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A GENERALIZATION OF STEENROD'S APPROXIMATION THEOREM

Christoph Wockel

Abstract. In this paper we aim for a generalization of the Steenrod Approximation Theorem from [16, Section 6.7], concerning a smoothing procedure for sections in smooth locally trivial bundles. The generalization is that we consider locally trivial smooth bundles with a possibly infinite-dimensional typical fibre. The main result states that a continuous section in a smooth locally trivial bundles can always be smoothed out in a very controlled way (in terms of the graph topology on spaces of continuous functions), preserving the section on regions where it is already smooth.

1. Introduction

This paper generalises a result of Steenrod on a very nice smoothing procedure for sections in locally trivial smooth bundles. It puts together ideas from [16, Section 6.7], [6, Chapter 2] and [15, Section A.3] and tries to produce a theorem of maximal generality out of them.

Theorem (Generalised Steenrod Approximation Theorem)**.** *Let M be a finite-dimensional connected manifold with corners, $\pi\colon E \to M$ be a locally trivial smooth bundle with a locally convex manifold N as typical fibre and $\sigma\colon M \to E$ be a continuous section. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that σ is smooth on a neighbourhood of $L \setminus U$, then for each open neighbourhood O of $\sigma(M)$ in E, there exists a section $\tau\colon M \to O$ which is smooth on a neighbourhood of L and equals σ on $M \setminus U$. Furthermore, there exists a homotopy $F\colon [0,1] \times M \to O$ between σ and τ such that each $F(t,\cdot)$ is a section of π and $F(t,x) = \sigma(x) = \tau(x)$ if $(t,x) \in [0,1] \times (M \setminus U)$.*

This theorem is of maximal generality in the sense that the proof depends heavily on the local compactness of M an the local convexity of N. Also, the topology kept in mind is the graph topology on spaces of continuous function, which is rather fine, e.g., in comparison to the compact-open topology. Thus there seems to be no result of greater generality (e.g., for arbitrary base-spaces and arbitrary fibres) which can be shown with the same method of proof.

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The paper is organized as follows. The first definitions and remarks introduce the setting of calculus on locally convex vector spaces and manifolds (with and without corners) modelled on such spaces. We then recall some basic constructions on the smoothing procedure for continuous functions with values in locally convex spaces, which we shall need in the proof of the main theorem. After having proved the main theorem, we formulate some immediate consequences of it, concerning the relation of smooth and continuous sections and functions.

Eventually, we arrive at the analogue result from [8] in the locally convex setting, stating that smooth and continuous homotopies into locally convex manifolds agree. However, our method of proof is different from the one used in [8] since it uses heavily the existence of charts onto convex subsets, which are not available for a convenient manifold in general. This result is quite interesting, because it has nice applications in bundle theory [13].

Definition 1 (cf. [5], [12] and [4])**.** Let X and Y be locally convex spaces and $U \subseteq X$ be open. Then $f\colon U \to Y$ is called *continuously differentiable* or C^1 if it is continuous, for each $v \in X$ the differential quotient

$$df(x) \cdot v := \lim_{h \to 0} \frac{1}{h}\big(f(x+hv) - f(x)\big)$$

exists and the map $df\colon U \times X \to Y$ is continuous. For $n > 1$ we, recursively define

$$d^n f(x)\colon (v_1, \ldots, v_n) := \lim_{h \to 0} \frac{1}{h}\left(d^{n-1} f(x+h) \cdot (v_1, \ldots, v_{n-1}) - d^{n-1} f(x) \cdot (v_1, \ldots, v_n)\right)$$

and say that f is C^n if $d^k f\colon U \times X^k \to Y$ exists for all $k = 1, \ldots, n$ and is continuous. We say that f is C^∞ or *smooth* if it is C^n for all $n \in \mathbb{N}$.

From this definition, the notion of a *locally convex manifold* is clear, i.e., a Hausdorff space such that each point has neighbourhood that is homeomorphic to an open subset of some locally convex space such that the corresponding coordinate changes are smooth. Together with such a fixed differentiable structure on M, we speak of M as a locally convex manifold.

Remark 2. In order to relate our results to other frequently used concepts of differential calculus on infinite-dimensional vector spaces and infinite-dimensional manifolds, we shortly line out the relation to our setting (cf. [7] for a more exhaustive comparison, where smooth maps in our setting are called C^∞_c-maps). In the case of Banach-spaces X and Y, a map is called *Fréchet differentiable* if it is differentiable in the sense of Definition 1 and the differential $x \mapsto df(x)$ is a continuous map into the space of bounded linear operators $B(X,Y)$, endowed with the norm topology. Thus, Fréchet differentiable (resp. smooth) maps are differentiable (resp. smooth) in our setting.

