ARCHIVUM MATHEMATICUM (BRNO)
Tomus 45 (2009), 203–212

LINEARIZED OSCILLATION OF NONLINEAR DIFFERENCE EQUATIONS WITH ADVANCED ARGUMENTS

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Abstract. This paper is concerned with the nonlinear advanced difference equation with constant coefficients

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i f_i(x_{n-k_i}) = 0\,, \quad n = 0, 1, \dots$$

where $p_i \in (-\infty, 0)$ and $k_i \in \{\dots, -2, -1\}$ for $i = 1, 2, \dots, m$. We obtain sufficient conditions and also necessary and sufficient conditions for the oscillation of all solutions of the difference equation above by comparing with the associated linearized difference equation. Furthermore, oscillation criteria are established for the nonlinear advanced difference equation with variable coefficients

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} f_i(x_{n-k_i}) = 0\,, \quad n = 0, 1, \dots$$

where $p_{in} \le 0$ and $k_i \in \{\dots, -2, -1\}$ for $i = 1, 2, \dots, m$.

1. Introduction

Oscillation theory of difference equations has attracted many researchers. In recent years there has been much research activity concerning the oscillation of solutions of difference equations. For these oscillatory results, we refer, for instance, [1]–[8]. Ladas [7] gave some criteria for the oscillatory behavior of the difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0\,, \quad n = 0, 1, \dots$$

where $p_i \in \mathbb{R}$, the set of all real numbers, and $k_i \in \mathbb{Z}$, the set of all integers, for $i = 1, 2, \dots, m$. Györi and Ladas [4] obtained necessary and sufficient conditions for the oscillatory behavior of the nonlinear delay difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i f_i(x_{n-k_i}) = 0\,, \quad n = 0, 1, \dots \tag{1.1}$$

2000 *Mathematics Subject Classification*: primary 39A10.

Key words and phrases: advanced difference equation, delay difference equation, nonlinear, oscillation.

Received January 24, 2008, revised May 2009. Editor O. Došlý.

where $p_i \in (0,\infty)$ and $k_i \in \mathbb{N}$, the set of all natural numbers, for $i = 1,2,\dots,m$ (see also [5]). Recently, the author [8] has investigated the oscillatory behavior of every solution of the nonlinear delay difference equation with variable coefficients

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} f_i(x_{n-k_i}) = 0\,, \quad n = 0,1,\dots \tag{1.2}$$

where $p_{in} \geq 0$ and $k_i \in \mathbb{N}$ for $i = 1,2,\dots,m$, and the following sufficient conditions for oscillation of solutions to (1.2) is obtained: Let $p_{in} \geq 0$, $\liminf\limits_{n\to\infty} p_{in} = p_i$ and $k_i \in \{0,1,\dots\}$ for $i = 1,2,\dots,m$. If each $f_i (i = 1,2,\dots,m)$ is a continuous function on $\mathbb{R}$ and satisfies

(i) $uf_i(u) > 0$, for $u \neq 0$,

(ii) $\liminf\limits_{u\to 0} f_i(u)/u = M_i$, where $0 < M_i < +\infty$,

(iii) $\sum\limits_{i=1}^{m} p_i M_i (k_i+1)^{k_i+1}/k_i^{k_i} > 1$,

then every solution of equation (1.2) oscillates.

Let $k = \max\{k_1,k_2,\dots,k_m\}$. If $k_i \in \mathbb{N}$ for $i = 1,2,\dots,m$, then, we recall that a sequence $\{x_n\}$ which is defined for $n \geq -k$ is said to be a solution of equation (1.1) if it satisfies (1.1) for $n \geq 0$. Similarly, if $k_i \in \{\dots,-2,-1\}$ for $i = 1,2,\dots,m$, then a sequence $\{x_n\}$ satisfying (1.1) for $n \geq 0$ is said to be a solution of (1.1). A solution $\{x_n\}$ of equation (1.1) is called oscillatory if the terms x_n of the sequence $\{x_n\}$ are neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory.

