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ON SOME PROPERTIES OF THE PICARD OPERATORS

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Abstract. We consider the Picard operators $\mathcal{P}_n$ and $\mathcal{P}_{n;r}$ in exponential weighted spaces. We give some elementary and approximation properties of these operators.

1. Introduction

1.1. The Picard operators

$$\mathcal{P}_n(f;x) := \frac{n}{2}\int_{\mathbb{R}} f(x-t)e^{-n|t|}\,dt = \frac{n}{2}\int_{\mathbb{R}} f(x+t)e^{-n|t|}\,dt\,, \tag{1}$$

$x \in \mathbb{R}$ and $n \in \mathbb{N}$, ($\mathbb{N} = \{1,2,\dots\}$, $\mathbb{R} = (-\infty,+\infty)$) are investigated for functions $f\colon \mathbb{R} \to \mathbb{R}$ from various classes in many monographs and papers (e.g. [2]–[8] [10, 11]).

G. H. Kirov in the paper [9] introduced the generalized Bernstein polynomials $\mathcal{B}_{n;r}$ for r-times differentiable functions $f \in C^r\big([0,1]\big)$ and he showed that $\mathcal{B}_{n;r}$ have better approximation properties than classical Bernstein polynomials $\mathcal{B}_n$.

The Kirov method was used in [12] to the generalized Picard operators

$$\mathcal{P}_{n;r}(f;x) := \mathcal{P}_n(F_r(t,x);x)\,, \qquad x \in \mathbb{R},\ n \in \mathbb{N},\ r \in \mathbb{N}_0\,, \tag{2}$$

$$F_r(t,x) := \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j\,, \tag{3}$$

($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) of r-times differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. Obviously $\mathcal{P}_{n;0}(f) \equiv \mathcal{P}_n(f)$.

In this paper we examine the Picard operators $\mathcal{P}_n$ (in Section 2) and $\mathcal{P}_{n;r}$ (in Section 3) for functions f belonging to the exponential weighted spaces $L_q^p(\mathbb{R})$ and $L_q^{p,r}(\mathbb{R})$ which definition is given below. We present some elementary properties, the orders of approximation and the Voronovskaya – type theorems for these operators.

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1.2. Let $q > 0$ and $1 \leq p \leq \infty$ be fixed,

$$v_q(x) := e^{-q|x|} \quad \text{for} \quad x \in \mathbb{R}\,, \tag{4}$$

and let $L_q^p \equiv L_q^p(\mathbb{R})$ be the space of all functions $f\colon \mathbb{R} \to \mathbb{R}$ for which $v_q f$ is Lebesgue integrable with p-th power over $\mathbb{R}$ if $1 \leq p < \infty$ and uniformly continuous and bounded on $\mathbb{R}$ if $p = \infty$. The norm in L_q^p is defined by

$$\|f\|_{p,q} \equiv \|f(\cdot)\|_{p,q} := \begin{cases} \left(\int\limits_{\mathbb{R}} |v_q(x)f(x)|^p dx\right)^{1/p} & \text{if } \ 1 \leq p < \infty\,, \\ \sup\limits_{x\in\mathbb{R}} v_q(x)|f(x)| & \text{if } \ p = \infty\,. \end{cases} \tag{5}$$

Moreover, let $r \in \mathbb{N}_0$ and $L_q^{p,r} \equiv L_q^{p,r}(\mathbb{R})$ be the class of all r-times differentiable functions $f \in L_q^p$ having the derivatives $f^{(k)} \in L_q^p$, $1 \leq k \leq r$. The norm in $L_q^{p,r}$ is given by (5). ($L_q^{p,0} \equiv L_q^p$). The spaces L_q^p and $L_q^{p,r}$ are called exponential weighted spaces ([1]).