Next, we recall the basic definitions of the convenient calculus from [9]. Let X and Y be arbitrary locally convex spaces. A curve $f\colon \mathbb{R} \to X$ is called smooth if it is smooth in the sense of Definition 1. Then the c^∞-topology on X is the final topology induced from all smooth curves $f \in C^\infty(\mathbb{R}, X)$. If X is a Fréchet space, then the c^∞-topology is again a locally convex vector topology which coincides with the original topology [8, Theorem 4.11]. If $U \subseteq X$ is c^∞-open, then $f : U \to Y$ is said to be of class C^∞ or smooth if

$$f_* \left(C^\infty(\mathbb{R}, U)\right) \subseteq C^\infty(\mathbb{R}, Y),$$

i.e., if f maps smooth curves to smooth curves. The chain rule [3, Proposition 1.15] implies that each smooth map in the sense of Definition 1 is smooth in the convenient sense. On the other hand, [8, Theorem 12.8] implies that on a Fréchet space a smooth map in the convenient sense is smooth in the sense of Definition 1. Hence for Fréchet spaces, this notion coincides with the one from Definition 1.

Definition 3. A *d-dimensional manifold with corners* is a paracompact Hausdorff space such that each point has a neighborhood that is homeomorphic to an open subset of

$$\mathbb{R}^d_+ = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i = 1, \ldots, d\right\}$$

and such that the corresponding coordinate changes are smooth (cf. [10]). The crucial point here is the notion of smoothness for non-open domains. The usual notion is to define a map $f \colon A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ to be smooth if for each $x \in A$, there exists a neighborhood U_x of x which is open in $\mathbb{R}^n$, and a smooth map $f_x \colon U_x \to \mathbb{R}^m$ such that $f_x|_{A \cap U_x} = f|_{A \cap U_x}$.

A more general concept of manifolds with corners modeled on locally convex spaces can be found in [11], [17] and [4], along with the appropriate definitions of differentiable or smooth functions. Basically, in this setting, a map on a non-open domain with dense interior is defined to be smooth if it is smooth on the interior and differentials extend continuously to the boundary.

Remark 4. We recall some basic facts from general topology. A topological space X is called *paracompact* if each open cover has a locally finite refinement. If X is the union of countably many compact subsets, then it is called *σ-compact*, and if each open cover has a countable subcover, it is called *Lindelöf*.

Now, let M be a finite-dimensional manifold with corners, which is in particular locally compact and locally connected. For these spaces, [2, Theorems XI.7.2+3] imply that M is paracompact if and only if each component is σ-compact, equivalently, Lindelöf. Furthermore, since paracompact spaces are normal, M is normal in each of these cases.

One very important fact on M is that it permits smooth partitions of unity (c.f. [6, Theorem 2.1]). That means that for each locally finite open cover $(V_i)_{i \in I}$ we find smooth functions $\lambda_i \colon M \to [0, 1]$ such that $\operatorname{supp}(\lambda_i) \subseteq V_i$ and $\sum_{i \in I} \big(\lambda_i(x)\big) = 1$.

Definition 5. If X is a Hausdorff space and Y is a topological space, then $C(X, Y)_{\text{c.o.}}$ is the space of continuous functions from X to Y, endowed with the compact-open topology (cf. [1, Section X.3.4]). A basic open set in this topology is given by $\lfloor C_1, W_1 \rfloor \cap \cdots \cap \lfloor C_n, W_n \rfloor$ for $C_1, \ldots, C_n \subseteq X$ compact and $W_1, \ldots, W_n \subseteq Y$ open, where

$$\lfloor C, W \rfloor := \left\{f \in C(X, Y) : f(C) \subseteq W\right\}.$$

Remark 6. If Y happens to be a topological group, then this topology coincides with the topology of compact convergence [1, Theorem X.3.4.2] and thus $C(X,Y)_{\text{c.o.}}$ is a topological group itself. If Y is a locally convex space, then $C(X,Y)_{\text{c.o.}}$ is again a locally convex space space with respect to pointwise operations.

If Y is a locally compact space, then the exponential law yields that the canonical map $\varphi\colon C(X,C(Y,Z)) \to C(X\times Y,Z)$, $\varphi(f)(x,y)=f(x)(y)$ is a homeomorphism [1, Section X.3.4].

