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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF REAL TWO-DIMENSIONAL DIFFERENTIAL SYSTEM WITH NONCONSTANT DELAY

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Abstract. In this article, stability and asymptotic properties of solutions of a real two-dimensional system $x'(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)x(\tau(t)) + \mathbf{h}(t, x(t), x(\tau(t)))$ are studied, where $\mathbf{A}$, $\mathbf{B}$ are matrix functions, $\mathbf{h}$ is a vector function and $\tau(t) \le t$ is a nonconstant delay which is absolutely continuous and satisfies $\lim\limits_{t\to\infty} \tau(t) = \infty$. Generalization of results on stability of a two-dimensional differential system with one constant delay is obtained using the methods of complexification and Lyapunov-Krasovskii functional and some new corollaries and examples are presented.

1. Introduction

The investigation of the problem is based on the combination of the method of complexification and the method of Lyapunov-Krasovskii functional, which is to a great extent effective for two-dimensional systems. This combination was successfully used in [3] for two-dimensional system of ODE's, in [2] for system with one constant delay and in [4] and [6] for systems with a finite number of constant delays.

The article is related to the paper [1] where asymptotic properties of system with one nonconstant delay were studied. The aim is, under some special conditions, to improve the results presented in [1] and to illustrate the advancement with an example.

The subject of our study is the real two-dimensional system

$$x'(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)x\big(\tau(t)\big) + \mathbf{h}\big(t, x(t), x\big(\tau(t)\big)\big) \tag{0}$$

where $\mathbf{A}(t) = \big(a_{ik}(t)\big)$, $\mathbf{B}(t) = \big(b_{ik}(t)\big)$ $(i, k = 1, 2)$ are real square matrices and $\mathbf{h}(t, x, y) = \big(h_1(t, x, y), h_2(t, x, y)\big)$ is a real vector function. We suppose that the functions a_{ik} are locally absolutely continuous on $[t_0, \infty)$, b_{ik} are locally Lebesgue integrable on $[t_0, \infty)$ and the function $\mathbf{h}$ satisfies Carathéodory conditions on

$$[t_0, \infty) \times \big\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 < R^2\big\} \times \big\{(y_1, y_2) \in \mathbb{R}^2 \colon y_1^2 + y_2^2 < R^2\big\}\,,$$

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where $0 < R \leq \infty$ is a real constant.

The following notation will be used throughout the article:

$\mathbb{R}$	set of all real numbers
$\mathbb{R}_+$	set of all positive real numbers
$\mathbb{R}_+^0$	set of all nonnegative real numbers
$\mathbb{C}$	set of all complex numbers
$\mathbb{N}$	set of all positive integers
$\operatorname{Re} z$	real part of z
$\operatorname{Im} z$	imaginary part of z
$\overline{z}$	complex conjugate of z
$AC_{\mathrm{loc}}(I, M)$	class of all locally absolutely continuous functions $I \to M$
$L_{\mathrm{loc}}(I, M)$	class of all locally Lebesgue integrable functions $I \to M$
$K(I \times \Omega, M)$	class of all functions $I \times \Omega \to M$ satisfying Carathéodory conditions on $I \times \Omega$.

Introducing complex variables $z = x_1 + ix_2$, $w = y_1 + iy_2$, we can rewrite the system (0) into an equivalent equation with complex-valued coefficients

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \big[A(t)z(\tau(t)) + B(t)\overline{z}(\tau(t))\big] + g\big(t, z(t), z\big(\tau(t)\big)\big)\,,$$

where

$$\begin{aligned}
a(t) &= \frac{1}{2}\big(a_{11}(t) + a_{22}(t)\big) + \frac{i}{2}\big(a_{21}(t) - a_{12}(t)\big)\,,\\
b(t) &= \frac{1}{2}\big(a_{11}(t) - a_{22}(t)\big) + \frac{i}{2}\big(a_{21}(t) + a_{12}(t)\big)\,,\\
A(t) &= \frac{1}{2}\big(b_{11}(t) + b_{22}(t)\big) + \frac{i}{2}\big(b_{21}(t) - b_{12}(t)\big)\,,\\
B(t) &= \frac{1}{2}\big(b_{11}(t) - b_{22}(t)\big) + \frac{i}{2}\big(b_{21}(t) + b_{12}(t)\big)\,,\\
g(t, z, w) &= h_1\Big(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2i}(z - \overline{z}), \frac{1}{2}(w + \overline{w}), \frac{1}{2i}(w - \overline{w})\Big)\\
&\quad + ih_2\Big(t, \frac{1}{2}\big(z + \overline{z}\big), \frac{1}{2i}(z - \overline{z}), \frac{1}{2}\big(w + \overline{w}\big), \frac{1}{2i}(w - \overline{w})\Big)\,.
\end{aligned}$$

Conversely, the last equation can be written in the real form (0) as well, the relations are similar as in [3].

