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ON THE LIPSCHITZ OPERATOR ALGEBRAS

A. Ebadian and A. A. Shokri

Abstract. In a recent paper by H. X. Cao, J. H. Zhang and Z. B. Xu an α-Lipschitz operator from a compact metric space into a Banach space A is defined and characterized in a natural way in the sence that $F : K \to A$ is a α-Lipschitz operator if and only if for each $\sigma \in X^*$ the mapping $\sigma \circ F$ is a α-Lipschitz function. The Lipschitz operators algebras $L^\alpha(K, A)$ and $l^\alpha(K, A)$ are developed here further, and we study their amenability and weak amenability of these algebras. Moreover, we prove an interesting result that $L^\alpha(K, A)$ and $l^\alpha(K, A)$ are isometrically isomorphic to $L^\alpha(K)\check{\otimes}A$ and $l^\alpha(K)\check{\otimes}A$ respectively. Also we study homomorphisms on the $L^\alpha_A(X, B)$.

1. Introduction

Let (K, d) be compact metric space with at least two elements and $(X, \|\cdot\|)$ be a Banach space over the scalar field $\mathbf{F}$ (= R or C). For a constant $\alpha > 0$ and an operator $T\colon K \to X$, set

$$L_\alpha(T) := \sup_{s\neq t} \frac{\|T(t) - T(s)\|}{d(s,t)^\alpha}\,, \tag{1}$$

which is called the Lipschitz constant of T. Define

$$T_\alpha(x, y) = \frac{T(x) - T(y)}{d(x,y)^\alpha}\,, \quad x \neq y$$
$$L^\alpha(K, X) = \{T\colon K \to X : L_\alpha(T) < \infty\}$$

and

$$l^\alpha(K, X) = \{T\colon K \to X : \|T_\alpha(x, y)\| \to 0 \ \text{ as } \quad d(x, y) \to 0\}\,.$$

The elements of $L^\alpha(K, X)$ and $l^\alpha(K, X)$ are called big and little Lipschitz operators, respectively [1].

Let $C(K, X)$ be the set of all continuous operators from K into X and for each $T \in C(K, X)$, define

$$\|T\|_\infty = \sup_{x\in K} \|T(x)\|\,.$$

For S, T in $C(K, X)$ and λ in F, define

$$(S + T)(x) = S(x) + T(x)\,, \quad (\lambda T)(x) = \lambda T(x)\,, \quad (x \in X)\,.$$

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It is easy to see that $(C(K,X),\ \|\cdot\|_\infty)$ becomes a Banach space over F and $L^\alpha(K,X)$ is a linear subspace of $C(K,X)$. For each element T of $L^\alpha(K,X)$, define $\|T\|_\alpha = L_\alpha(T) + \|T\|_\infty$.

In their papers [3, 4], Cao, Zhang and Xu proved that $(L^\alpha(K,X),\ \|\cdot\|_\alpha)$ is a Banach space over F and $l^\alpha(K,X)$ is a closed linear subspace of $(L^\alpha(K,X),\ \|\cdot\|_\alpha)$. Now, let $(A,\ \|\cdot\|)$ be a unital Banach algebra with unit e. In this paper, we show that $(L^\alpha(K,A),\ \|\cdot\|_\alpha)$ is a Banach algebra under pointwise and scalar multiplication and $l^\alpha(K,A)$ is a closed linear subalgebra of $(L^\alpha(K,A),\ \|\cdot\|_\alpha)$ and study many aspects of these algebras. The spaces $L^\alpha(K,A)$ and $l^\alpha(K,A)$ are called big and little Lipschitz operators algebras. Note that Lipschitz operators algebras are, in fact, extensions of Lipschitz algebras. Sherbert [11, 12], Weaver [13, 14], Honary and Mahyar [7], Johnson [8, 9], Alimohammadi and Ebadian [1], Ebadian [6], Bade, Curtis and Dales [2], studied some properties of Lipschitz algebras. We will study (weak) amenability of Lipschitz operators algebras. Also we study homomorphisms on the $L^\alpha_A(X,B)$.

2. Characterizations of Lipschitz operators algebras

In this section, let (K,d) be a compact metric space which has at least two elements and $(A,\|\cdot\|)$ to denote a unital Banach algebra over the scalar field F (= R or C).