A finer topology on $C(X,Y)$ is the graph topology, which we term $C(X,Y)_\Gamma$ (cf. [14]). A basic open set in this topology is given by $\Gamma_U := \left\{f\in C(X,Y) : \Gamma(f)\subseteq U\right\}$, where $\Gamma(f)$ denotes the graph of f in $X\times Y$ and $U\subseteq X\times Y$ is open.

Proposition 7. *If M is a finite-dimensional σ-compact manifold with corners, then for each locally convex space Y the space $C^\infty(M,Y)$ is dense in $C(M,Y)_{\text{c.o.}}$. If $f\in C(M,Y)$ has compact support and U is an open neighbourhood of* $\operatorname{supp}(f)$*, then each neighbourhood of f in $C(M,Y)$ contains a smooth function whose support is contained in U.*

Proof. The proof of [15, Theorem A.3.1] carries over. □

Corollary 8. *If M is a finite-dimensional σ-compact manifold with corners and V is an open subset of the locally convex space Y, then $C^\infty(M,V)$ is dense in $C(M,V)_{\text{c.o.}}$.*

Lemma 9. *Let M be a finite-dimensional σ-compact manifold with corners, Y be a locally convex space, $W\subseteq Y$ be open and convex and $f\colon M\to W$ be continuous. If $L\subseteq M$ is closed and $U\subseteq M$ is open such that f is smooth on a neighbourhood of $L\setminus U$, then each neighbourhood of f in $C(M,W)_{\text{c.o.}}$ contains a continuous map $g\colon M\to W$, which is smooth on a neighbourhood of L and which equals f on $M\setminus U$.*

Proof. (cf. [6, Theorem 2.5])

Let $A\subseteq M$ be an open set containing $L\setminus U$ such that $f\big|_A$ is smooth. Then $L\setminus A\subseteq U$ is closed in M, and, since M is normal (cf. Remark 4), there exists $V\subseteq U$ open with $L\setminus A\subseteq V\subseteq \overline{V}\subseteq U$. Then $\{U, M\setminus\overline{V}\}$ is an open cover of M, and there exists a smooth partition of unity $\{\lambda_1,\lambda_2\}$ subordinated to this cover. Then

$$G_f\colon C(M,W)_{\text{c.o.}}\to C(M,W)_{\text{c.o.}}\,,\quad G_f(\gamma)(x)=\lambda_1(x)\gamma(x)+\lambda_2(x)f(x)$$

is continuous since $\gamma\mapsto\lambda_1\gamma$ and $\lambda_1\gamma\mapsto\lambda_1\gamma+\lambda_2 f$ are continuous.

If γ is smooth on $A\cup V$ then so is $G_f(\gamma)$, because λ_1 and λ_2 are smooth, f is smooth on A and $\lambda_2\big|_V\equiv 0$. Note that $L\subseteq A\cup(L\setminus A)\subseteq A\cup V$, so that $A\cup V$ is an open neighbourhood of L. Furthermore, we have $G_f(\gamma)=\gamma$ on V and $G_f(\gamma)=f$ on $M\setminus U$. Since $G_f(f)=f$, there is for each open neighbourhood O of f an open neighbourhood O' of f such that $G_f(O')\subseteq O$. By Corollary 8 there is a smooth function $h\in O'$ such that $g:=G_f(h)$ has the desired properties. □

Lemma 10. *Let M be a finite-dimensional σ-compact manifold with corners, N be a smooth manifold, modelled on a locally convex space, $W\subseteq N$ be diffeomorphic to an open convex subset of the modelling space of N and $f\colon M\to W$ be continuous.*

If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L \setminus U$, then each neighbourhood of f in $C(M,W)_{\mathrm{c.o.}}$ contains a map $g\colon M \to W$ which is smooth on a neighbourhood of L and which equals f on $M \setminus U$.

Proof. Let $\varphi\colon W \to \varphi(W)$ be a diffeomorphism. If $\lfloor C_1, V_1\rfloor \cap \ldots \cap \lfloor C_n, V_n\rfloor$ is an open neighbourhood of $f \in C(M,W)_{\mathrm{c.o.}}$, then $\lfloor C_1, \varphi(V_1)\rfloor \cap \ldots \cap \lfloor C_n, \varphi(V_n)\rfloor$ is an open neighbourhood of $\varphi \circ f$ in $C(M, \varphi(W))_{\mathrm{c.o.}}$. We apply Lemma 9 to this open neighbourhood to obtain a map h. Then $g := \varphi^{-1} \circ h$ has the desired properties. $\square$