As usual, for $f \in L_q^p$ and $k \in \mathbb{N}$ we define the k-th modulus of smoothness:

$$\omega_k(f; L_q^p; t) := \sup_{|h|\leq t} \|\Delta_h^k f(\cdot)\|_{p,q} \quad \text{for} \quad t \geq 0\,, \tag{6}$$

$$\Delta_h^k f(x) := \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x + jh)\,. \tag{7}$$

The above ω_k has the following properties:

$$\omega_k(f; L_q^p; t_1) \leq \omega_k(f; L_q^p; t_2) \quad \text{for} \quad 0 \leq t_1 < t_2, \tag{8}$$

$$\omega_k(f; L_q^p; \lambda t) \leq (1 + \lambda)^k e^{kq\lambda t} \omega_k(f; L_q^p; t) \quad \text{for} \quad \lambda, t \geq 0\,, \tag{9}$$

$$\lim_{t\to 0+} \omega_k(f; L_q^p; t) = 0\,, \tag{10}$$

for every $f \in L_q^p$ and $k \in \mathbb{N}$ (see [6, Chapter 6] and [13, Chapter 3]).

By ω_k we define the Lipschitz class

$$\mathrm{Lip}_M^k(L_q^p; \alpha) := \left\{f \in L_q^p : \omega_k\left(f; L_q^p; t\right) \leq M t^\alpha \quad \text{for} \quad t \geq 0\right\} \tag{11}$$

for fixed numbers: $1 \leq p \leq \infty$, $q > 0$, $k \in \mathbb{N}$, $M > 0$ and $0 < \alpha \leq k$.

2. Some properties of $\mathcal{P}_n$

2.1. By elementary calculations can be obtained the following two lemmas.

Lemma 1. *The equality*

$$\int_0^\infty t^r e^{-st}\, dt = \frac{r!}{s^{r+1}} \tag{12}$$

there holds for every $r \in \mathbb{N}_0$ and $s > 0$.

Lemma 2. *Let $e_0(x) = 1$, $e_1(x) = x$ and let $\varphi_x(t) = t - x$ for $x, t \in \mathbb{R}$. Then*

$$\mathcal{P}_n(e_i; x) = e_i(x) \quad \textit{for} \quad x \in \mathbb{R},\ n \in \mathbb{N},\ i = 0, 1\,, \tag{13}$$

and

$$\mathcal{P}_n\left(\varphi_x^k(t);x\right) = \frac{\left(1+(-1)^k\right)k!}{2n^k}\,, \tag{14}$$

$$\mathcal{P}_n\left(|\varphi_x(t)|^k \exp\left(q\,|\varphi_x(t)|\right);x\right) = \frac{k!n}{(n-q)^{k+1}}\,, \tag{15}$$

for $x \in \mathbb{R}$, $n \geq q+1$ *and* $k \in \mathbb{N}_0$.

Using the above results and arguing analogously to the proof of Lemma 2 in [10] we can obtain the following basic lemma.

Lemma 3. *Let* $f \in L_q^p$ *with fixed* $1 \leq p \leq \infty$ *and* $q > 0$. *Then*

$$\|\mathcal{P}_n(f)\|_{p,q} \leq (1+q)\|f\|_{p,q} \quad \text{for} \quad n \geq q+1\,. \tag{16}$$

The formula (1) *and* (16) *show that* $\mathcal{P}_n$, $n \geq q+1$, *is a positive linear operator acting from the space* L_q^p *to* L_q^p.

2.2. By (6), (7), (11) and (16) can be derived the following geometric properties of $\mathcal{P}_n$ given by (1).

Theorem 1. *Let* $f \in L_q^p$ *with fixed* $1 \leq p \leq \infty$ *and* $q > 0$ *and let* $q+1 \leq n \in \mathbb{N}$. *Then*

(i) *if* f *is non-decreasing (non-increasing) on* $\mathbb{R}$, *then* $\mathcal{P}_n(f)$ *is also non-decreasing (non-increasing) on* $\mathbb{R}$,

(ii) *if* f *is convex (concave) on* $\mathbb{R}$, *then* $\mathcal{P}_n(f)$ *is also convex (concave) on* $\mathbb{R}$,

(iii) *for every* $k \in \mathbb{N}$ *there holds the inequality*

$$\omega_k\left(\mathcal{P}_n(f);L_q^p;t\right) \leq (1+q)\omega_k\left(f;L_q^p;t\right)\,, \quad t \geq 0\,,$$

(iv) *if* $f \in \mathrm{Lip}_M^k\left(L_q^p;\alpha\right)$ *with fixed* $k \in \mathbb{N}$, $0 < \alpha \leq k$ *and* $M > 0$, *then also* $\mathcal{P}_n(f) \in \mathrm{Lip}_{M^*}^k\left(L_q^p;\alpha\right)$ *with the same* k *and* α *and* $M^* = (1+q)M$,