2. Results

We study the equation

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + A(t)z\big(\tau(t)\big) + B(t)\overline{z}\big(\tau(t)\big) + g\big(t, z(t), z\big(\tau(t)\big)\big)\,, \tag{1}$$

where $\tau \in AC_{\mathrm{loc}}(J, \mathbb{R})$, $A, B \in L_{\mathrm{loc}}(J, \mathbb{C})$, $a, b \in AC_{\mathrm{loc}}(J, \mathbb{C})$, $g \in K(J \times \Omega, \mathbb{C})$, where $J = [t_0, \infty)$, $\Omega = \big\{(z, w) \in \mathbb{C}^2 \colon |z| < R, |w| < R, \big\}$, $R > 0$.

Without the loss of generality, we will assume that $R = \infty$, thus $\Omega = \mathbb{C}^2$.

In this article we consider the case

$$\text{(2')} \qquad \liminf_{t\to\infty}\big(|\operatorname{Im} a(t)| - |b(t)|\big) > 0, \quad \tau(t) \le t, \quad \lim_{t\to\infty} \tau(t) = \infty$$

and study the behavior of solutions of (1) under this assumptions.

Obviously, this case is included in the case $\liminf_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0$ considered in [1], but in this special case we are able to derive more useful results as we will see later in an example. The idea is based upon the well known result that the condition $|a| > |b|$ in an autonomous equation $z' = az + b\overline{z}$ ensures that zero is a focus, a centre or a node while under the condition $|\operatorname{Im} a| > |b|$ zero can be just a focus or a centre. Details are contained in [3].

The conditions (2') imply the existence of $T_1 \ge t_0$, $T \ge T_1$ and $\mu > 0$ such that

$$\text{(2)} \qquad |\operatorname{Im} a(t)| > |b(t)| + \mu \quad \text{for} \quad t \ge T_1, \quad t \ge \tau(t) \ge T_1 \quad \text{for} \quad t \ge T\,.$$

Denote

$$\text{(3)} \qquad \gamma(t) = \operatorname{Im} a(t) + \sqrt{(\operatorname{Im} a(t))^2 - |b(t)|^2}\, \operatorname{sgn}\big(\operatorname{Im} a(t)\big)\,, \quad c(t) = -ib(t)\,.$$

Since $|\gamma(t)| > |\operatorname{Im} a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$\text{(4)} \qquad |\gamma(t)| > |c(t)| + \mu$$

is true for all $t \ge T_1$. It is easy to verify that $\gamma, c \in AC_{\mathrm{loc}}([T_1,\infty),\mathbb{C})$.

For the purpose of this paper we denote

$$\text{(5)} \qquad \vartheta(t) = \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \overline{c}(t)c'(t)) + |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2}\,.$$

In the text we will consider following conditions:

(i) The numbers $T_1 \ge t_0$, $T \ge T_1$ and $\mu > 0$ are such that (2) holds.

(ii) There are functions $\kappa_0, \kappa_1, \lambda\colon [T,\infty) \to \mathbb{R}$ such that

$$|\gamma(t)g(t,z,w) + c(t)\overline{g}(t,z,w)| \le \kappa_0(t)|\gamma(t)z + c(t)\overline{z}| + \kappa_1(t)|\gamma\big(\tau(t)\big)w + c\big(\tau(t)\big)\overline{w}| + \lambda(t)$$

for $t \ge T$, $z \in \mathbb{C}$ and $w \in \mathbb{C}$, where $\kappa_0, \lambda \in L_{\mathrm{loc}}\big([T,\infty),\mathbb{R}\big)$.

(iii) $\beta \in AC_{\mathrm{loc}}\big([T,\infty),\mathbb{R}^0_+\big)$ is a function satisfying

$$\text{(6)} \qquad \tau'(t)\beta(t) \ge \psi(t) \quad \text{a.e. on } [T,\infty)\,,$$

where ψ is defined for every $t \ge T$ by

$$\text{(7)} \qquad \psi(t) = \kappa_1(t) + \big(|A(t)| + |B(t)|\big)\frac{|\gamma(t)| + |c(t)|}{|\gamma(\tau(t))| - |c(\tau(t))|}\,.$$

(iv) The function $\Lambda \in L_{\mathrm{loc}}\big([T,\infty),\mathbb{R}\big)$ satisfies the inequalities $\beta'(t) \le \Lambda(t)\beta(t)$, $\theta(t) \le \Lambda(t)$ for almost all $t \in [T,\infty)$, where the function θ is defined by

$$\text{(8)} \qquad \theta(t) = \operatorname{Re} a(t) + \vartheta(t) + \kappa_0(t) + \beta(t)\,.$$

Clearly, if A, B, κ_1 are absolutely continuous on $[T,\infty)$ and $\psi(t) \geq 0$ on $[T,\infty)$, we may choose $\beta(t) = \psi(t)$.