Theorem 11 (Generalised Steenrod Approximation Theorem)**.** *Let M be a finite-dimensional connected manifold with corners, $\pi\colon E \to M$ be a locally trivial smooth bundle with a locally convex manifold N as typical fibre and $\sigma\colon M \to E$ be a continuous section. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that σ is smooth on a neighbourhood of $L \setminus U$, then for each open neighbourhood O of $\sigma(M)$ in E, there exists a section $\tau\colon M \to O$ which is smooth on a neighbourhood of L and equals σ on $M \setminus U$. Furthermore, there exists a homotopy $F\colon [0,1] \times M \to O$ between σ and τ such that each $F(t,\cdot)$ is a section of π and $F(t,x) = \sigma(x) = \tau(x)$ if $(t,x) \in [0,1] \times (M \setminus U)$.*

Proof. (cf. [16, Section 6.7]) We describe roughly how the proof is going to work. After choosing an appropriate cover $(V_i)_{i\in\mathbb{N}}$ of M in the beginning, we shall inductively construct sections τ_i of π, that become smooth on increasing subsets of M. To avoid convergence considerations, we construct τ stepwise from the τ_i in the end.

We claim that there exist locally finite open covers $(V_i)_{i\in\mathbb{N}_0}$, $(V_i')_{i\in N_0}$ of M, $(W_i)_{i\in\mathbb{N}_0}$ of $\sigma(M)$ and a collection $(Z_i)_{i\in\mathbb{N}_0}$ of open subsets of N which are diffeomorphic to convex open subsets of the modelling space of N, such that we have

- $\overline{V_i'}$ and $\overline{V_i}$ are compact
- $\overline{V_i'} \subseteq V_i$ and $W_i \subseteq O$
- $\sigma(\overline{V_i}) \subseteq W_i$ (which is equivalent to $\overline{V_i} \subseteq \pi(W_i)$ and implies $\overline{V_i} \subseteq \pi(W_i)$)
- the restricted bundle $\pi\big|_{\pi(W_i)}$ is trivial and there exist smooth trivializations $\varphi_i\colon \pi^{-1}\big(\pi(W_i)\big) \to \pi(W_i) \times Y$ such that $\varphi_i\big(\pi^{-1}(V_i)\big) = V_i \times Z_i$

for each $\in \mathbb{N}_0$. First, we recall the properties of the topology on M from Remark 4. Now, let $(S_t)_{t\in T}$ be a trivialising cover of M and $\varphi_t\colon \pi^{-1}(S_t) \to S_t \times N$ be the corresponding local trivializations. That means, each φ_t is a diffeomorphism satisfying $\mathrm{pr}_1\big(\varphi_t(x)\big) = \pi(x)$ for all $x \in \pi^{-1}(S_t)$. Then each $x \in M$ is in $S_{t(x)}$ for some map $M \ni x \mapsto t(x) \in T$. Furthermore, there exist open neighbourhoods $U_x \subseteq S_{t(x)}$ of x and $Z_x \subseteq N$ of $\mathrm{pr}_2\big(\varphi_{t(x)}(\sigma(x))\big) \in N$ such that Z_x is diffeomorphic to an open convex subset of the modelling space of N and $\varphi^{-1}_{(t(x))}(U_x \times Z_x) \subseteq O$. Since M is normal, each $x \in M$ has a relatively compact open neighbourhood V_x such that $\overline{V_x} \subseteq U_x$ and $\mathrm{pr}_2\big(\varphi_{t(x)}(\sigma(\overline{V_x}))\big) \subseteq Z_x$. Furthermore, let V_x' be an open neighbourhood of x such that $\overline{V_x'} \subseteq V_x$. As M is paracompact, $(V_x)_{x\in M}$ has a locally finite refinement $(V_j)_{j\in J}$. Since each $\overline{V_j}$ is covered by finitely many

$V'_{x_{j,1}}, \ldots, V'_{x_{j,k_j}}$, we deduce that

$$(V_j \cap V'_{x_{j,1}}, \ldots, V_j \cap V'_{x_{j,k_j}})_{j \in J}$$

is also a locally finite open cover of M. By re-defining the index set we thus get two locally finite open covers $(V_i)_{i\in I}$ and $(V'_i)_{i\in I}$ such that $\overline{V'_i}$ and $\overline{V_i}$ are compact and we have $\overline{V'_i} \subseteq V_i$ for each $i \in I$. In addition, we may assume that $I = \mathbb{N}^+$ for M is Lindelöf.