(v) *If* $f \in L_q^{\infty,r}$ *with a fixed* $r \in \mathbb{N}$, *then* $\mathcal{P}_n(f) \in L_q^{\infty,r}$ *and for derivatives of* $\mathcal{P}_n(f)$ *there holds*

$$\left\|\mathcal{P}_n^{(k)}(f)\right\|_{\infty,q} = \left\|\mathcal{P}_n\left(f^{(k)}\right)\right\|_{\infty,q} \leq (1+q)\left\|f^{(k)}\right\|_{\infty,q}$$

Proof. For example we prove (iii). From the formulas (1) and (7) there results that

$$\Delta_h^k \mathcal{P}_n(f;x) = \mathcal{P}_n\left(\Delta_h^k f;x\right) \quad \text{for} \quad x,h \in \mathbb{R}, k \in \mathbb{N}\,.$$

Next, by (5) and (16), we have

$$\left\|\Delta_h^k \mathcal{P}_n(f;\cdot)\right\|_{p,q} = \left\|\mathcal{P}_n\left(\Delta_h^k f,\cdot\right)\right\|_{p,q} \leq (1+q)\left\|\Delta_h^k f(\cdot)\right\|_{p,q}$$

for $h \in \mathbb{R}$ and $n \geq q+1$, and using (6), we get the statement (iii). □

2.3. Arguing similarly to [5] and [10] and applying (6)–(9), (12) and (16) we can prove the following approximation theorem.

Theorem 2. *Suppose that $f \in L_q^p$ with fixed $1 \le p \le \infty$ and $q > 0$. Then*

$$\|\mathcal{P}_n(f) - f\|_{p,q} \le \frac{5}{2}(1+3q)^3 \omega_2\Big(f; L_q^p; \frac{1}{n}\Big)$$

for every $n \ge 3q+1$.

From Theorem 2 and (8), (10) and (11) there results the following

Corollary 1. *If $f \in L_q^p$, $1 \le p \le \infty$, $q > 0$, then*

$$\lim_{n\to\infty} \|\mathcal{P}_n(f) - f\|_{p,q} = 0\,. \tag{17}$$

In particular, if $f \in \mathrm{Lip}_M^2\left(L_q^p;\alpha\right)$ with fixed $0 < \alpha \le 2$ and $M > 0$, then

$$\|\mathcal{P}_n(f) - f\|_{p,q} = O(n^{-\alpha}) \quad \text{as} \quad n \to \infty\,.$$

Applying Corollary 1, we shall prove the Voronovskaya-type theorem for $\mathcal{P}_n$.

Theorem 3. *Let $f \in L_q^{\infty,2}$ with a fixed $q > 0$. Then*

$$\lim_{n\to\infty} n^2[\mathcal{P}_n(f;x) - f(x)] = f''(x) \tag{18}$$

for every $x \in \mathbb{R}$.

Proof. Choose $f \in L_q^{\infty,2}$ and $x \in \mathbb{R}$. Then, by the Taylor formula, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \psi(t;x)(t-x)^2 \quad \text{for} \quad t \in \mathbb{R}\,,$$

where $\psi(t) \equiv \psi(t,x)$ is a function belonging to L_q^∞ and $\lim_{t\to x}\psi(t;x) = \psi(x) = 0$. Using operator $\mathcal{P}_n$, $n \ge 2q+1$, and (13) and (14), we get

$$\mathcal{P}_n(f(t);x) = f(x) + n^{-2}f''(x) + \mathcal{P}_n\left(\psi(t)\varphi_x^2(t);x\right) \tag{19}$$

and by the Hölder inequality and (14):

$$\begin{aligned}\left|\mathcal{P}_n\left(\psi(t)\varphi_x^2(t);x\right)\right| &\le \left(\mathcal{P}_n\left(\psi^2(t);x\right)\mathcal{P}_n\left(\varphi_x^4(t);x\right)\right)^{1/2} \\ &= n^{-2}\left(24\mathcal{P}_n\left(\psi^2(t);x\right)\right)^{1/2}\,.\end{aligned}$$

From properties of ψ and (17) there results that $\lim\limits_{n\to\infty}\mathcal{P}_n\left(\psi^2(t);x\right) = \psi^2(x) = 0$. Consequently,

$$\lim_{n\to\infty} n^2\mathcal{P}_n\left(\psi(t)\varphi_x^2(t);x\right) = 0 \tag{20}$$

and by (19) and (20) follows (18). □

Now we estimate the rate of convergence given by (18).