Theorem 2.1. *$(L^\alpha(K,A),\ \|\cdot\|_\alpha)$ is a Banach algebra over* F *and $l^\alpha(K,A)$ is a closed linear subspace of $(L^\alpha(K,A),\ \|\cdot\|_\alpha)$.*

Proof. As we have already $L^\alpha(K,A)$ is a Banach space and $l^\alpha(K,A)$ is a closed linear subspace if it. Now let $T,\ S \in L^\alpha(K,A)$, and define

$$(TS)(t) = T(t)S(t) \quad (t \in K)\,.$$

Then

$$\begin{aligned}\|TS\|_\alpha &= \|TS\|_\infty + L_\alpha(TS)\\ &\le \|T\|_\infty\|S\|_\infty + \sup_{t\neq s}\frac{\|T(t)S(t)-T(s)S(s)\|}{d(t,s)^\alpha}\\ &\le \|T\|_\infty\|S\|_\infty + \|T\|_\infty L_\alpha(S) + \|S\|_\infty L_\alpha(T)\\ &\le \big(\|T\|_\infty + L_\alpha(T)\big)\big(\|S\|_\infty + L_\alpha(S)\big)\\ &= \|T\|_\alpha\|S\|_\alpha\,.\end{aligned}$$

So that we see that $(L^\alpha(K,A),\ \|\cdot\|_\alpha)$ is a Banach algebra and $l^\alpha(K,A)$ is a closed linear subspace of $(L^\alpha(K,A),\ \|\cdot\|_\alpha)$. □

Theorem 2.2. *Let (K,d) be a compact metric space. Then $L^\alpha(K,A)$ is uniformly dense in $C(K,A)$.*

Proof. Let $f \in C(K,A)$. Then for every $\sigma \in A^*$ we have $\sigma \circ f \in C(K)$, so that there is $g \in L^\alpha(K)$ such that $\|g - \sigma\circ f\|_\infty < \varepsilon$. We define, the map $\eta\colon \mathrm{C} \to A$ by

$\eta(\lambda) = \lambda \cdot e$. It is easy to see that $\eta \circ g \in L^\alpha(K, A)$, and for every $\sigma \in A^*$, we have

$$|\sigma\big(g(x) \cdot e - f(x)\big)| = |g(x) - (\sigma \circ f)(x)| < \varepsilon\,, \quad (x \in K)\,.$$

Therefore $|\sigma(\eta \circ g - f)(x)| < \varepsilon$ for every $\sigma \in A^*$ and $x \in K$. This implies that $\|(\eta \circ g - f)(x)\| < \varepsilon$ for every $x \in K$. Therefore, $\|\eta \circ g - f\|_\infty < \varepsilon$ and the proof is complete. □

Remark 2.3. Let A, B be unital Banach algebras over F. Then the injective tensor $A \check{\otimes} B$ is a unital Banach algebra under norm $\|\cdot\|_\epsilon$, [10].

Theorem 2.4. $L^\alpha(K, A) = \{F \colon K \to A \mid \sigma \circ F \in L^\alpha(K, \mathrm{C}), (\forall \sigma \in A^*)\}$

Proof. Use the principle of Uniform Boundedness. □

Lemma 2.5. *Let $(E_1, \|\cdot\|_1), (E_2, \|\cdot\|_2)$ be Banach spaces. Then for $G \in E_1 \check{\otimes} E_2$*

$$\|G\|_\varepsilon = \sup\big\{\|\big(\operatorname{id}\otimes\phi\big)(G)\|_1 \,:\, \phi \in E_2^*,\ \|\phi\| \le 1\big\}\,.$$

Proof. See [10]. □

Theorem 2.6. *Let (K, d) be a compact metric space and A be a unital commutative Banach algebra. Then $L^\alpha(K, A)$ is isometrically isomorphic to $L^\alpha(K)\check{\otimes}A$.*