In the present paper, we investigate the oscillatory properties of equation (1.1) for the case $p_i \in (-\infty,0)$ and $k_i \in \{\dots,-2,-1\}$ for $i = 1,2,\dots,m$ by comparing with the associated linearized difference equation. We also deal with the oscillatory behavior of equation (1.2) for the case $p_{in} \leq 0$ and $k_i \in \{\dots,-2,-1\}$ for $i = 1,2,\dots,m$.

2. Linearized Oscillation of Equation (1.1)

Consider the nonlinear advanced difference equation (1.1) where, for $i = 1,2,\dots,m$,

$$p_i \in (-\infty,0) \quad \text{and} \quad k_i \in \{\dots,-2,-1\} \quad \text{with} \quad \sum_{i=1}^{m}(p_i+k_i) \neq -(m+1)\,, \tag{2.1}$$

$$f_i \in C(\mathbb{R},\mathbb{R}) \quad \text{and} \quad uf_i(u) > 0 \quad \text{for} \quad u \neq 0\,. \tag{2.2}$$

In this section, we will use the following condition:

$$\limsup_{u\to\infty} \frac{f_i(u)}{u} \leq 1 \quad \text{for} \quad i = 1,2,\dots,m\,. \tag{2.3}$$

If condition (2.3) is satisfied, then the linearized equation associated with equation (1.1) is given by

$$b_{n+1} - b_n + \sum_{i=1}^{m} p_i b_{n-k_i} = 0\,, \quad n = 0,1,\dots \tag{2.4}$$

Lemma 2.1. *For each $i = 1, 2, \ldots, m$, assume that* (2.1) *holds. Assume further that (p_{in}) is a sequence of real numbers such that*

$$\limsup_{n\to\infty} p_{in} \le p_i \quad \textit{for} \quad i = 1, 2, \ldots, m\,. \tag{2.5}$$

If the linear difference inequality

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} \ge 0\,, \quad n = 0, 1, \ldots \tag{2.6}$$

has an eventually positive solution, then so does equation (2.4).

Proof. Let $k = \min\{k_1, k_2, \ldots, k_m\}$. Assume first $k = -1$. Observe that $k_i = -1$ for each $i = 1, 2, \ldots, m$. Then, by (2.6) and (2.4) we get respectively that

$$x_{n+1}(1 + \sum_{i=1}^{m} p_{in}) \ge x_n \tag{2.7}$$

and

$$b_{n+1}(1 + \sum_{i=1}^{m} p_i) = b_n\,. \tag{2.8}$$

Since x_n is eventually positive in (2.7), it follows, for n sufficiently large, that

$$\sum_{i=1}^{m} p_{in} > -1\,. \tag{2.9}$$

By condition (2.5), for a given $\varepsilon > 0$, there is positive integer n_0 such that

$$0 > p_i \ge p_{in} - \frac{\varepsilon}{m} \quad \text{for} \quad n \ge n_0 \quad \text{and} \quad i = 1, 2, \ldots, m\,.$$

Hence, by using (2.9) and (2.1) we have

$$0 > \sum_{i=1}^{m} p_i \ge \sum_{i=1}^{m} p_{in} - \varepsilon > -1 - \varepsilon \quad \text{for} \quad n \ge n_0\,.$$

So, this yields that

$$-1 \le \sum_{i=1}^{m} p_i < 0\,. \tag{2.10}$$

Since $k_i = -1$ for each $i = 1, 2, \ldots, m$, the last condition in (2.1) reduces to $\sum_{i=1}^{m} p_i \ne -1$. Then, it follows from (2.10) that

$$-1 < \sum_{i=1}^{m} p_i < 0\,.$$

This condition guarantees that the solution of equation (2.8) with $b_0 = 1$ is positive.

