ARCHIVUM MATHEMATICUM (BRNO)
Tomus 45 (2009), 137–146

APPROXIMATION OF ENTIRE FUNCTIONS OF SLOW GROWTH ON COMPACT SETS

G. S. Srivastava and Susheel Kumar

Abstract. In the present paper, we study the polynomial approximation of entire functions of several complex variables. The characterizations of generalized order and generalized type of entire functions of slow growth have been obtained in terms of approximation and interpolation errors.

1. Introduction

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [5] and Shah [6]. Hence, let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii) $\lim\limits_{x\to\infty} \dfrac{h[\{1+1/\psi(x)\}x]}{h(x)} = 1$ for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions $h(x)$ satisfying conditions (i) and

(iii) $\lim\limits_{x\to\infty} \dfrac{h(cx)}{h(x)} = 1$ for every $c > 0$, that is $h(x)$ is slowly increasing.

For an entire transcendental function $f(z) = \sum\limits_{n=1}^{\infty} b_n z^n$, $M(r) = \max\limits_{|z|=r} |f(z)|$ and functions $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, the generalized order is given by

$$\rho(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r)]}{\beta(\log r)} .$$

Further, for $\alpha(x)$, $\beta^{-1}(x)$ and $\gamma(x) \in L^0$, generalized type of an entire transcendental function $f(z)$ is given as

$$\sigma(\alpha,\beta,\rho,f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r)]}{\beta[\{\gamma(r)\}^{\rho}]} ,$$

2000 *Mathematics Subject Classification*: primary 30B10; secondary 30D20, 32K05.

Key words and phrases: entire function, Siciak extremal function, generalized order, generalized type, approximation errors, interpolation errors.

Received June 17, 2008, revised May 2009. Editor O. Došlý.

where $0 < \rho < \infty$ is a fixed number.
Let $g\colon C^N \to C$, $N \geq 1$, be an entire transcendental function. For $z = (z_1, z_2, \ldots, z_N) \in C^N$, we put

$$S(r,g) = \sup\left\{|g(z)| \,:\, |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 = r^2\right\}, \quad r > 0\,.$$

Then we define the generalized order and generalized type of $g(z)$ as

$$\rho(\alpha,\beta,g) = \limsup_{r\to\infty} \frac{\alpha[\log S(r,g)]}{\beta(\log r)}$$

and

$$\sigma(\alpha,\beta,\rho,g) = \limsup_{r\to\infty} \frac{\alpha[\log S(r,g)]}{\beta[\{\gamma(r)\}^\rho]}\,.$$

Let K be a compact set in C^N and let $\|\cdot\|_K$ denote the sup norm on K. The function $\Phi_K(z) = \sup\left(|p(z)|^{1/n} \,:\, p-\text{ polynomial, } \deg p \leq n\,, \|p\|_K \leq 1\,,\ n \in N\right)$, $z \in C^N$ is called the Siciak extremal function of the compact set K (see [2] and [3]). Given a function f defined and bounded on K, we put for $n = 1, 2, \ldots$

$$E_n^1(f,K) = \|f - t_n\|_K\,;$$

$$E_n^2(f,K) = \|f - l_n\|_K\,;$$

$$E_{n+1}^3(f,K) = \|l_{n+1} - l_n\|_K\,;$$

where t_n denotes the n^{th} Chebyshev polynomial of the best approximation to f on K and l_n denotes the n^{th} Lagrange interpolation polynomial for f with nodes at extremal points of K (see [2] and [3]).

The generalized order of an entire function of several complex variables has been characterized by Janik [3]. His characterization of order in terms of the above errors has been obtained under the condition

$$\left|\frac{d(\beta^{-1}[c\alpha(x)])}{d(\log x)}\right| \leq b\,; \quad x \geq a\,. \tag{1.1}$$

Clearly (1.1) is not satisfied for $\alpha(x) = \beta(x)$. Thus in this case, the corresponding result of Janik is not applicable. In the present paper we define generalized order and generalized type of entire functions of several complex variables in a new way. Our results apply satisfactorily to entire functions of slow growth and generalize many previous results.