Theorem 4. *Let $f \in L_q^{\infty,2}$ with a fixed $q > 0$. Then*

$$\left\|n^2\left[\mathcal{P}_n(f) - f\right] - f''\right\|_{\infty,q} \le 4(1+q)^4\omega_1\Big(f''; L_q^\infty; \frac{1}{n}\Big) \tag{21}$$

for $n \ge q+1$.

Proof. For $f \in L_q^{\infty,2}$ and $x, t \in \mathbb{R}$ there holds the Taylor-type formula

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + (t-x)^2 I(t,x)\,,$$

where

$$I(t,x) := \int_0^1 (1-u)\left[f''(x+u(t-x)) - f''(x)\right] du. \tag{22}$$

Using operator $\mathcal{P}_n$, $n \geq q+1$, and (13)-(15), we get

$$\mathcal{P}_n(f(t);x) = f(x) + n^{-2} f''(x) + \mathcal{P}_n\left(\varphi_x^2(t) I(t,x);x\right)\,,$$

which implies that

$$\left| n^2 \left[\mathcal{P}_n(f;x) - f(x)\right] - f''(x)\right| \leq n^2 \mathcal{P}_n\left(\varphi_x^2(t)|I(t,x)|;x\right)\,, \tag{23}$$

for $x \in \mathbb{R}$ and $n \geq q+1$. Now, applying (6), (8) and (9), we get from (22):

$$\begin{aligned}
|I(t,x)| &\leq \int_0^1 (1-u)\omega_1\left(f''; L_q^\infty; u|t-x|\right) e^{q|x|}\, du \\
&\leq \frac{1}{2}\omega_1\left(f''; L_q^\infty; |t-x|\right) e^{q|x|} \\
&\leq \frac{1}{2}\omega_1\left(f''; L_q^\infty; \frac{1}{n}\right)(1+n|t-x|)\, e^{q|x|+q|t-x|}\,.
\end{aligned}$$

and next by (4) and (15) we can write

$$\begin{aligned}
n^2 v_q(x)\mathcal{P}_n\left(\varphi_x^2(t)|I(t,x)|;x\right) &\leq \frac{n^2}{2}\omega_1\Big(f''; L_q^\infty; \frac{1}{n}\Big) \\
&\quad \times\left\{\mathcal{P}_n\big((t-x)^2 e^{q|t-x|};x\big) + n\mathcal{P}_n\big(|t-x|^3 e^{q|t-x|};x\big)\right\} \\
&= \omega_1\Big(f''; L_q^\infty; \frac{1}{n}\Big)\Big(\frac{n^3}{(n-q)^3} + \frac{3n^4}{(n-q)^4}\Big) \\
&\leq 4(1+q)^4 \omega_1\Big(f''; L_q^\infty; \frac{1}{n}\Big) \quad \text{for} \quad x \in \mathbb{R}\,,\ n \geq q+1\,.
\end{aligned}$$

Now the estimate (21) is obvious by (23), the last inequality and (5). □

Theorem 5. *Suppose that $f \in L_q^{\infty,r}$ with fixed $q > 0$ and $r \in \mathbb{N}$. Then*

$$\|\mathcal{P}_n^{(r)}(f) - f^{(r)}\|_{\infty,q} \leq \frac{5}{2}(1+3q)^3 \omega_2\Big(f^{(r)}; L_q^\infty; \frac{1}{n}\Big) \tag{24}$$

for $n \geq 3q+1$.