Proof. It is straightforward to prove that the mapping $V \colon L^\alpha(K) \times A \to L^\alpha(K, A)$ defined by

$$\begin{aligned} V(f, a) &= fa \qquad (f \in L^\alpha(K),\ a \in A)\,,\\ (fa)(x) &:= f(x)a \qquad (x \in K)\,, \end{aligned}$$

is bilinear. Therefore there exists a unique linear map $T \colon L^\alpha(K)\check{\otimes}A \to L^\alpha(K, A)$ such that $T(f \otimes a) = V(f, a) = fa$, [10]. For every $G \in L^\alpha(K)\check{\otimes}A$, there is $m \in \mathrm{N}$, $f_j \in L^\alpha(K)$ and $a_j \in A$ $(1 \le j \le m)$ such that $G = \sum_{j=1}^m f_j \otimes a_j$, so we have

$$\begin{aligned} \|G\|_\varepsilon &= \sup_{\phi \in A^*, \|\phi\| \le 1} \|(\operatorname{id}\otimes\phi)(G)\|_\alpha = \sup_{\phi \in A^*, \|\phi\| \le 1} \Big\|(\operatorname{id}\otimes\phi)\Big(\sum_{j=1}^m f_j \otimes a_j\Big)\Big\| \\ &= \sup_{\phi \in A^*, \|\phi\| \le 1} \Big\|\sum_{j=1}^m f_j \phi(a_j)\Big\|_\alpha = \sup_{\phi \in A^*, \|\phi\| \le 1} \Big[\sup_{x \in K}\Big|\sum_{j=1}^m f_j(x)\phi(a_j)\Big| \\ &\quad + \sup_{x \ne y} \frac{|\sum_{j=1}^m f_j(x)\phi(a_j) - \sum_{j=1}^m f_j(y)\phi(a_j)|}{d^\alpha(x, y)}\Big] \\ &= \sup_{\phi \in A^*, \|\phi\| \le 1} \Big[\sup_{x \in K}\Big|\phi\Big(\sum_{j=1}^m f_j(x)a_j\Big)\Big| \\ &\quad + \sup_{x \ne y} \frac{\big|\phi\big(\sum_{j=1}^m (f_j(x)a_j - f_j(y)a_j)\big)\big|}{d^\alpha(x, y)}\Big] \end{aligned}$$

$$
\begin{aligned}
&\le \sup_{\phi\in A^*,\|\phi\|\le 1}\Big[\sup_{x\in K}\|\phi\|\Big\|\sum_{j=1}^m f_j(x)a_j\Big\| \\
&\quad + \sup_{x\neq y}\|\phi\|\frac{\|\sum_{j=1}^m f_j(x)a_j-\sum_{j=1}^m f_j(y)a_j\|}{d^\alpha(x,y)}\Big] \\
&\le \sup_{x\in K}\Big\|\sum_{j=1}^m f_j(x)a_j\Big\|+\sup_{x\neq y}\frac{\|\sum_{j=1}^m f_j(x)a_j-\sum_{j=1}^m f_j(y)a_j\|}{d^\alpha(x,y)} \\
&= \Big\|\sum_{j=1}^m f_ja_j\Big\|_\infty + p_\alpha\Big(\sum_{j=1}^m f_ja_j\Big) = \Big\|\sum_{j=1}^m f_ja_j\Big\|_\alpha = \Big\|T\Big(\sum_{j=1}^m f_j\otimes a_j\Big)\Big\|_\alpha \\
&= \|TG\|_\alpha \quad \Longrightarrow \|G\|_\varepsilon \le \|TG\|_\alpha\,.
\end{aligned}
$$

Now let $\gamma>0$ be arbitrary, such that $\|TG\|_\alpha>\gamma$. Then $\|\sum_{j=1}^m f_ja_j\|_\alpha>\gamma$, and so we have

$$
\Big\|\sum_{j=1}^m f_ja_j\Big\|_\infty + p_\alpha\Big(\sum_{j=1}^m f_ja_j\Big)>\gamma
$$