Let Ω be the class of functions $h(x)$ satisfying conditions (i) and

(iv) there exist a function $\delta(x) \in \Lambda$ and constants x_0, c_1 and c_2 such that

$$0 < c_1 \leq \frac{d\{h(x)\}}{d\{\delta(\log x)\}} \leq c_2 < \infty \quad \text{for all} \quad x > x_0\,.$$

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(v) $\displaystyle\lim_{x\to\infty} \frac{d\{h(x)\}}{d(\log x)} = c_3\,, \quad 0 < c_3 < \infty.$

Kapoor and Nautiyal [4] showed that classes Ω and $\overline{\Omega}$ are contained in Λ and $\Omega \bigcap \overline{\Omega} = \phi$. They defined the generalized order $\rho(\alpha, \alpha, f)$ for entire functions as

$$\rho(\alpha, \alpha, f) = \lim_{r\to\infty} \sup \frac{\alpha[\log M(r)]}{\alpha(\log r)} ,$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$. Ganti and Srivastava [1] defined the generalized type $\sigma(\alpha, \alpha, \rho, f)$ of an entire function $f(z)$ having finite generalized order $\rho(\alpha, \alpha, f)$ as

$$\sigma(\alpha, \alpha, \rho, f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r)]}{[\alpha(\log r)]^{\rho}} .$$

2. Main results

Theorem 2.1. *Let K be a compact set in C^N. If $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$ then the function f defined and bounded on K, is a restriction to K of an entire function g of finite generalized order $\rho(\alpha, \alpha, g)$ if and only if*

$$\rho(\alpha, \alpha, g) = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left\{-\frac{1}{n}\log E_n^s(f, K)\right\}} ; \quad s = 1, 2, 3 .$$

Proof. Let g be an entire transcendental function. Write $\rho = \rho(\alpha, \alpha, g)$ and

$$\theta_s = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left\{-\frac{1}{n}\log E_n^s\right\}} ; \quad s = 1, 2, 3 .$$

Here E_n^s stands for $E_n^s\big(g|_K, K\big)$, $s = 1, 2, 3$. We claim that $\rho = \theta_s$, $s = 1, 2, 3$. It is known (see e.g. [7]) that

$$E_n^1 \le E_n^2 \le (n_* + 2)E_n^1 , \quad n \ge 0 , \tag{2.1}$$

$$E_n^3 \le 2(n_* + 2)E_{n-1}^1 , \quad n \ge 1 , \tag{2.2}$$

where $n_* = \binom{n+N}{n}$. Using Stirling formula for the approximate value of

$$n! \approx e^{-n} n^{n+1/2}\sqrt{2\pi} ,$$

we get $n_* \approx \frac{n^N}{N!}$ for all large values of n. Hence for all large values of n, we have

$$E_n^1 \le E_n^2 \le \frac{n^N}{N!}[1 + o(1)]E_n^1$$

and

$$E_n^3 \le 2\frac{n^N}{N!}[1 + o(1)]E_n^1 .$$

Thus $\theta_3 \le \theta_2 = \theta_1$ and it suffices to prove that $\theta_1 \le \rho \le \theta_3$. First we prove that $\theta_1 \le \rho$. Using the definition of generalized order, for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$\log S\,(r, g) \ \le\ \alpha^{-1}\big\{\overline{\rho}\,\alpha(\log r)\big\} ,$$

where $\overline{\rho} = \rho + \varepsilon$ provided r is sufficiently large. Without loss of generality, we may suppose that

$$K \subset B = \{z \in C^N : |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 \le 1\} .$$

Then

$$E_n^1 \le E_n^1(g, B)\,.$$

Now following Janik ([3, p.324]), we get

$$E_n^1(g, B) \le r^{-n} S(r, g)\,, \quad r \ge 2\,, \;\; n \ge 0$$

or

$$\log E_n^1 \le -n \log r + \alpha^{-1}\big\{\overline{\rho}\,\alpha(\log r)\big\}\,.$$

Putting $r = \exp\big[\alpha^{-1}\{\alpha(n)/\overline{\rho}\}\big]$ in the above inequality, we obtain

$$\log E_n^1 \le -n\big[\alpha^{-1}\{\alpha(n)/\overline{\rho}\}\big] + n$$

or

$$\frac{\alpha(n)}{\alpha\left\{1 - \frac{1}{n}\log E_n^1\right\}} \le \overline{\rho}\,.$$

Taking limits as $n \to \infty$, we get

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left\{-\frac{1}{n}\log E_n^1\right\}} \le \overline{\rho}\,.$$