Assume now that $k \leq -2$. Dividing by x_n on the both sides of (2.6) and letting $z_n = \dfrac{x_{n+1}}{x_n}$, we conclude, for all n sufficiently large, that

$$z_n - 1 + \sum_{i=1}^{m} p_{in}\Big(\frac{x_{n-k_i}}{x_n}\Big) \geq 0\,. \tag{2.11}$$

Since

$$\begin{aligned}\frac{x_{n-k_i}}{x_n} &= \frac{x_{n-k_i}}{x_{n-k_i-1}} \cdot \frac{x_{n-k_i-1}}{x_{n-k_i-2}} \cdots \frac{x_{n+1}}{x_n} \\ &= z_{n-k_i-1} \cdot z_{n-k_i-2} \cdots z_n\,,\end{aligned}$$

we get from (2.11) that

$$z_n - 1 + \sum_{i=1}^{m} p_{in}(z_{n-k_i-1} \cdot z_{n-k_i-2} \cdots z_n) \geq 0\,. \tag{2.12}$$

Let $z = \limsup_{n\to\infty} z_n$. Then, by (2.12) observe that $z_n > 1$ and that $z > 1$. We now claim that

$$z - 1 + \sum_{i=1}^{m} p_i z^{-k_i} \geq 0\,. \tag{2.13}$$

Indeed, by using (2.1) and (2.5), for a given ε such that $0 < \varepsilon < 1$, there is a positive integer n_1 such that $p_{in} \leq (1-\varepsilon)p_i$ for $i = 1, 2, \ldots, m$ and $n \geq n_1$. Hence, for all $n \geq n_1$, from (2.12) we may write

$$z_n \geq 1 - (1-\varepsilon)\sum_{i=1}^{m} p_i(z_{n-k_i-1} \cdot z_{n-k_i-2} \cdots z_n)\,, \tag{2.14}$$

choose n_2 such that $n_2 \geq n_1 - k$ and that

$$z_n \geq (1+\varepsilon)z \quad \text{for} \quad n \geq n_2 + k\,.$$

Then, for $n \geq n_2 + k$, we obtain from (2.14) that

$$z_n \geq 1 - (1-\varepsilon)\sum_{i=1}^{m} p_i z^{-k_i}(1+\varepsilon)^{-k_i}\,. \tag{2.15}$$

Taking limit superior as $n \to \infty$ on the both sides of (2.15), we have

$$z \geq 1 - (1-\varepsilon)\sum_{i=1}^{m} p_i z^{-k_i}(1+\varepsilon)^{-k_i}\,.$$

Since ε is arbitrary, the inequality above implies (2.13), which proves our claim. Define

$$F(\lambda) = \lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i}\,.$$

Then, it easy to see that $F(0+) = -1$ and $F(z) \geq 0$ .This guarantees that the characteristic equation of equation (2.4) has a positive root λ_0. Therefore, $b_n = \lambda_0^n$ is a positive solution of equation (2.4), which completes the proof. □

Using Lemma 2.1 we have the following main result.

Theorem 2.2. *Assume that* (2.1)*,* (2.2) *and* (2.3) *hold. If every solution of the linearized difference equation* (2.4) *oscillates, then every solution of the non-linear difference equation* (1.1) *oscillates.*

Proof. Assume, for the sake of contradiction, that equation (1.1) has an eventually positive solution $\{x_n\}$. The case in which $\{x_n\}$ is eventually negative is similar and is omitted. By (1.1), $\{x_n\}$ is eventually increasing sequence. We claim that $\{x_n\}$ is not bounded above. Otherwise, there would be finite number $L > 0$ such that $\lim_{n\to\infty} x_n = L$. Since each f_i is continuous on $\mathbb{R}$, we get $\lim_{n\to\infty} f_i(x_{n-k_i}) = f_i(L) > 0$ for $i = 1, 2, \ldots, m$. So, taking limit as $n \to \infty$ on the both sides of equation (1.1) we have $\sum_{i=1}^{m} p_i f_i(L) = 0$, which contradicts the first condition of (2.1). Therefore, $\{x_n\}$ is increasing and unbounded above, which implies that

$$\lim_{n\to\infty} x_n = +\infty\,. \tag{2.16}$$

We can now rewrite (1.1) in the form

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_{in} x_{n-k_i} = 0\,,$$

where

$$p_{in} = p_i \frac{f_i(x_{n-k_i})}{x_{n-k_i}} \quad \text{for} \quad i = 1, 2, \ldots, m\,.$$

From (2.16) and (2.3), it is clear that

$$\limsup_{n\to\infty} p_{in} \le p_i \quad \text{for} \quad i = 1, 2, \ldots, m\,.$$

So, the hypotheses of Lemma 2.1 are satisfied. This yields that the linearized difference equation (2.4) has an eventually positive solution which contradicts the hypothesis. □

We now obtain the oscillatory conditions of the linearized difference equation (2.4) whenever every solution of equation (1.1) oscillates. We first need the following lemma.