$$
\begin{aligned}
&\Rightarrow \sup_{x\in K}\Big\|\sum_{j=1}^m f_j(x)a_j\Big\|+\sup_{x\neq y}\frac{\|\sum_{j=1}^m f_j(x)a_j-\sum_{j=1}^m f_j(y)a_j\|}{d^\alpha(x,y)}>\gamma \\
&\Rightarrow \sup_{\phi\in A^*,\|\phi\|\le 1}\Big[\sup_{x\in K}\Big|\sum_{j=1}^m f_j(x)\phi(a_j)\Big| \\
&+\sup_{x\neq y}\frac{|\sum_{j=1}^m f_j(x)\phi(a_j)-\sum_{j=1}^m f_j(y)\phi(a_j)|}{d^\alpha(x,y)}\Big]>\gamma \\
&\Rightarrow \sup_{\phi\in A^*,\|\phi\|\le 1}\Big[\Big\|\sum_{j=1}^m f_j\phi(a_j)\Big\|_\infty + p_\alpha\Big(\sum_{j=1}^m f_j\phi(a_j)\Big)\Big]>\gamma \\
&\Rightarrow \sup_{\phi\in A^*,\|\phi\|\le 1}\Big[\Big\|(\mathrm{id}\otimes\phi)\Big(\sum_{j=1}^m f_j\otimes a_j\Big)\Big\|_\infty + p_\alpha\Big((\mathrm{id}\otimes\phi)\Big(\sum_{j=1}^m f_j\otimes a_j\Big)\Big)\Big]>\gamma \\
&\Rightarrow \sup_{\phi\in A^*,\|\phi\|\le 1}\Big\|(\mathrm{id}\otimes\phi)\Big(\sum_{j=1}^m f_j\otimes a_j\Big)\Big\|_\alpha>\gamma \\
&\Rightarrow \Big\|\sum_{j=1}^m f_j\otimes a_j\Big\|_\varepsilon>\gamma \quad \Rightarrow \|G\|_\varepsilon>\gamma\,.
\end{aligned}
$$

Since $\gamma>0$ is arbitrary, then we have $\|TG\|_\alpha\le\|G\|_\varepsilon$. Therefore $\|TG\|_\alpha=\|G\|_\varepsilon$, and this implies that T is a linear isometry map. So T one-one and continuous map. Now, we show that T is a onto map. For this, we show that the range of T, R_T is a closed and dense subset of $L^\alpha(K,A)$. It is easy to see that R_T is closed. Let $f\in L^\alpha(K,A)$ and $\gamma>0$. There exist $a_1,\dots,a_n\in A$ such that $X:=f(K)\subset\bigcup_{i=1}^n B(a_i,\gamma)$. Set $U_j=f^{-1}(B(a_j,\gamma))$ where $j=1,\dots,n$. Then there exist $f_1, \dots, f_n \in L^\alpha(K, A)$ and $\sigma \in A^*$such that $\mathrm{supp}(f_j) \subset U_j$ for $j = 1, \dots, n$ and $\sigma \circ (f_1 + \ldots + f_n) = 1$. For every $x \in K$ we have,

$$\begin{aligned} &\big\|f(x) - \big((\sigma \circ f_1)a_1 + \cdots + (\sigma \circ f_n)a_n\big)(x)\big\| \\ &\quad = \big\|f(x)\big((\sigma \circ f_1)(x) + \cdots + (\sigma \circ f_n)(x)\big) \\ &\qquad - \big((\sigma \circ f_1)(x)a_1 + \cdots + (\sigma \circ f_n)(x)a_n\big)\big\| \\ &\quad = \big\|(\sigma \circ f_1)(x)\big(f(x) - a_1\big) + \cdots + (\sigma \circ f_n)(x)\big(f(x) - a_n\big)\big\| \\ &\quad \le \sum_{i=1}^{n} \big|(\sigma \circ f_i)(x)\big|\,\big\|f(x) - a_i\big\| < \gamma\,, \end{aligned}$$

since $\mathrm{supp}\, f_j \subset U_j$. Therefore,

$$\big\|f - \big((\sigma \circ f_1)a_1 + \cdots + (\sigma \circ f_n)a_n\big)\big\|_\alpha < \gamma\,.$$

This implies that

$$\Big\|f - \sum_{i=1}^{n} T(\sigma \circ f_i \otimes a_i)\Big\|_\alpha < \gamma\,.$$