Since each $\overline{V_i}$ is contained in some $\overline{V_{x(i)}}$ of M and the values of $\mathrm{pr}_2 \circ \varphi_{t(x(i))} \circ \sigma$ on $\overline{V_{x(i)}}$ are contained in $Z_{x(i)}$, we get local trivializations $\varphi_i := \varphi_{t(x(i))}\big|_{\pi^{-1}(U_{x(i)})}$ and open subsets $Z_i := Z_{x(i)}$ of N and $W_i := \varphi^{-1}_{t(x(i))}(U_{x(i)} \times Z_i)$ satisfying all requirements.

We set $V_0 := \emptyset$ and $V'_0 := \emptyset$, and observe that $(V_i)_{i\in\mathbb{N}_0}$ and $(V'_i)_{i\in\mathbb{N}_0}$ are locally finite covers by their construction. Furthermore, we assume that σ is smooth on the open neighbourhood A of $L \setminus U$ and that A' is another open neighbourhood of $L \setminus U$ with $\overline{A'} \subseteq A$. Define

$$L_i := L \cap \overline{V'_i} \setminus (V'_0 \cup \ldots \cup V'_{i-1})$$

which is closed and contained in V_i. Since $L \setminus A' \subseteq U$ we have $L_i \setminus A' \subseteq V_i \cap U$ and there exist open subsets $U_i \subseteq V_i \cap U$ such that $L_i \setminus A' \subseteq U_i \subseteq \overline{U_i} \subseteq V_i \cap U$. We claim that there exist continuous sections $\tau_i \in C(M,E)$, $i \in \mathbb{N}_0$, satisfying

(a) $\tau_i = \tau_{i-1}$ on $M \setminus \overline{U_i}$ for all $i \in \mathbb{N}^+$,
(b) $\tau_i(M) \subseteq O$ and $\tau_i(\overline{V_j}) \subseteq W_j$ for all $i, j \in \mathbb{N}_0$,
(c) τ_i is smooth on a neighbourhood of $L_0 \cup \ldots \cup L_i \cup \overline{A'}$ for all $i \in \mathbb{N}_0$ and
(d) for each $i \in \mathbb{N}^+$ there exists a homotopy $F_{i-1}\colon [0,1] \times M \to O$ such that each $F(t,\cdot)$ is a section of π, $F_{i-1}(0,\cdot) = \tau_{i-1}$ and $F_{i-1}(1,\cdot) = \tau_i$, which is constantly $\tau_{i-1} = \tau_i$ on $[0,1] \times (M \setminus \overline{U_i})$.

Condition (a) will ensure that we can construct τ stepwise from the τ_i, and condition (b) will ensure that we can view $\tau_{i-1}\big|_{V_i}$ as a Z_i-valued function on V_i and thus can apply Corollary 10 to $\tau_{i-1}\big|_{V_i}$ in order to construct τ_i. Finally, condition (c) will ensure the asserted smoothness property of τ, and condition (d) will enable us to construct the asserted homotopy.

For $i = 0$ we set $\tau_0 = \sigma$, which clearly satisfies conditions (a)-(c). Hence we assume that the τ_i are defined for $i < a$. We consider the set

$$Q := \left\{\gamma \in C(V_a, W_a) : \gamma = \tau_{a-1} \text{ on } V_a \setminus \overline{U_a}\right\},$$

which is a closed subspace of $C(V_a, W_a)_{\text{c.o.}}$. Then we have a well-defined map

$$G\colon Q \to C(M, W_a), \quad G(\gamma)(x) = \begin{cases} \gamma(x) & \text{if } x \in \overline{U_a} \\ \tau_{a-1}(x) & \text{if } x \in M \setminus \overline{U_a}. \end{cases}$$

Note that, by condition (b), we have $\tau_{a-1}(V_a) \subseteq W_a$, whence $\tau_{a-1}\big|_{V_a} \in Q$. Furthermore, $\tau_{a-1}(M) \subseteq O$ ensures

$$G(\gamma)(M) \subseteq O \quad \text{if} \quad \gamma(V_a) \subseteq O. \tag{1}$$

Since $(V_j)_{j\in\mathbb{N}_0}$ is locally finite and $\overline{V_j}$ is compact, the set $\{j \in \mathbb{N}_0 : \overline{U_a} \cap \overline{V_j} \neq \emptyset\}$ is finite and hence

$$O' = \bigcap_{j\in\mathbb{N}_0} \lfloor \overline{U_a} \cap \overline{V_j}, W_a \cap W_j \rfloor$$

is an open neighbourhood of $\tau_{a-1}\big|_{V_a}$ in $C(V_a, W_a)_{\text{c.o.}}$ by condition (b).