Lemma 2.3. *For each $i = 1, 2, \ldots, m$, assume that* (2.1) *holds and that λ_0 is a positive root of the characteristic equation*

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0 \tag{2.17}$$

of equation (2.4)*. Let $k = \min\{k_1, k_2, \ldots, k_m\}$ and $n_1 \in \mathbb{N}$ such that $n_1 \ge -k$ and let $q \in (-\infty, 0)$. If $\{x_n\}$ is a solution of the difference inequality*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} \le 0\,, \quad n = -k-1, -k, \ldots, n_1 - 1 \tag{2.18}$$

with the initial conditions

$$x_n = q\lambda_0^n\,, \quad n = 0, 1, \dots, -k-1\,, \tag{2.19}$$

then we have

$$x_n leqq\lambda_0^n\,, \quad n = -k, -k+1, \dots, n_1\,.$$

Proof. Since each $p_i < 0$, observe that $\lambda_0 > 1$. So, the case where $k = -1$ is clear. Assume now that $k \leq -2$. Let $z_n = \dfrac{x_n}{x_{n-1}}$ $(n = 1, 2, \dots, -k-1, -k, \dots)$ provided that $x_{n-1} \neq 0$. By using (2.17), (2.18) and (2.19), we have

$$0 \leq z_{-k} - 1 + \sum_{i=1}^{m} p_i \frac{x_{-k-1-k_i}}{x_{-k-1}} = z_{-k} - 1 + \sum_{i=1}^{m} p_i \lambda_0^{-k_i} \Rightarrow z_{-k} \geq \lambda_0\,,$$

which implies that $x_{-k} \leq q\lambda_0^{-k}$. In a similar manner,

$$0 \leq z_{-k+1} - 1 + \sum_{i=1}^{m} p_i \frac{x_{-k-k_i}}{x_{-k}} = z_{-k+1} - 1 + \sum_{i=1}^{m} p_i \lambda_0^{-k_i} \Rightarrow z_{-k+1} \geq \lambda_0\,,$$

which implies that $x_{-k+1} \leq q\lambda_0^{-k+1}$. So, the proof follows from induction. □

Theorem 2.4. *Assume that* (2.1) *and* (2.2) *hold. Assume further that there exists a positive constant* δ *such that one of the following items is satisfied:*

$$f_i(u) \leq u \quad \textit{for} \quad u \leq -\delta \quad \textit{and} \quad i = 1, 2, \dots, m\,, \tag{2.20}$$

$$f_i(u) \geq u \quad \textit{for} \quad u \geq \delta \quad \textit{and} \quad i = 1, 2, \dots, m\,. \tag{2.21}$$

If every solution of (1.1) *oscillates, then every solution of the linearized equation* (2.4) *also oscillates.*

Proof. Suppose that (2.20) holds. The case of (2.21) is similar and is omitted. Assume now, for the sake of contradiction, that (2.4) has an eventually negative solution (b_n). Then, from [6, Lemma 7.1.1], we conclude that the characteristic equation of (2.4)

$$\lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i} = 0$$

has a positive root λ_0. Since $p_i \in (-\infty, 0)$, it is clear that $\lambda_0 > 1$. Let $\{x_n\}$ be the solution of (1.1) with the initial conditions

$$x_n = q\lambda_0^n\,, \quad n = 0, 1, \dots, -k-1$$

where $k = \min\{k_1, k_2, \dots, k_m\}$ and $q = -\delta\lambda_0^{k+1}$. Note that if we prove

$$x_n < 0 \quad \text{for} \quad n = -k, -k+1, \dots \tag{2.22}$$

then we get a contradiction for the oscillatory of equation (1.1).