We conclude that $\bar{R}_T = L^\alpha(K, A)$. So $R_T = L^\alpha(K, A)$, since R_T is closed. Hence T is a onto map. Also by product $\bullet$ on $L^\alpha(K)\check{\otimes}A$

$$(f\otimes a) \bullet (g\otimes b) = fg \otimes ab \quad (f, g \in L^\alpha(K),\ a, b \in A)\,,$$

clearly T is homomorphism. □

Furthermore T is open map, for this purpose, let τ and τ' be topologies on $L^\alpha(K)\check{\otimes}A$ and $L^\alpha(K, A)$ respectively. Let $U \in \tau$, we show that $T(U) \in \tau'$. Let p be a limit point in $L^\alpha(K, A)\backslash T(U)$. Then there exists a sequence $\{p_n\}$ in $L^\alpha(K, A)\backslash T(U)$ converges to p. Since T is onto, there is a sequence $\{q_n\}$ and q in $L^\alpha(K)\check{\otimes}A$ such that $T(q_n) = p_n$ and $Tq = p$. Therefore $T(q_n)$ converges to p in $L^\alpha(K)$. Since $q_n \in L^\alpha(K)\check{\otimes}A$, we can find $m \in \mathrm{N}$, $f_j^{(n)} \in L^\alpha(K)$ and $a_j^{(n)} \in A$ such that whenever $1 \le j \le m$ we have

$$T(q_n) = \sum_{j=1}^{m} f_j^{(n)} a_j^{(n)}\,. \tag{1}$$

Also, since $q \in L^\alpha(K)\check{\otimes}A$ there exist $r \in \mathrm{N}$, $g_i \in L^\alpha(K)$ and $b_i \in A$ such that

$$p = T(q) = \sum_{i=1}^{r} g_i b_i\,. \tag{2}$$

Since $\|T(q_n) - p\|_\alpha \to 0$ as $n \to \infty$, for every positive number γ there exists a positive integer N such that

$$\Big\|\sum_{j=1}^{m} f_j^{(n)} a_j^{(n)} - \sum_{i=1}^{r} g_i b_i\Big\|_\alpha < \gamma\,, \tag{3}$$

when $n \geq N$. By applying (3), we have

$$\sup_{(x\in K)} \Big\| \sum_{j=1}^{m} f_j^{(n)}(x) a_j^{(n)} - \sum_{i=1}^{r} g_i(x) b_i \Big\| + \sup_{(x\neq y)} \frac{1}{d(x,y)^{\alpha}}$$
$$\times \Big\| \sum_{j=1}^{m} f_j^{(n)}(x) a_j^{(n)} - \sum_{i=1}^{r} g_i(x) b_i - \sum_{j=1}^{m} f_j^{(n)}(y) a_j^{(n)} + \sum_{i=1}^{r} g_i(y) b_i \Big\| < \gamma \,.$$

Therefore if $\sigma \in A^*$ with $\|\sigma\| \leq 1$ then

$$\sup_{(x\in K)} \Big\| \sum_{j=1}^{m} f_j^{(n)}(x) \sigma(a_j^{(n)}) - \sum_{i=1}^{r} g_i(x) \sigma(b_i) \Big\| + \sup_{(x\neq y)} \frac{1}{d(x,y)^{\alpha}}$$
$$\times \Big\| \sum_{j=1}^{m} f_j^{(n)}(x) \sigma(a_j^{(n)}) - \sum_{i=1}^{r} g_i(x) \sigma(b_i) \sum_{j=1}^{m} f_j^{(n)}(y) \sigma(a_j^{(n)}) + \sum_{i=1}^{r} g_i(y) \sigma(b_i) \Big\| < \gamma \,.$$

This implies that

$$\Big\| \sum_{j=1}^{m} f_j^{(n)} \sigma(a_j^{(n)}) - \sum_{i=1}^{r} g_i \sigma(b_i) \Big\|_{\alpha} < \gamma \tag{4}$$

Now by using (4), for every $\phi \in L^{\alpha}(K)^*$ with $\|\phi\|_{\alpha} \leq 1$ we have,

$$\Big| \phi\Big(\sum_{j=1}^{m} f_j^{(n)} \sigma(a_j^{(n)}) - \sum_{i=1}^{r} g_i \sigma(b_i) \Big) \Big| < \gamma \,,$$

hence

$$\Big| \sum_{j=1}^{m} \phi(f_j^{(n)}) \sigma(a_j^{(n)}) - \sum_{i=1}^{r} \phi(g_i) \sigma(b_i) \Big| < \gamma \,, \tag{5}$$