Since $\varepsilon > 0$ is arbitrary small. Therefore finally we get

$$\theta_1 \le \rho\,. \tag{2.3}$$

Now we will prove that $\rho \le \theta_3$. If $\theta_3 = \infty$, then there is nothing to prove. So let us assume that $0 \le \theta_3 < \infty$. Therefore for a given $\varepsilon > 0$ there exist $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \le \frac{\alpha(n)}{\alpha\left\{-\frac{1}{n}\log E_n^3\right\}} \le \theta_3 + \varepsilon = \overline{\theta_3}$$

or

$$E_n^3 \le \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big]\,.$$

Now from the property of maximum modulus, we have

$$S(r, g) \le \sum_{n=0}^{\infty} E_n^3 r^n\,,$$

$$S(r, g) \le \sum_{n=0}^{n_0} E_n^3 r^n + \sum_{n=n_0+1}^{\infty} r^n \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big]\,.$$

Now for $r > 1$, we have

$$S(r, g) \le A_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big]\,, \tag{2.4}$$

where A_1 is a positive real constant. We take

$$N(r) = \alpha^{-1}\big(\overline{\theta_3}\alpha[\log\{(N+1)r\}]\big)\,. \tag{2.5}$$

Now if r is sufficiently large, then from (2.4) and (2.5) we have

$$S(r,g) \le A_1 r^{n_0} + r^{N(r)} \sum_{n_0+1\le n\le N(r)} \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big] + \sum_{n>N(r)} r^n \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big]$$

or

$$S(r,g) \le A_1 r^{n_0} + r^{N(r)} \sum_{n=1}^{\infty} \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big] + \sum_{n>N(r)} r^n \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big]. \tag{2.6}$$

Now we have

$$\limsup_{n\to\infty} \big(\exp[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}]\big)^{1/n} = 0\,.$$

Hence the first series in (2.6) converges to a positive real constant A_2. So from (2.6) we get

$$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} r^n \exp\big[-n\alpha^{-1}\{\alpha(n)/\overline{\theta_3}\}\big]\,,$$

$$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} r^n \exp\big[-n\log\{(N+1)r\}\big]\,,$$

$$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n>N(r)} \Big(\frac{1}{N+1}\Big)^n,$$

$$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n=1}^{\infty} \Big(\frac{1}{N+1}\Big)^n. \tag{2.7}$$

It can be easily seen that the series in (2.7) converges to a positive real constant A_3. Therefore from (2.7), we get

$$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + A_3\,,$$

$$S(r,g) \le A_2 r^{N(r)}\big[1+o(1)\big]\,,$$

$$\log S(r,g) \le \big[1+o(1)\big] N(r)\log r\,,$$

$$\log S(r,g) \le [1+o(1)]\alpha^{-1}(\overline{\theta_3}\alpha\big[\log\{(N+1)r\}\big])\log r\,,$$

$$\log S(r,g) \le \big[1+o(1)\big]\big[\alpha^{-1}\{(\overline{\theta_3}+\delta)\alpha[\log\{(N+1)r\}]\}\big]\,,$$

where $\delta > 0$ is suitably small.
Hence

$$\alpha[\log S(r,g)] \le (\overline{\theta_3}+\delta)\alpha[\log\{(N+1)r\}]$$

$$\frac{\alpha\,[\log S(r,g)]}{\alpha[\log r]} \le (\overline{\theta_3}+\delta)\,\big[1+o(1)\big]\,.$$

Proceedings to limits as $r \to \infty$, since δ is arbitrarily small, we get

$$\rho \le \theta_3 .$$

Now let f be a function defined and bounded on K and such that for $s = 1, 2, 3$

$$\theta_s = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left\{-\frac{1}{n}\log E_n^s\right\}}$$

is finite. We claim that the function

$$g = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1})$$

is the required entire continuation of f and $\rho(\alpha,\alpha,g) = \theta_s$. Indeed, for every $d_1 > \theta_s$

$$\frac{\alpha(n)}{\alpha\left\{-\frac{1}{n}\log E_n^s\right\}} \le d_1$$

provided n is sufficiently large. Hence

$$E_n^s \le \exp\left[-n\alpha^{-1}\{\alpha(n)/d_1\}\right].$$

Using the inequalities (2.1), (2.2) and converse part of theorem, we find that the function g is entire and $\rho(\alpha,\alpha,g)$ is finite. So by (2.3), we have $\rho(\alpha,\alpha,g) = \theta_s$, as claimed. This completes the proof of Theorem 2.1. □

