 - \varepsilon A \ge 1 - \varepsilon\lambda_\delta \int_0^b \frac{ds}{f(s)} \ge 1 - \frac{\varepsilon D}{\delta^2}\int_0^b \frac{ds}{f(s)}$$

because $0 < \lambda \le \lambda_\delta \le \frac{D}{\delta^2}$. Due to the fact that $Q = D\int_0^b \frac{ds}{f(s)}$, we deduce that

$$\frac{1}{\sqrt{2}}\int_0^b \frac{d\sigma}{\sqrt{F(\sigma)}} \le T \le \frac{1}{\sqrt{2(1-\frac{\varepsilon}{\delta^2}Q)}}\int_0^b \frac{d\sigma}{\sqrt{F(\sigma)}}\,. \tag{24}$$

We observe that

$$\frac{1}{\sqrt{2(1-\frac{\varepsilon Q}{\delta^2})}} = \frac{1}{\sqrt{2}} + \frac{\varepsilon Q}{2\sqrt{2}\delta^2} + o(\frac{1}{\delta^2}).$$

It follows from (24) that

$$T_e \le T \le T_e + \frac{\varepsilon Q}{2\sqrt{2}\delta^2} \int_0^b \frac{d\sigma}{\sqrt{F(\sigma)}} + o(\frac{1}{\delta^2})\,.$$

Taking into account the expression of T_e, we arrive at

$$T_e \le T \le T_e + \frac{\varepsilon Q T_e}{2\delta^2} + o(\frac{1}{\delta^2})\,.$$

Use the fact that δ=dist(a, $\partial\Omega$) to complete the rest of the proof. □

Remark 2.1. If $f(s) = (1-s)^{-1}$ then $F(s) = -ln(1-s)$. Consequently $T_e = \frac{1}{\sqrt{2}} \int_0^1 \frac{ds}{\sqrt{-ln(1-s)}}$ and its value is approximately equal 1.25.

Remark 2.2. From Theorem 2.2, we see that if dist$(a, \partial\Omega)$ tends to infinity then the quenching time T of the solution u of (1)–(4) tends to $T_e = \frac{1}{\sqrt{2}} \int_0^b \frac{ds}{F(s)}$. A direct consequence of this fact is that if $\Omega = \mathbb{R}^N$ then $T = T_e$.

3. Numerical results

In this section, we give some computational experiments to confirm the theory developed in the previous section. We consider the radial symmetric solution of (1)–(4) when $\Omega = B(0,1)$, $L = \Delta$ and $f(u) = (1-u)^{-p}$ with $p > 0$. Hence the problem (1)–(4) may be rewritten as follows

$$u_{tt} = \varepsilon\Big(u_{rr} + \frac{N-1}{r}u_r\Big) + (1-u)^{-p}\,, \quad r \in (0,1)\,, \quad t \in (0,T)\,, \tag{25}$$

$$u_r(0,t) = 0\,, \quad u(1,t) = 0\,, \quad t \in (0,T)\,, \tag{26}$$

$$u(r,0) = 0\,, \quad u_t(r,0) = 0\,, \quad r \in (0,1)\,. \tag{27}$$

We start by the construction of some adaptive schemes as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$ and approximate the solution u of (25)–(27) by the solution $U_h^{(n)} = (U_0^{(n)}, \ldots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - 2U_0^{(n)} + U_0^{(n-1)}}{\Delta t_n^2} = \varepsilon N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (1 - U_0^{(n)})^{-p}\,,$$

$$\frac{U_i^{(n+1)} - 2U_i^{(n)} + U_i^{(n-1)}}{\Delta t_n^2} = \varepsilon\Big(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}\Big)$$
$$+ (1 - U_i^{(n)})^{-p}\,, \quad 1 \le i \le I-1\,,$$

$$U_I^{(n)} = 0\,,$$

$$U_i^{(0)} = 0\,, \quad U_i^{(1)} = 0\,, \quad 0 \le i \le I\,,$$

where $n \ge 1$. In order to permit the discrete solution to reproduce the properties of the continuous solution, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min(h^2, (1 - \|U_h^{(n)}\|_\infty)^p)$$

with $\|U_h^{(n)}\|_\infty = \sup_{0\le i\le I} |U_i^{(n)}|$. We also approximate the solution u of (25)–(27) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - 2U_0^{(n)} + U_0^{(n-1)}}{\Delta t_n^2} = \varepsilon N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (1 - U_0^{(n)})^{-p},$$

$$\frac{U_i^{(n+1)} - 2U_i^{(n)} + U_i^{(n-1)}}{\Delta t_n^2} = \varepsilon\Big(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h}\Big) + (1 - U_i^{(n)})^{-p}, \quad 1 \le i \le I-1,$$

$$U_I^{(n+1)} = 0, \quad U_i^{(0)} = 0, \quad U_i^{(1)} = 0, \ 0 \le i \le I,$$

where $n \ge 1$. As in the case of the explicit scheme, we also take here

$$\Delta t_n = \min\left\{h^2, (1 - \|U_h^{(n)}\|_\infty)^p\right\}.$$

For our time step, we remark that if the norm of the discrete solution approaches one, the time step tends to zero. This is the general idea of adaptive schemes. Let us notice that it is possible to choose other time steps. For instance, one may take $\Delta t_n = \min\{h^2, (1 - \|U_h^{(n-1)}\|_\infty)^p\}$. In this last case, we see that if $\|U_h^{(n-1)}\|_\infty$ tends to one for the large values of n, then the time step approaches zero. This time step does not perturb the final result on the numerical quenching time.
We need the following definition.

Definition 3.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to\infty} \|U_h^{(n)}\|_\infty = 1$ and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges where $\|U_h^{(n)}\|_\infty = \sup_{0\le i\le I} |U_i^{(n)}|$. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical quenching time of the discrete solution $U_h^{(n)}$.

In the tables 1 and 2, in rows, we present the numerical quenching times, the numbers of iterations n, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}.$$

The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $N = 3$; $\varepsilon = 1/50$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPU_t	s
16	1.247227	642	2	-
32	1.251864	2567	12	-
64	1.253119	10267	98	1.88
128	1.253450	41069	769	1.92

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPU_t	s
16	1.247227	642	3	-
32	1.251864	2567	10	-
64	1.253119	10267	96	1.88
128	1.253450	41069	759	1.92

Remark 3.1. To obtain the above computational results, we have used MATLAB. Let us notice that in MATLAB, implicit schemes and explicit schemes have practically the same importance because one transforms all operations in linear systems. For this fact, we remark that the CPU time of the explicit scheme is approximately equal to that of the implicit scheme. We also notice that the CPU time of the implicit scheme is slightly better than that of the explicit scheme. If we had used for instance C++, we would see that the CPU time of the explicit scheme would be better than that of the implicit scheme because in this last case, for the implicit scheme, C++ needs to solve linear systems which is not the case for the explicit scheme.

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