By (5), we conclude

$$\sup \Big| \sum_{j=1}^{m} \phi(f_j^{(n)}) \sigma(a_j^{(n)}) - \sum_{i=1}^{r} \phi(g_i) \sigma(b_i) \Big| < \gamma \,, \qquad \|\sigma\| \leq 1 \,,\ \|\phi\|_{\alpha} \leq 1 \,. \tag{6}$$

Therefore $\|q_n - q\|_{\epsilon} \leq \gamma$ and hence $q_n \to q$ or $q_n \to T^{-1}p$ in $L^{\alpha}(K)\check{\otimes}A$. This show that $p \in T(U)^c$.

Remark 2.7. By using the above theorem we can prove that $l^{\alpha}(K, A) \cong l^{\alpha}(K)\check{\otimes}A$.

3. (Weak) Amenability of $L^{\alpha}(K, A)$

Let A be a Banach algebra and X be a Banach A-module over F. The linear map $D\colon A \to X$ is called an X-derivation on A, if $D(ab) = D(a)\cdot b + a \cdot D(b)$, for every $a, b \in A$. The set of all continues X-derivations on A is a vector space over F which is denoted by $Z^1(A, X)$. For each $x \in X$ the map $\delta_x\colon A \to X$, defined by $\delta_x(a) = a \cdot x - x \cdot a$, is a continues X-derivation on A. The X-derivation $D\colon A \to X$ is called an inner derivation on A if there exists an $x \in X$ such that $D = \delta_x$. The set of all inner X-derivations on A is a linear subspace of $Z^1(A, X)$

which is denoted by $B^1(A,X)$. The quotient space $Z^1(A,X)/B^1(A,X)$ is denoted by $H^1(A,X)$ and is called the first cohomology group of A with coefficients in X.

Definition 3.1. The Banach algebra A over F is called amenable if for every Banach A-module X over F, $H^1(A,X^*)=\{0\}$. The Banach algebra A over F is called weakly amenable if $H^1(A,A^*)=\{0\}$.

The notion of amenability of Banach algebras were first introduced by B. E. Johnson in 1972 [8]. Bade, Curtis and Dales [2], studied the (weak) amenability of Lipschitz algebras in 1987 [2]. In this section, we study the (weak) amenability of $L^\alpha(K,A)$.

For every Banach algebra B, let Φ_B be the space of maximal ideal of B.

Definition 3.2. Let A be a commutative Banach algebra and let $\phi\in\Phi_A\cup\{0\}$. The non-zero linear functional D on A is called point derivation at ϕ if

$$D(ab)=\phi(a)D(b)+\phi(b)D(a), \quad (a,b\in A)\,.$$

Lemma 3.3. *For each non-isolated point $x\in K$ and $\sigma\in A^*$, if the map $\phi\colon L^\alpha(K,A)\to \mathrm{C}$ is given by*

$$\phi(f)=(\sigma\circ f)(x), \quad \big(f\in L^\alpha(K,A)\big)$$

then $\phi\in\Phi_{L^\alpha(K,A)}$.

Proof. Obvious. ☐

Let (K,d) be a fixed non-empty compact metric space, set

$$\Delta=\big\{(x,y)\in K\times K : x=y\big\}\,, \quad W=K\times K-\Delta\,.$$

We now examine the amenability and weak amenability of Lipschitz operators algebras $L^\alpha(K,A)$ and $l^\alpha(K,A)$.

Theorem 3.4. *Let (K,d) be an infinite compact metric space and take $\alpha\in(0,1]$. Then $L^\alpha(K,A)$ is not weakly amenable.*

Proof. Let x be a non-isolated point in K. We define

$$W_x:=\big\{\{(x_n,y_n)\}_{n=1}^\infty : (x_n,y_n)\in W,\ (x_n,y_n)\to(x,x) \text{ as } n\to\infty\big\}\,.$$