Next we prove

Theorem 2.2. *Let K be a compact set in C^N such that Φ_K is locally bounded in C^N. Set $G(x,\sigma,\rho) = \alpha^{-1}\left[\{\sigma\,\alpha(x)\}^{1/\rho}\right]$, where ρ is a fixed number, $1 < \rho < \infty$. Let $\alpha(x) \in \overline{\Omega}$ and $\frac{dG(x,\sigma,\rho)}{d\log x} = O(1)$ as $x \to \infty$ for all $0 < \sigma < \infty$. Then the function f defined and bounded on K, is a restriction to K of an entire function g of generalized type $\sigma(\alpha,\alpha,\rho,g)$ if and only if*

$$\sigma(\alpha,\alpha,\rho,g) = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\left\{\alpha\left(\frac{\rho}{\rho-1}\log[E_n^s(f,K)]^{-1/n}\right)\right\}^{\rho-1}}, \quad s = 1,2,3.$$

Before proving the Theorem 2.2 we state and prove a lemma.

Lemma 2.1. *Let K be a compact set in C^N such that Φ_K is locally bounded in C^N. Set $G(x,\mu,\lambda) = \alpha^{-1}\left[\{\mu\,\alpha(x)\}^{1/\lambda}\right]$, where λ is a fixed number, $1 < \lambda < \infty$. Let $\alpha(x) \in \overline{\Omega}$ and $\frac{dG(x,\mu,\lambda)}{d\log x} = O(1)$ as $x \to \infty$ for all $0 < \mu < \infty$. Let $(p_n)_{n\in N}$ be a sequence of polynomials in C^N such that*

(i) $\deg p_n \le n$, $n \in N$;

(ii) *for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that*

$$\|p_n\|_K \le \exp\left(-\frac{\lambda-1}{\lambda}n\,\alpha^{-1}\left[\left\{\frac{1}{\mu}\alpha(n/\lambda)\right\}^{1/(\lambda-1)}\right]\right).$$

Then $\sum_{n=0}^{\infty} p_n$ is an entire function and $\sigma(\alpha,\alpha,\lambda,\sum_{n=0}^{\infty} p_n) \le \mu$ provided $\sum_{n=0}^{\infty} p_n$ is not a polynomial.

Proof. By assumption, we have

$$\|p_n\|_K r^n \le r^n \exp\Big(-\frac{\lambda-1}{\lambda} n\alpha^{-1}\Big[\Big\{\frac{1}{\mu}\alpha(n/\lambda)\Big\}^{1/(\lambda-1)}\Big]\Big), \quad n \ge n_0\,,\ r>0\,.$$

If $\alpha(x) \in \overline{\Omega}$, then by assumptions of lemma, there exists a number $b>0$ such that for $x>a$, we have

$$\Big|\frac{dG(x,\mu,\lambda)}{d\log x}\Big| < b\,.$$

Let us consider the function

$$\phi(x) = r^x \exp\Big(-\frac{\lambda-1}{\lambda}\, x\alpha^{-1}\Big[\Big\{\frac{1}{\mu}\alpha(x/\lambda)\Big\}^{1/(\lambda-1)}\Big]\Big)\,.$$

Using the technique of Seremeta [5], it can be easily seen that the maximum value of $\phi(x)$ is attained for a value of x given by

$$x^*(r) = \lambda\alpha^{-1}\big\{\mu[\alpha\{\log r - a(r)\}]^{\lambda-1}\big\}\,,$$

where

$$a(r) = \frac{dG(x/\lambda, 1/\mu, \lambda-1)}{d\log x}\,.$$

Thus

$$\|p_n\|_K r^n \le \exp\big\{b\lambda\alpha^{-1}[\mu\{\alpha(\log r + b)\}^{\lambda-1}]\big\}\,, \quad n\ge n_0\,,\ r>0\,. \tag{2.8}$$

Let us write $K_r = \{z \in C^N : \Phi_K(z) < r,\ r>1\}$, then for every polynomial p of degree $\le n$, we have (see [3])

$$|p_n(z)| \le \|p_n\|_K \Phi_K^n(z)\,, \quad z\in C^N\,.$$

So the series $\sum_{n=0}^{\infty} p_n$ is convergent in every K_r, $r>1$, whence $\sum_{n=0}^{\infty} p_n$ is an entire function. Put

$$M^*(r) = \sup\big\{\|p_n\|_K\ r^n\ :\ n\in N, r>0\big\}\,.$$

On account of (2.8), for every $r>0$, there exists a positive integer $\nu(r)$ such that

$$M^*(r) = \|p_{\nu(r)}\|_K\ r^{\nu(r)}$$

and

$$M^*(r) > \|p_n\|_K\ r^n\,, \quad n>\nu(r)\,.$$

It is evident that $\nu(r)$ increases with r. First suppose that $\nu(r)\to\infty$ as $r\to\infty$. Then putting $n=\nu(r)$ in (2.8), we get for sufficiently large r

$$M^*(r) \le \exp\big\{b\lambda\alpha^{-1}[\mu\{\alpha(\log r + b)\}^{\lambda-1}]\big\}\,. \tag{2.9}$$