Since τ_{a-1} is a section, $\tau_{a-1}\big|_{V_a}$ is also a section of the restricted bundle $\pi_a := \pi\big|_{V_a}$. For π_a has the smooth trivialization φ_a, the space of sections of π_a is homeomorphic to $C(V_a, N)$ by the homeomorphism $\sigma' \mapsto H(\sigma') := \mathrm{pr}_2 \circ \varphi_a \circ \sigma'$ with inverse given by $f \mapsto \varphi_a^{-1} \circ (\mathrm{id} \times f)$
circ diag. This shows in particular that $H(\sigma')$ is smooth in a neighbourhood of $x \in V_a$ if and only if σ' is so.

We want to apply Lemma 10 to $g := H\big(\tau_{a-1}\big|_{V_a}\big) \in C(V_a, N)$ and claim for this reason that g takes values in some subset of N, diffeomorphic to a convex neighbourhood of its modelling space. This in turn is true, as $\varphi_a(W_a) \subseteq V_a \times Z_a$ and thus $g = \mathrm{pr}_2 \circ \varphi_a \circ \tau_{a-1}\big|_{V_a}$ takes values in Z_a by condition (b).

In order to construct τ_i, we now apply Lemma 10 to the manifold with corners V_a, its closed subset $L'_a := (L \cap \overline{V'_a}) \cup (\overline{A'} \cap V_a) \subseteq V_a$, the open set $U_a \subseteq V_a$, $g \in C(V_a, Z_a)$ and the open neighbourhood $H(O')$ of g. Due to the construction, we have $L_a \setminus U_a \subseteq A' \cap V_a$ and, furthermore, $L \cap \overline{V'_a} \subseteq L_0 \cup \ldots \cup L_a$. Hence we have

$$L'_a \setminus U_a \subseteq \big(L_0 \cup \cdots \cup L_{a-1} \cup (L_a \setminus U_a)\big) \cup (\overline{A'} \cap V_a \setminus U_a) \subseteq L_1 \cup \ldots \cup L_{a-1} \cup \overline{A'}$$

so that by condition (c), $\tau_{a-1}\big|_{V_a}$ and, consequently, g are smooth on a neighbourhood of $L'_a \setminus U_a$. We thus obtain a map $h \in H(O') \subseteq C(V_a, Z_a)$ which is smooth on a neighbourhood of L'_a. Furthermore, $H^{-1}(h)$ coincides with $\tau_{a-1}\big|_{V_a}$ on $V_a \setminus U_a \supseteq V_a \setminus \overline{U_a}$, because there h coincides with g and $h(x) = g(x)$ implies $H^{-1}(h)(x) = H^{-1}(g)(x) = \tau_{a-1}(x)$ for $x \in V_a$. As a consequence, $H^{-1}(h)$ is contained in $O' \cap Q$, and we set $\tau_a := G\big(H^{-1}(h)\big)$.

It remains to check that τ_a satisfies conditions (a)–(d). Since $H^{-1}(h)$ coincides with $\tau_{a-1}\big|_{V_a}$ on $V_a \setminus \overline{U_a}$, condition (a) is satisfied. From the construction we know that $H^{-1}(h)(V_a) \subseteq W_a \subseteq O$, which implies $G(H^{-1}(h))(M) \subseteq O$ by (1). In addition $H^{-1}(h) \in O'$, which implies in turn $H^{-1}(h)(\overline{U_a} \cap \overline{V_j}) \subseteq W_j$ and, furthermore, $\tau_a(\overline{V_j}) \subseteq W_j$. Eventually, condition (b) is fulfilled. Furthermore, τ_a inherits the smoothness properties from τ_{a-1} on $M \setminus \overline{U_a}$, from h on V_a and since $L_a \subseteq L \cap \overline{V'_a}$, condition (c) also holds. To construct F_{a-1}, we set

$$F_{a-1} \colon [0,1] \times M \to O\,,$$

$$(t,x) \mapsto \begin{cases} \tau_{a-1}(x) & \text{if} \quad x \notin V_a \\ \varphi_a^{-1}(x, (1-t)\cdot g(x) + t \cdot h(x)) & \text{if} \quad x \in V_a\,, \end{cases}$$

where the convex combination between $g(x)$ and $h(x)$ in Z_a has to be understood in local coordinates in the convex set which Z_a is diffeomorphic to. Since g equals h on $V_a \setminus \overline{U_a}$ and we have $\varphi_a^{-1}(x, g(x)) = \tau_{a-1}(x)$ for $x \in V_a$, this defines a continuous map which satisfies the requirements of condition (d). This finishes the construction of τ_a and thus the induction.