For the net $w=\{(x_n,y_n)\}_{n=1}^\infty$ in W_x and $\sigma\in A^*$, we put

$$\overline{w}(f)=\frac{(\sigma\circ f)(x_n)-(\sigma\circ f)(y_n)}{d(x_n,y_n)^\alpha}\,, \quad \big(f\in L^\alpha(K,A)\big)$$

then $\|\overline{w}(f)\|_\infty\le\|\sigma\|\ \|f\|_\alpha$. Hence, $\overline{w}$ is continues. Now set

$$D_w(f)=\mathrm{LIM}\big(\overline{w}(f)\big)\,, \quad \big(f\in L^\alpha(K,A)\big)\,,$$

where LIM$(\cdot)$ is Banach limit [12]. We show that the linear map D_w is a non-zero point derivation at ϕ, which ϕ is given by Lemma 6. We have,

$$\begin{aligned} D_w(fg) &= \mathrm{LIM}(\overline{w}(fg)) \\ &= \mathrm{LIM}\, \frac{(\sigma \circ fg)(x_n) - (\sigma \circ fg)(y_n)}{d(x_n, y_n)^\alpha} \\ &= \mathrm{LIM}\, \frac{1}{d(x_n, y_n)^\alpha} \left[\sigma \circ \big(f(x_n)g(x_n) - f(x_n)g(y_n)\big)\right] \\ &= \mathrm{LIM}\, \frac{1}{d(x_n, y_n)^\alpha} \left[\sigma \circ \big(f(x_n)\big(g(x_n) - g(y_n)\big) + g(y_n)\big(f(x_n) - f(y_n)\big)\big)\right] \\ &= (\sigma \circ f)(x)\, \mathrm{LIM}\, \big(\overline{w}(g)\big) + (\sigma \circ g)(x)\, \mathrm{LIM}\, \big(\overline{w}(g)\big) \\ &= \phi(f) D_w(g) + \phi(g) D_w(f) \end{aligned}$$

Therefore, by the continuity f, g and properties of Banach limit we conclude D_w is a non-zero, continues point derivation at ϕ on $L^\alpha(K, A)$, an so by [5], $L^\alpha(K, A)$ is not weakly amenable. □

Corollary 3.5. *$L^\alpha(K, A)$ is not amenable.*

Theorem 3.6. *Let $K \subseteq \mathrm{C}$ be an infinite compact set, and take $\alpha \in (0, 1)$. Then $l^\alpha(K, A)$ is not amenable.*

Proof. Let $x_0 \in K$. We define

$$M_{x_0} := \{f \in l^\alpha(K, A) : \ (\sigma \circ f)(x_0) = 0 \quad \forall \sigma \in A^*\}.$$

If $\sigma \in A^*$, then for each $f \in M_{x_0}^2$ we have

$$\frac{(\sigma \circ f)(x)}{d(x, x_0)^{2\alpha}} \longrightarrow 0 \quad \text{as} \quad d(x, x_0) \longrightarrow 0.$$

For $\beta \in (\alpha, 2\alpha)$, set $f_\beta(x) := \eta\big(d(x, x_0)^\beta\big)$, $x \in K$ where, the map $\eta\colon \mathrm{C} \to A$ defined by $\eta(\lambda) = \lambda \cdot e$. Then $f_\beta \in M_{x_0}$ and $\{f_\beta + M_{x_0}^2 : \ \beta \in (\alpha, 2\alpha)\}$ is a linearly independent set in $\frac{M_{x_0}}{M_{x_0}^2}$ because x_0 is non-isolated in K. Therefore $M_{x_0}^2$ has infinite codimension in M_{x_0}, and so $M_{x_0} \neq M_{x_0}^2$ then by [5] M_{x_0} has not a bounded approximate identity, and since M_{x_0} is closed ideal in $l^\alpha(K, A)$, then $l^\alpha(K, A)$ is not amenable. □

Theorem 3.7. *Let (K, d) be a compact metric space and A be a unital commutative Banach algebra. If $\frac{1}{2} < \alpha < 1$, then $l^\alpha(\mathrm{T}, A)$ is not weakly amenable, where T is unit circle in complex plane.*

Proof. By Remark 2.7, we have $l^\alpha(\mathrm{T}, A) \cong l^\alpha(\mathrm{T}) \check{\otimes} A$. Since by [5], $l^\alpha(\mathrm{T})$ is not weakly amenable, hence $l^\alpha(\mathrm{T}, A)$ is not weakly amenable. □