Put

$$F_r = \{z\in C^N : \Phi_K(z) = r\}\,, \quad r>1$$

and

$$M(r) = \sup\Big\{\Big|\sum_{n=0}^{\infty} p_n(z)\Big| \,:\, z \in F_r\Big\}, \quad r > 1\,.$$

Now following Janik ([3] p.323), we have for some positive constant k

$$S\Big(r, \sum_{n=0}^{\infty} p_n\Big) \le M(kr) \le 2M^*(2kr)\,. \tag{2.10}$$

Combining (2.9) and (2.10), we get

$$S\Big(r, \sum_{n=0}^{\infty} p_n\Big) \le 2\exp\big\{b\lambda\alpha^{-1}[\mu\{\alpha(\log r + b)\}^{\lambda-1}]\big\}$$

or

$$\frac{\alpha[\frac{1}{b\lambda}\log\{\frac{1}{2}S(r, \sum_{n=0}^{\infty} p_n)\}]}{[\alpha(\log 2kr + b)]^{\lambda-1}} \le \mu\,.$$

Since $\alpha(x) \in \overline{\Omega}$, we get on using (v)

$$\limsup_{r\to\infty} \frac{\alpha\,[\log S(r, \sum_{n=0}^{\infty} p_n)]}{[\alpha(\log r)]^{\lambda}} \le \mu$$

or

$$\sigma(\alpha, \alpha, \lambda, \sum_{n=0}^{\infty} p_n) \le \mu\,.$$

In the case when $\nu(r)$ is bounded then $M^*(r)$ is also bounded, whence $\sum_{n=0}^{\infty} p_n$ reduces to a polynomial. Hence the Lemma 2.1 is proved. $\square$

Proof of Theorem 2.2. Let g be an entire transcendental function. Write $\sigma = \sigma(\alpha, \alpha, \rho, g)$ and

$$\eta_s = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\Big\{\alpha\Big(\frac{\rho}{\rho-1}\log[E_n^s]^{-1/n}\Big)\Big\}^{\rho-1}}\,, \quad s = 1, 2, 3\,.$$

Here E_n^s stands for $E_n^s\,(g|_K, K)$, $s = 1, 2, 3$. We claim that $\sigma = \eta_s$, $s = 1, 2, 3$. Now following Theorem 2.1, here we prove that $\eta_1 \le \sigma \le \eta_3$. First we prove that $\eta_1 \le \sigma$. Using the definition of generalized type, for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$S\,(r, g) \le \exp\big(\alpha^{-1}[\overline{\sigma}\,\{\alpha(\log r)\}^{\rho}]\big)\,,$$

where $\overline{\sigma} = \sigma + \varepsilon$ provided r is sufficiently large. Thus following Theorem 2.1, here we have

$$E_n^1 \le r^{-n}\exp\big(\alpha^{-1}[\overline{\sigma}\,\{\alpha(\log r)\}^{\rho}]\big)$$

$$E_n^1 \le \exp\big[-n\log r + (\alpha^{-1}[\overline{\sigma}\,\{\alpha(\log r)\}^{\rho}])\big]. \tag{2.11}$$

Let $r = r(n)$ be the unique root of the equation

$$\alpha\Big[\frac{n\,\log r}{\rho}\Big] = \overline{\sigma}\,\{\alpha(\log r)\}^{\rho}\,. \tag{2.12}$$

Then

$$\log r \simeq \alpha^{-1}\Big\{\frac{1}{\overline{\sigma}}\alpha(n/\rho)\Big\}^{1/(\rho-1)} = G(n/\rho, 1/\overline{\sigma}, \rho-1)\,. \tag{2.13}$$