We next construct τ. First we set $m(x) := \max\big\{i : x \in \overline{V_i}\big\}$ and $n(x) := \max\big\{i : x \in V_i\big\}$. Then obviously $n(x) \leq m(x)$ and each $x \in M$ has a neighbourhood on which $\tau_{n(x)}, \ldots, \tau_{m(x)}$ coincide since $\overline{U_i} \subseteq V_i$ and $\tau_i = \tau_{i-1}$ on $M \backslash U_i$. Hence $\tau(x) := \tau_{n(x)}(x)$ defines a continuous function on M. If $x \in L$, then $x \in L_0 \cup \ldots \cup L_{n(x)}$ and thus τ is smooth on a neighbourhood of x. If $x \in M \setminus U$, then $x \notin U_1 \cup \ldots \cup U_{n(x)}$ and thus $\tau(x) = \sigma(x)$.

We finally construct the homotopy F. First observe that if $x \in M$ and $n > n(x)$, then $x \notin V_n \supseteq \overline{U_n}$ and $F_n(t,x) = \tau_{n-1}(x) = \tau_{n-2}(x) = \cdots = \tau_{n(x)}(x) = \tau(x)$ by condition (d). We set

$$F\colon (0,1] \times M \to O\,, \quad (t,x) \mapsto F_{n-1}\big((n+1)\cdot(1-nt), x\big) \quad \text{if} \quad t \in \left[\frac{1}{n}, \frac{1}{n+1}\right].$$

This is well-defined and continuous since $\big(\big[\frac{1}{n}, \frac{1}{n+1}\big] \times M\big)_{n \in \mathbb{N}^+}$ covers $(0,1] \times M$, and for $n \geq 2$ and

$$(t,x) \in \left(\left[\frac{1}{n-1}, \frac{1}{n}\right] \times M\right) \cap \left(\left[\frac{1}{n}, \frac{1}{n+1}\right] \times M\right)$$

we have $t = \frac{1}{n}$ and thus $F(t,x) = F_{n-2}(1,x) = \tau_{n-1}(x) = F_{n-1}(0,x)$ by condition (d). Furthermore, $F\big|_{\{1\}\times M} = F(0,\cdot) = \sigma$. We extend F to $[0,1] \times M$ by setting $F(0,x) = \tau(x)$. This is in fact a continuous extension since each x is contained in its open neighbourhood $V_{m(x)}$ and $F(t,x') = \tau(x')$ for $(t,x') \in \big[\frac{1}{m(x)}, 0\big] \times V_{m(x)}$ by the first observation of this paragraph. Clearly, each $F\big|_{\{t\}\times M}$ is a section, because each $F_n\big|_{\{t\}\times M}$ and τ are so. Furthermore, if $(t,x) \in [0,1] \times (M \setminus U)$, then $x \notin U_1 \cup \ldots \cup U_{n(x)}$ and thus $F_0(t,x) = \cdots = F_{n(x)}(t,x) = F(t,x)$. □

Corollary 12. *Let M be a finite-dimensional connected manifold with corners, N a be a locally convex manifold and $f \in C(M,N)$. If $A \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $A \setminus U$, then each open neighbourhood O of f in $C(M,N)_\Gamma$ contains a map g, homotopic to f in O, which is smooth on a neighbourhood of A and equals f on $M \setminus U$. In particular, $C^\infty(M,N)$ is dense in $C(M,N)_\Gamma$.*

Furthermore, the same statement holds if we replace the graph topology $C(X,Y)_\Gamma$ with the compact-open topology $C(X,Y)_{\text{c.o.}}$.

Proof. We consider the globally trivial bundle $\mathrm{pr}_1\colon M \times N \to M$. Then the space of (continuous or smooth) mappings from M to N is isomorphic to the space of (continuous or smooth) sections by $f \mapsto \sigma_f$ with $\sigma_f(x) = \big(x, f(x)\big)$. Then $\Gamma(f) = \sigma_f(M)$ and the assertion follows directly from Theorem 11 and the observation that the graph topology is finer than the compact-open topology. □

Proposition 13. *Let M be a finite-dimensional connected manifold with corners and $\pi\colon E \to M$ be a locally trivial smooth bundle with a locally convex manifold N as typical fibre. Then each continuous section is homotopic to a smooth section. Furthermore, if there exists a continuous homotopy between the smooth sections σ and τ, then there exists a smooth homotopy between σ and τ.*

Furthermore, we have that each base-point preserving continuous section is homotopic, by a base-point preserving homotopy, to a base-point preserving smooth *section. Furthermore, if there exists a continuous base-point preserving homotopy between the smooth sections σ and τ, then there exists a smooth base-point preserving homotopy between σ and τ.*