If condition (2.22) were not true, then there would be an integer n_1 such that $n_1 \geq -k$ and that $x_n < 0$ for $n = 0, 1, = \dots, n_1 - 1$ but $x_{n_1} \geq 0$ holds. By (1.1) we have

$$x_{n+1} < x_n \quad \text{for} \quad n = -k-1, -k, \dots, n_1 - 1\,.$$

This yields that

$$x_n < x_{-k-1} = q\lambda_0^{-k-1} = -\delta < 0 \quad \text{for} \quad n = -k, -k+1, \dots, n_1 \,.$$

Hence, we get

$$x_n < -\delta \quad \text{for} \quad n = -k, -k+1, \dots, n_1 \,. \tag{2.23}$$

By using (2.20) and (2.23), it follows from (1.1) that

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} \le 0 \quad \text{for} \quad n = -k-1, -k, \dots, n_1 - 1 \,.$$

Since the hypotheses of Lemma 2.3 hold, we obtain that $x_{n_1} \le q\lambda_0^{n_1} < 0$. This contradiction completes the proof. □

By combining Theorem 2.2 with Theorem 2.4 we obtain the following necessary and sufficient conditions for every solution of the non-linear difference equation (1.1).

Corollary 2.5. *Assume that* (2.1) *and* (2.2) *hold. Assume further that either* (2.20) *or* (2.21) *is satisfied and let*

$$\lim_{u\to\infty} \frac{f_i(u)}{u} = 1 \quad \textit{for} \quad i = 1, 2, \dots, m \,.$$

Then, every solution of equation (1.1) *oscillates if and only if every solution of the associated linearized equation* (2.4) *oscillates.*

3. Oscillation conditions for equation (1.2)

Consider the nonlinear advanced difference equation (1.2) such that, for $i = 1, 2, \dots, m$, $k_i \in \{\dots, -2, -1\}$ and the condition

$$\liminf_{u\to\infty} \frac{f_i(u)}{u} = M_i \,, \quad 0 < M_i < +\infty \tag{3.1}$$

holds. In this section, we will use the convention $0^0 = 1$.

Then we have the following

Theorem 3.1. *For each $i = 1, 2, \dots, m$, let $k_i \in \{\dots, -2, -1\}$, $p_{in} \le 0$ $(n \in \mathbb{N})$ and $\limsup_{n\to\infty} p_{in} = p_i < 0$. Assume that* (2.2) *and* (3.1) *hold. If the condition*

$$\sum_{i=1}^{m} p_i M_i \frac{(k_i + 1)^{k_i+1}}{k_i^{k_i}} > 1 \tag{3.2}$$

is satisfied, then every solution of equation (1.2) *oscillates.*

Proof. Assume that $\{x_n\}$ is an eventually positive solution of (1.2). Then, it is easy to see that $\{x_n\}$ is eventually increasing sequence. As in the proof of Theorem 2.2 we claim that $\{x_n\}$ is unbounded above. Otherwise, there exists $L > 0$ such that

$\lim_{n\to\infty} x_n = L$. This implies that $\lim_{n\to\infty} f_i(x_{n-k_i}) = f_i(L) > 0$. Taking limit inferior as $n \to \infty$ in (1.2) we have

$$\liminf_{n\to\infty} \left\{ \sum_{i=1}^{m} (-p_{in}) f_i(x_{n-k_i}) \right\} = 0\,. \tag{3.3}$$

It follows from (3.3) that

$$\sum_{i=1}^{m} \liminf_{n\to\infty} \{(-p_{in}) f_i(x_{n-k_i})\} \le 0\,,$$

or

$$\sum_{i=1}^{m} f_i(L) \liminf_{n\to\infty} (-p_{in}) = -\sum_{i=1}^{m} p_i f_i(L) \le 0\,,$$

which is impossible since $p_i < 0$ and $f_i(L) > 0$ for $i = 1, 2, \ldots, m$. So, $\{x_n\}$ is eventually increasing and unbounded above, which gives $\lim_{n\to\infty} x_n = +\infty$. On the other hand, dividing equation (1.2) by x_n and letting $z_n = \dfrac{x_{n+1}}{x_n}$ we get eventually that

$$z_n = 1 - \sum_{i=1}^{m} p_{in} \frac{f_i(x_{n-k_i})}{x_{n-k_i}} \left(z_{n-k_i-1} \cdot z_{n-k_i-2} \cdots z_n\right)\,. \tag{3.4}$$