Using (2.12) and (2.13) in (2.11), we get

$$E_n^1 \le \exp\Big[-n\,G(n/\rho, 1/\overline{\sigma}, \rho-1) + \frac{n}{\rho}\,G(n/\rho, 1/\overline{\sigma}, \rho-1)\Big]\,,$$

$$\frac{\rho}{\rho-1}\log[E_n^1]^{-1/n} \ge \alpha^{-1}\Big\{\Big(\frac{1}{\overline{\sigma}}\alpha(n/\rho)\Big)^{1/(\rho-1)}\Big\}\,,$$

$$\frac{\alpha(n/\rho)}{\Big\{\alpha\Big(\frac{\rho}{\rho-1}\log[E_n^1]^{-1/n}\Big)\Big\}^{\rho-1}} \le \overline{\sigma}\,.$$

Proceedings to limits, we get

$$\limsup_{n\to\infty}\frac{\alpha(n/\rho)}{\Big\{\alpha\Big(\frac{\rho}{\rho-1}\log[E_n^1]^{-1/n}\Big)\Big\}^{\rho-1}} \le \overline{\sigma}$$

or

$$\eta_1 \le \overline{\sigma}\,.$$

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\eta_1 \le \sigma\,. \tag{2.14}$$

Now we will prove that $\sigma \le \eta_3$. Suppose that $\eta_3 < \sigma$, then for every μ_1, $\eta_3 < \mu_1 < \sigma$, we have

$$\frac{\alpha(n/\rho)}{\Big\{\alpha\Big(\frac{\rho}{\rho-1}\log[E_n^3]^{-1/n}\Big)\Big\}^{\rho-1}} \le \mu_1$$

provided n is sufficiently large. Thus

$$E_n^3 \le \exp\Big(-\frac{\rho-1}{\rho}n\,\alpha^{-1}\Big[\Big\{\frac{1}{\mu_1}\alpha(n/\rho)\Big\}^{1/(\rho-1)}\Big]\Big)\,.$$

Also by previous lemma, $\sigma \le \mu_1$. Since μ_1 has been chosen less than σ, we get a contradiction. Hence

$$\sigma \le \eta_3\,.$$

Now let f be a function defined and bounded on K such that for $s = 1, 2, 3$

$$\eta_s = \limsup_{n\to\infty}\frac{\alpha(n/\rho)}{\Big\{\alpha\Big(\frac{\rho}{\rho-1}\log[E_n^s]^{-1/n}\Big)\Big\}^{\rho-1}}$$

is finite. We claim that the function

$$g = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1})$$

is the required entire continuation of f and $\sigma(\alpha, \alpha, \rho, g) = \eta_s$. Indeed, for every $d_2 > \eta_s$

$$\frac{\alpha(n/\rho)}{\left\{\alpha\left(\frac{\rho}{\rho-1}\log[E_n^s]^{-1/n}\right)\right\}^{\rho-1}} \leq d_2$$

provided n is sufficiently large. Hence

$$E_n^s \leq \exp\Big(-\frac{\rho-1}{\rho} n\,\alpha^{-1}\Big[\Big\{\frac{1}{d_2}\alpha(n/\rho)\Big\}^{1/(\rho-1)}\Big]\Big).$$

Using the inequalities (2.1), (2.2) and previous lemma, we find that the function g is entire and $\sigma(\alpha, \alpha, \rho, g)$ is finite. So by (2.14), we have $\sigma(\alpha, \alpha, \rho, g) = \eta_s$, as claimed. This completes the proof of the Theorem 2.2. □

Acknowledgement. The authors are very thankful to the referee for his valuable comments and observations which helped in improving the paper.

References

[1] Ganti, R., Srivastava, G. S., *Approximation of entire functions of slow growth*, General Math. **14** (2) (2006), 65–82.

[2] Janik, A., *A characterization of the growth of analytic functions by means of polynomial approximation*, Univ. Iagel. Acta Math. **24** (1984), 295–319.

[3] Janik, A., *On approximation of entire functions and generalized order*, Univ. Iagel. Acta Math. **24** (1984), 321–326.

[4] Kapoor, G. P., Nautiyal, A., *Polynomial approximation of an entire function of slow growth*, J. Approx. Theory **32** (1981), 64–75.

[5] Seremeta, M. N., *On the connection between the growth of the maximum modulus of an entire function and the moduli of the coefficients of its power series expansion*, Amer. Math. Soc. Transl. Ser. 2 **88** (1970), 291–301.

[6] Shah, S. M., *Polynomial approximation of an entire function and generalized order*, J. Approx. Theory **19** (1977), 315–324.

[7] Winiarski, T., *Application of approximation and interpolation methods to the examination of entire functions of n complex variables*, Ann. Polon. Math. **28** (1973), 97–121.

Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee – 247667, India
E-mail: girssfma@iitr.ernet.in