Proof. If $f \in L_q^{\infty,r}$, then for the r-th derivative of $\mathcal{P}_n(f)$ we have by Theorem 1, (13) and (7):

$$\begin{aligned}
\mathcal{P}_n^{(r)}(f;x) - f^{(r)}(x) &= \frac{n}{2}\int_{\mathbb{R}} \left[f^{(r)}(x+t) - f^{(r)}(x)\right] e^{-n|t|}\, dt \\
&= \frac{n}{2}\int_0^\infty \left[\Delta_t^2 f^{(r)}(x-t)\right] e^{-nt}\, dt\,.
\end{aligned}$$

From this and by (6), (9) and (12) we deduce that

$$\begin{aligned}\|\mathcal{P}_n^{(r)}(f)-f^{(r)}\|_{\infty,q} &\leq \frac{n}{2}\int_0^\infty \omega_2\left(f^{(r)};L_q^\infty;t\right)e^{-(n-q)t}\,dt\\ &\leq \omega_2\Big(f^{(r)};L_q^\infty;\frac{1}{n}\Big)\frac{n}{2}\int_0^\infty (1+nt)^2 e^{-(n-3q)t}\,dt\\ &= \omega_2\Big(f^{(r)};L_q^\infty;\frac{1}{n}\Big)\Big\{\frac{n}{2(n-3q)}+\frac{n^2}{(n-3q)^2}+\frac{n^3}{(n-3q)^3}\Big\}\end{aligned}$$

for $n\geq 3q+1$, which yields the estimate (24). □

3. Some properties of $\mathcal{P}_{n;r}$

3.1. The formulas (1)–(3) show that the operators $\mathcal{P}_{n;r}$, $r\in\mathbb{N}_0$, generalize $\mathcal{P}_n$ and $\mathcal{P}_{n;0}(f)\equiv\mathcal{P}_n(f)$ for $f\in L_q^{p,0}$. By this fact and Section 1, we shall consider $\mathcal{P}_{n;r}$ for $r\in\mathbb{N}$ only.

Lemma 4. *Let $1\leq p\leq\infty$, $q>0$ and $k\in\mathbb{N}$ be fixed numbers. Then for every $f\in L_q^{p,r}$ and $n\geq q+1$ there holds*

$$\|\mathcal{P}_{n;r}(f)\|_{p,q}\leq (1+q)\sum_{j=0}^{r}\|f^{(j)}\|_{p,q}\,. \tag{25}$$

The formulas (1)–(3) and the inequality (24) show that $\mathcal{P}_{n;r}$, $n\geq q+1$, is a linear operator acting from $L_q^{p,r}$ to L_q^p.

Proof. Let $1\leq p<\infty$. Then, by (1)–(3), the Minkowski inequality and (12), we get for $f\in L_q^{p,r}$ and $n\geq q+1$:

$$\begin{aligned}\|\mathcal{P}_{n;r}(f)\|_{p,q} &\leq \sum_{j=0}^{r}\frac{1}{j!}\|\mathcal{P}_n\left(f^{(j)}(t)\varphi_x^j(t);\cdot\right)\|_{p,q}\\ &\leq \sum_{j=0}^{r}\frac{1}{j!}\Big(\int_{\mathbb{R}}\Big|e^{-q|x|}\frac{n}{2}\int_{\mathbb{R}}t^j f^{(j)}(x+t)e^{-n|t|}\,dt\Big|^p dx\Big)^{1/p}\\ &\leq \sum_{j=0}^{r}\frac{n}{2j!}\int_{\mathbb{R}}|t|^j e^{-n|t|}\Big(\int_{\mathbb{R}}\Big|e^{-q|x|}f^{(j)}(x+t)\Big|^p dx\Big)^{1/p}dt\\ &\leq \sum_{j=0}^{r}\frac{n}{2j!}\|f^{(j)}\|_{p,q}\int_{\mathbb{R}}|t|^j e^{-(n-q)|t|}dt\\ &= \sum_{j=0}^{r}\|f^{(j)}\|_{p,q}\frac{n}{(n-q)^{j+1}}\leq (1+q)\sum_{j=0}^{r}\|f^{(j)}\|_{p,q}\,.\end{aligned}$$

The proof of (25) for $p=\infty$ is similar. □

3.2. First we shall prove an analogy of Theorem 2.

Theorem 6. *Suppose that $f \in L_q^{p,r}$ with fixed $1 \le p \le \infty$, $q > 0$ and $r \in \mathbb{N}$. Then*

$$\|\mathcal{P}_{n;r}(f) - f\|_{p,q} \le M_1 n^{-r} \omega_1\Big(f^{(r)}; L_q^p; \frac{1}{n}\Big) \tag{26}$$

for every $n \ge q+1$, where $M_1 = (r+2)(1+2q)^{r+2}$.