Proof. The first assertion is already covered by Theorem 11. For the second assertion let $F'\colon [0,1]\times M\to E$ be a homotopy with $F'(0,\cdot)=\sigma$ and $F'(1,\cdot)=\tau$. Then we can construct a new homotopy F'' between σ and τ which is smooth on a neighbourhood of the closed subset $\{0,1\}\times M$ of the manifold with corners $[0,1]\times M$. In fact, taking a smooth map $\gamma\colon [0,1]\to[0,1]$ with $\gamma([0,\varepsilon])=\{0\}$ and $\gamma([1-\varepsilon,1])=\{1\}$ for some $\varepsilon\in(0,\frac{1}{2})$, $F''(t,x)=F'(\gamma(t),x)$ defines such a homotopy. Now, $\sigma'\colon [0,1]\times M\to[0,1]\times M\times E$, $(t,x)\mapsto(t,x,F''(t,x))$ defines a section in the pull-back bundle $\mathrm{pr}_2^*(E)$ of E along the projection $\mathrm{pr}_2\colon [0,1]\times M\to M$. Furthermore, σ' inherits the smoothness properties of F''. Applying Theorem 11 to the manifold with corners $[0,1]\times M$, the closed subset $\{0,1\}\times M$, the open subset $(0,1)\times M$ and σ' yields in the third component a smooth map $F\colon [0,1]\times M\to E$ with $F(0,\cdot)=F''(0,\cdot)=F'(0,\cdot)=f$ and $F(1,\cdot)=F''(1,\cdot)=F'(1,\cdot)=g$.

In the case of a base-point preserving section σ, we first claim that there exists a base-point preserving homotopy to a section which is constantly $f(x_0)$ on a neighbourhood of the base-point x_0 of M. In fact, if x_0 denotes the base-point of M, then it has a neighbourhood U such that there exists a local trivialisation $\phi\colon \pi^{-1}(U)\to U\times N$ and that $\mathrm{pr}_2\left(\varphi(\sigma(U))\right)\subseteq Z$ for some open subset Z of N which is diffeomorphic to the modelling space of N. We set $f:=\mathrm{pr}_2\circ\varphi\circ\sigma\big|_U$ and take a smooth map $\lambda\colon M\to[0,1]$ which is constantly 1 on some neighbourhood of x_0 and with $\mathrm{supp}(\lambda)\subseteq U$. Then we define a homotopy

$$G\colon [0,1]\times M\to E\,,$$

$$(t,x)\mapsto\begin{cases}\sigma(x) & \text{if}\quad x\notin U\\ \varphi^{-1}\big(x,\big(1-t\lambda(x)\big)\cdot f(x)+t\lambda(x)\cdot f(x_0)\big) & \text{if}\quad x\in U\,,\end{cases}$$

where the convex combination between $f(x)\in Z$ and $f(x_0)\in Z$ has to be understood in local coordinates in the convex set which Z is diffeomorphic to. It is easily verified that G is continuous and has the desired properties, so we may assume that σ is already smooth on some neighbourhood of x_0. Again, interpreting G as a section in the pull-back bundle $\mathrm{pr}_2^*(E)$ as in the first part of the proof and applying Theorem 11 yields a homotopy which is constantly $\sigma(x_0)$ on $[0,1]\times\{x_0\}$.

Similarly, if σ and τ are smooth and homotopic by a continuous and base-point preserving homotopy F', then the construction of the first part of the proof yields a homotopy F'' which is yet base-point preserving. By a partition of unity argument similar to construction on G, we may also assume that F'' is continuously $\sigma(x_0)=\tau(x_0)$ on a neighbourhood of $[0,1]\times\{x_0\}$. Once more, interpreting F'' as a section in the pull-back bundle $\mathrm{pr}_2^*(E)$ as in the first part of the proof and applying Theorem 11 yields a smooth homotopy which coincides with F'' on $\{0,1\}\times M\cup[0,1]\times\{x_0\}$ and thus meets all requirements. □

Corollary 14 (cf. [8])**.** *Let M be a finite-dimensional connected manifold with corners and N a be a locally convex manifold. Then each continuous map is* *homotopic to a smooth map. Furthermore, two smooth maps f and g are homotopic if and only if they are smoothly homotopic, i.e., there exists a smooth map $F\colon [0,1]\times M \to N$ with $F(0,\cdot)=f$ and $F(1,\cdot)=g$.*

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Fachbereich Mathematik, Technische Universität Darmstadt
Schlossgartenstrasse 7, D-64289 Darmstadt, Germany
E-mail: christoph@wockel.eu