Let $\liminf_{n\to\infty} z_n = z$. Observe that $z_n > 1$ and $z \ge 1$. Taking limit inferior as $n \to \infty$ on the both sides of (3.4) we may write

$$\begin{aligned} z &\ge 1 + \sum_{i=1}^{m} \liminf_{n\to\infty} (-p_{in}) \liminf_{n\to\infty} \left(\frac{f_i(x_{n-k_i})}{x_{n-k_i}} \right) \liminf_{n\to\infty} z_{n-k_i-1} \cdots \liminf_{n\to\infty} z_n \\ &= 1 - \sum_{i=1}^{m} M_i z^{-k_i} \limsup_{n\to\infty} p_{in} = 1 - \sum_{i=1}^{m} p_i M_i z^{-k_i}\,. \end{aligned}$$

Therefore,

$$\sum_{i=1}^{m} p_i M_i z^{-k_i} \ge 1 - z\,,$$

which implies that $z \ne 1$ and that

$$\sum_{i=1}^{m} p_i M_i \frac{z^{-k_i}}{1-z} \le 1\,. \tag{3.5}$$

Now consider the function g defined by $g(z) = \dfrac{z^{-k_i}}{1-z}$. Then, it is not difficult to see that $g'\Big(\dfrac{k_i}{k_i+1}\Big) = 0$ and $g''\Big(\dfrac{k_i}{k_i+1}\Big) < 0$. Since $p_i < 0$ for $i = 1, 2, \ldots, m$, we conclude that

$$\sum_{i=1}^{m} p_i M_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} = \sum_{i=1}^{m} p_i M_i g\left(\frac{k_i}{k_i+1}\right) \le \sum_{i=1}^{m} p_i M_i \frac{z^{-k_i}}{1-z}\,.$$

Hence by (3.5)

$$\sum_{i=1}^{m} p_i M_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} \leq 1\,,$$

which contradicts (3.2).

In a similar manner, one can easily show that equation (1.2) has no eventually negative solution. So, the proof is completed. □

Finally, using Theorem 3.1 we have the following result.

Corollary 3.2. *Let k_i and p_i be the same as in Theorem 3.1. Assume that* (2.2) *and* (3.1) *hold. If*

$$m\Big(\prod_{i=1}^{m} |p_i|\, M_i\Big)^{1/m} \Big|\frac{(\bar{k}+1)^{\bar{k}+1}}{\bar{k}^{\bar{k}}}\Big| > 1\,, \tag{3.6}$$

where $\bar{k} = \dfrac{1}{m}\sum\limits_{i=1}^{m} k_i$, *then every solution of equation* (1.2) *oscillates.*

Proof. Assume that $\{x_n\}$ is an eventually positive solution of (1.2). By using (3.5) and (3.6), and also applying the arithmetic-geometric mean inequality, we have

$$\begin{aligned} 1 &\geq \sum_{i=1}^{m} p_i M_i \frac{z^{-k_i}}{1-z} \geq m\Big[\prod_{i=1}^{m} p_i M_i \frac{z^{-k_i}}{1-z}\Big]^{1/m} \\ &= m\frac{z^{-\bar{k}}}{z-1}\Big[\prod_{i=1}^{m} (-p_i) M_i\Big]^{1/m} \geq m\Big|\frac{(\bar{k}+1)^{\bar{k}+1}}{(\bar{k})^{\bar{k}}}\Big|\Big(\prod_{i=1}^{m} |p_i|\, M_i\Big)^{1/m}, \end{aligned}$$

which contradicts (3.6) and completes the proof. □

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