Proof. For every $f \in L_q^{p,r}$ and $x, t \in \mathbb{R}$ there holds the following Taylor-type formula:

$$f(x) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^r}{(r-1)!} I_r(t,x)\,, \tag{27}$$

where

$$I_r(t,x) := \int_0^1 (1-u)^{r-1} \left[f^{(r)}(t + u(x-t)) - f^{(r)}(t) \right] du\,. \tag{28}$$

From (27), (28) and (3) there results that

$$F_r(t,x) = f(x) - \frac{(x-t)^r}{(r-1)!} I_r(t,x)\,,$$

and next by (2), (13) and (7) it follows that

$$\begin{aligned} \mathcal{P}_{n;r}(f;x) - f(x) &= \frac{(-1)^{r+1}}{(r-1)!}\mathcal{P}_n\left((t-x)^r I_r(t,x); x\right) \\ &= \frac{(-1)^{r+1} n}{2(r-1)!} \int_{\mathbb{R}} \Big(t^r \int_0^1 (1-u)^{r-1} \Delta_{-ut}^1 f^{(r)}(x+t) du \Big) e^{-n|t|} dt \end{aligned} \tag{29}$$

for $x \in \mathbb{R}$ and $n \ge 2q+1$.

If $1 \le p < \infty$, then using the Minkowski inequality and (5)–(9) and (12), we get from (29):

$$\begin{aligned} &\|\mathcal{P}_{n;r}(f) - f\|_{p,q} \\ &= \frac{n}{2(r-1)!} \Big(\int_{\mathbb{R}} |e^{-q|x|} \int_{\mathbb{R}} t^r e^{-n|t|} \Big(\int_0^1 (1-u)^{r-1} \Delta_{-ut}^1 f^{(r)}(x+t)\, du \Big)\, dt|^p dx \Big)^{1/p} \\ &\le \frac{n}{2(r-1)!} \int_{\mathbb{R}} |t|^r e^{-(n-q)|t|} \Big(\int_0^1 (1-u)^{r-1} \|\Delta_{-ut}^1 f^{(r)}(\cdot)\|_{p,q} du \Big)\, dt \\ &\le \frac{n}{2(r-1)!} \int_{\mathbb{R}} |t|^r e^{-(n-q)|t|} \Big(\int_0^1 (1-u)^{r-1} \omega_1\left(f^{(r)}; L_q^p; u|t|\right) du \Big)\, dt \\ &\le \frac{n}{2r!} \int_{\mathbb{R}} |t|^r e^{-(n-q)|t|} \omega_1\left(f^{(r)}; L_q^p; |t|\right)\, dt \\ &\le \frac{n}{r!} \omega_1\Big(f^{(r)}; L_q^p; \frac{1}{n}\Big) \int_0^\infty t^r (1+nt) e^{-(n-2q)t}\, dt \\ &= \omega_1\Big(f^{(r)}; L_q^p; \frac{1}{n}\Big) \Big(\frac{n}{(n-2q)^{r+1}} + \frac{(1+r)n^2}{(n-2q)^{r+2}} \Big) \end{aligned}$$

for $n \ge 2q+1$, which implies (26) for $1 \le p < \infty$.

The proof of (26) for $f \in L_q^{\infty,r}$ is analogous. $\square$

From Theorem 6 we can derive the following

Corollary 2. *If $f \in L_q^{p,r}$, $1 \le p \le \infty$, $q > 0$ and $r \in \mathbb{N}$, then*

$$\lim_{n\to\infty} n^r \left\|\mathcal{P}_{n;r}(f) - f\right\|_{p,q} = 0\,.$$

Moreover, if $f^{(r)} \in \mathrm{Lip}_M^1\left(L_q^p;\alpha\right)$ with some $0 < \alpha \le 1$ and $M > 0$ then

$$\|\mathcal{P}_{n;r}(f) - f\|_{p,q} = O\left(n^{-r-\alpha}\right) \quad as \quad n \to \infty\,.$$

Arguing analogously to the proof of Theorem 2 given in paper [12] and applying Corollary 1, we can obtain the following Voronovskaya-type theorem for operators $\mathcal{P}_{n;r}$.