Corollary 3.8. *Let A be a finite-dimensional weakly amenable Banach algebra. If $0 < \alpha < \frac{1}{2}$, then $l^\alpha(K, A)$ is weakly amenable.*

Proof. By [10], $l^\alpha(K) \hat{\otimes} A$ is weakly amenable. Now by [10], we have $l^\alpha(K) \hat{\otimes} A \cong l^\alpha(K) \check{\otimes} A$ and this implies that $l^\alpha(K) \check{\otimes} A$ is weakly amenable and so $l^\alpha(K, A)$ is weakly amenable. □

4. Homomorphisms on the $L_A^\alpha(X,B)$

Definition 4.1. Let (X,d) be a compact metric space in C, $\alpha\in(0,1]$, $(B,\|\cdot\|)$ be a commutative Banach algebra with unit $\mathbf{e}$, and B^* be the dual space of B, define

$$A(X,B)=\left\{f\in C(X,B):\Lambda\circ f \text{ is analytic in interior of } X,\Lambda\in B^*\right\}$$

$$L_A^\alpha(X,B)=\left\{f\in L^\alpha(X,B):\Lambda\circ f \text{ is analytic in interior of } X,\ \Lambda\in B^*\right\}$$

$$l_A^\alpha(X,B)=\left\{f\in l^\alpha(X,B):\Lambda\circ f \text{ is analytic in interior of } X,\ \Lambda\in B^*\right\}$$

In this case, we have

$$L_A^\alpha(X,B)=L^\alpha(X,B)\cap A(X,B)$$

and

$$l_A^\alpha(X,B)=l^\alpha(X,B)\cap A(X,B)\,.$$

So $L_A^\alpha(X,B)\cong L_A^\alpha(X)\check{\otimes}B$ and $l_A^\alpha(X,B)\cong l_A^\alpha(X)\check{\otimes}B$.

Theorem 4.2. *Every character χ on $L_A^\alpha(X,B)$ (and $l_A^\alpha(X,B)$) is of form $\chi=\psi\circ\delta_z$ for some character ψ on B and some $z\in X$.*

Proof. Since $L_A^\alpha(X,B)\cong L_A^\alpha(X)\check{\otimes}B$, let $j\colon L^\alpha(X)\to L_A^\alpha(X,B)$, $h\mapsto h\otimes\mathbf{e}$, be the canonical embedding. Then there is $z\in X$ such that $\chi\circ j$ is the evaluation in z, that is $\chi\circ j=\delta_z$ where $\delta_z(\varphi)=\varphi(z)$. Consider the ideal

$$I:=\left\{f\in L_A^\alpha(X,B):f(z)=0\right\}.$$

We will show that I is contained in the kernel of χ. Given $f\in I$ we define,

$$\varphi(\omega):=\begin{cases}\omega-z & \text{if}\quad \omega\neq z;\\ 0 & \text{if}\quad \omega=z.\end{cases}$$

and

$$g(\omega):=\begin{cases}\dfrac{f(\omega)}{\omega-z} & \text{if}\quad \omega\neq z;\\ f'(z) & \text{if}\quad \omega=z.\end{cases}$$

Since f has a Taylor series expansion

$$f(\omega)=\sum_{n=1}^{\infty}\frac{f^{(n)}(z)}{n!}(\omega-z)^n$$

around z, it is easy to see that $\Lambda\circ g$ is holomorphic ($\Lambda\in B^*$), and hence $g\in L_A^\alpha(X,B)$. We have

$$\chi(f)=\chi\big(j(\varphi)g\big)=(\chi\circ j)(\varphi)\chi(g)=\delta_z(\varphi)\chi(g)=\varphi(z)\chi(g)=0\,.$$

The evaluation δ_z is an epimorphism and since $\ker\delta_z=I\subset\ker\chi$, we obtain the desired factorization $\chi=\psi\circ\delta_z$ for some character ψ on B. $\square$

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Faculty of Basic Science, Science and Research Brunch
Islamic Azad University (IAU), Tehran, Iran
E-mail: `a-shokri@iau-ahar.ac.ir`

Department of Mathematics, Faculty of Science
Urmia University, Urmia, Iran
E-mail: `a.ebadian@urmia.ac.ir`