Theorem 7. *Let $f \in L_q^{\alpha,r}$ with fixed $r \in \mathbb{N}$ and $q > 0$. Then*

$$\begin{aligned}\mathcal{P}_{n;r}(f;x) - f(x) = {}& \frac{(-1)^r - 1}{2n^{r+1}} f^{(r+1)}(x) \\ & + \frac{(r+1)[1+(-1)^r]}{2n^{r+2}} f^{(r+2)}(x) + o(n^{-r-2}) \quad as \quad n \to \infty\,,\end{aligned}$$

at every $x \in \mathbb{R}$. In particular, if r is even number, then

$$\lim_{n\to\infty} n^{r+2}\left[\mathcal{P}_{n;r}(f;x) - f(x)\right] = (r+1)f^{(r+2)}(x) \tag{30}$$

at every $x \in \mathbb{R}$.

Similarly to Theorem 4 now we shall estimate the rate of convergence given by (30).

Theorem 8. *Let $q > 0$ and even number $r \in \mathbb{N}$ be fixed. Then for every $f \in L_q^{\infty,r+2}$ and $n \ge 2q+1$ there holds*

$$\|n^{r+2}\left[\mathcal{P}_{n;r}(f) - f\right] - (r+1)f^{(r+2)}\|_{p,q} \le M_2\omega_1\Big(f^{(r+2)}; L_q^\infty; \frac{1}{n}\Big)\,, \tag{31}$$

where $M_2 = (1+2q)^{r+4}(r+4)^2$.

Proof. Similarly to the proof of Theorem 6 we use the Taylor-type formula of $f \in L_q^{\infty,r+2}$:

$$f(x) = \sum_{j=0}^{r+2} \frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^{r+2}}{(r+1)!} I_1(t,x)\,, \tag{32}$$

for $x, t \in \mathbb{R}$, where

$$I_1(t,x) := \int_0^1 (1-u)^{r+1}\left[f^{(r+2)}(t+u(x-t)) - f^{(r+2)}(t)\right] du\,. \tag{33}$$

Analogously for $f^{(r+1)} \in L_q^{\infty,1}$ and $x, t \in \mathbb{R}$ we have

$$f^{(r+1)}(t) = f^{(r+1)}(x) + f^{(r+2)}(x)(t-x) + (t-x)I_2(t,x) \tag{34}$$

with

$$I_2(t,x) := \int_0^1 \left[f^{(r+2)}(x+u(t-x)) - f^{(r+2)}(x) \right] du\,. \tag{35}$$

By (3) and (34) the formula (32) can be rewritten in the form:

$$\begin{aligned} f(x) = F_r(t,x) &+ \frac{(x-t)^{r+1}}{(r+1)!} f^{(r+1)}(x) \\ &+ \Big(\frac{1}{(r+2)!} - \frac{1}{(r+1)!} \Big) f^{(r+2)}(x)(x-t)^{r+2} \\ &- \frac{(x-t)^{r+2}}{(r+1)!} I_2(t,x) + \frac{(x-t)^{r+2}}{(r+2)!} \left[f^{(r+2)}(t) - f^{(r+2)}(x) \right] \\ &+ \frac{(x-t)^{r+2}}{(r+1)!} I_1(t,x) \quad \text{for} \quad x,t \in \mathbb{R}\,. \end{aligned} \tag{36}$$

Let now $x \in \mathbb{R}$ be a fixed point. Using operator $\mathcal{P}_n$ and (1)–(3) and (13)–(15), we get from (36):

$$f(x) = \mathcal{P}_{n;r}(f;x) - \frac{r+1}{n^{r+2}} f^{(r+2)}(x) + \sum_{i=1}^{3} T_i(x) \quad \text{for} \quad n \ge 2q+1\,,$$

where

$$\begin{aligned} T_1(x) :=& \frac{1}{(r+1)!} \mathcal{P}_n \left((t-x)^{r+2} I_2(t,x); x \right)\,, \\ T_2(x) :=& \frac{1}{(r+2)!} \mathcal{P}_n \left((t-x)^{r+2} \left[f^{(r+2)}(t) - f^{(r+2)}(x) \right]; x \right)\,, \\ T_3(x) :=& \frac{1}{(r+1)!} \mathcal{P}_n \left((t-x)^{r+2} I_1(t,x); x \right)\,. \end{aligned}$$

Consequently we have

$$\| n^{r+2} \left[\mathcal{P}_{n,r}(f) - f \right] - (r+1) f^{(r+2)} \|_{\infty,q} \le n^{r+2} \sum_{i=1}^{3} \|T_i\|_{\infty,q}\,. \tag{37}$$

From (35) and (6)–(9) it follows that

$$\begin{aligned} v_q(x)|T_1(x)| \le& \frac{e^{-q|x|}}{(r+1)!} \mathcal{P}_n \left(|t-x|^{r+2} |I_2(t,x)|; x \right) \\ \le& \frac{1}{(r+1)!} \mathcal{P}_n \left(|t-x|^{r+2} \omega_1 \left(f^{(r+2)}; L_q^\infty; |t-x| \right); x \right) \\ \le& \frac{1}{(r+1)!} \omega_1 \Big(f^{(r+2)}; L_q^\infty; \frac{1}{n} \Big) \\ & \times \left[\mathcal{P}_n \left(|t-x|^{r+1} e^{q|t-x|}; x \right) + n \mathcal{P}_n \left(|t-x|^{r+2} e^{q|t-x|}; x \right) \right] \end{aligned}$$

and further by (15) we have

$$\|T_1\|_{\infty,q} \le \frac{1}{(r+1)!}\omega_1\Big(f^{(r+2)};L_q^{\infty};\frac{1}{n}\Big)\Big[\frac{n(r+2)!}{(n-q)^{r+3}}+\frac{n^2(r+3)!}{(n-q)^{r+4}}\Big]. \tag{38}$$

Analogously, by (6)–(9) there results that

$$\begin{aligned}\big|f^{(r+2)}(t)-f^{(r+2)}(x)\big| &\le \omega_1\big(f^{(r+2)};L_q^{\infty};|t-x|\big)e^{q|x|}\\ &\le e^{q|x|+q|t-x|}\big(1+n|t-x|\big)\omega_1\Big(f^{(r+2)};L_q^{\infty};\frac{1}{n}\Big)\end{aligned}$$

and from (33):

$$\begin{aligned}|I_1(t,x)| &\le \int_0^1 (1-u)^{r+1}\omega_1\left(f^{(r+2)};L_q^{\infty};u|t-x|\right)e^{q|t|}\,du\\ &\le e^{q|t|}\omega_1\left(f^{(r+2)};L_q^{\infty};|t-x|\right)\int_0^1(1-u)^{r+1}\,du\\ &\le \frac{1}{r+2}e^{q|t|+q|t-x|}\left(1+n|t-x|\right)\omega_1\Big(f^{(r+2)};L_q^{\infty};\frac{1}{n}\Big).\end{aligned}$$

Using the above inequalities and (15), we deduce that

$$\|T_2\|_{\infty,q} \le \omega_1\Big(f^{(r+2)};L_q^{\infty};\frac{1}{n}\Big)\Big[\frac{n}{(n-q)^{r+3}}+\frac{(r+3)n^2}{(n-q)^{r+4}}\Big] \tag{39}$$

and

$$\|T_3\|_{\infty,q} \le \omega_1\Big(f^{(r+2)};L_q^{\infty};\frac{1}{n}\Big)\Big[\frac{n}{(n-2q)^{r+3}}+\frac{(r+3)n^2}{(n-2q)^{r+4}}\Big], \tag{40}$$

for $n \ge 2q+1$. Summarizing (37)–(40), we immediately obtain the desired inequality (31). □

Remarks 1. Theorem 6 shows that the order of approximation of function $f \in L_q^{p,r}$ by $\mathcal{P}_{n;r}(f)$ is dependent on r and it improves if r grows. Moreover, Theorem 6 and Theorem 2 show that the operators $\mathcal{P}_{n;r}$ with $r \ge 2$ have better approximation properties than $\mathcal{P}_n$ for $f \in L_q^{p,r}$.

We mention also that the similar theorems can be obtained for the Gauss-Weierstrass operators

$$W_n(f;x) := \sqrt{n/\pi}\int_{\mathbb{R}} f(x-t)e^{-nt^2}dt, \quad x\in\mathbb{R},\ n\in\mathbb{N},$$

defined in exponential weighted spaces $L_q^p(\mathbb{R})$ with the weighted function $v_q(x) = e^{-qx^2}, q>0$.

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