Under the assumption (i), we can estimate

$$\begin{aligned} |\vartheta| &\leq \frac{|\operatorname{Re}(\gamma\gamma' - \bar{c}c')| + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{(|\gamma'| + |c'|)(|\gamma| + |c|)}{\gamma^2 - |c|^2} \\ &= \frac{|\gamma'| + |c'|}{|\gamma| - |c|} \leq \frac{1}{\mu}(|\gamma'| + |c'|), \end{aligned}$$

hence the functions ϑ and θ are locally Lebesgue integrable on $[T,\infty)$. Moreover, if $\beta \in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$, then in (iv) we may choose

$$\Lambda(t) = \max\Big(\theta(t), \frac{\beta'(t)}{\beta(t)}\Big) .$$

Finally, if $\lambda(t) \equiv 0$ in (ii), then equation (1) has the trivial solution $z(t) \equiv 0$. Notice that in this case the condition (ii) implies that the functions $\kappa_j(t)$ are nonnegative on $[T,\infty)$ for $j = 0, 1$, and due to this, $\psi(t) \geq 0$ on $[T,\infty)$.

In the proof of the main theorem we will need

Lemma 1. *Let a_1, a_2, b_1, $b_2 \in \mathbb{C}$ and $|a_2| > |b_2|$. Then*

$$\operatorname{Re} \frac{a_1 z + b_1 \bar{z}}{a_2 z + b_2 \bar{z}} \leq \frac{\operatorname{Re}(a_1 \overline{a_2} - b_1 \overline{b_2}) + |a_1 b_2 - a_2 b_1|}{|a_2|^2 - |b_2|^2}$$

for $z \in \mathbb{C}$, $z \neq 0$.

For the proof see [4] or [3].

Theorem 1. *Let the conditions* (i), (ii), (iii) *and* (iv) *hold and $\lambda(t) \equiv 0$.*

a) *If*

$$\limsup_{t\to\infty} \int^t \Lambda(s)\, ds < \infty\,, \tag{9}$$

then the trivial solution of (1) *is stable on $[T,\infty)$;*

b) *if*

$$\lim_{t\to\infty} \int^t \Lambda(s)\, ds = -\infty\,, \tag{10}$$

then the trivial solution of (1) *is asymptotically stable on $[T,\infty)$.*

Proof. The proof is similar to that of Theorem 1 from [1].

Choose arbitrary $t_1 \geq T$. Let $z(t)$ be any solution of (1) satisfying the condition $z(t) = z_0(t)$ for $t \in [T_1, t_1]$, where $z_0(t)$ is a continuous complex-valued initial function defined on $t \in [T_1, t_1]$. Consider Lyapunov function

$$V(t) = U(t) + \beta(t) \int_{\tau(t)}^t U(s)\, ds\,, \tag{11}$$

where

$$U(t) = |\gamma(t) z(t) + c(t) \bar{z}(t)|\,.$$

To simplify the following computation, denote $w(t) = z\big(\tau(t)\big)$ and write the functions of variable t without brackets, for example, z instead of $z(t)$.

From (11) we get

$$V' = U' + \beta' \int_{\tau(t)}^{t} U(s)\,ds + \beta|\gamma z + c\overline{z}| - \tau'\beta|\gamma(\tau)w + c(\tau)\overline{w}| \tag{12}$$

for almost all $t \ge t_1$ for which $z(t)$ is defined and $U'(t)$ exists.

Denote $\mathcal{K} = \{t \ge t_1 \colon z(t) \text{ exists}, U(t) \ne 0\}$ and $\mathcal{M} = \{t \ge t_1 \colon z(t) \text{ exists}, U(t) = 0\}$. It is clear that the derivative $U'(t)$ exists for almost all $t \in \mathcal{K}$, and the existence of the derivative which is zero almost everywhere in the set $\mathcal{M}$ can be proved in the same way as in [1] or [2].

In particular, the derivative U' exists for almost all $t \ge t_1$ for which $z(t)$ is defined, thus (12) holds for almost all $t \ge t_1$ for which $z(t)$ is defined.

Now turn our attention to the set $\mathcal{K}$. For almost all $t \in \mathcal{K}$ it holds that $UU' = U\big(\sqrt{(\gamma z + c\overline{z})(\overline{\gamma z} + \overline{c}z)}\big)' = \operatorname{Re}\big[(\gamma\overline{z} + \overline{c}z)(\gamma' z + \gamma z' + c'\overline{z} + c\overline{z}')\big]$. As $z(t)$ is a solution of (1), we have

$$\begin{aligned} UU' &= \operatorname{Re}\big\{(\gamma\overline{z} + \overline{c}z)\big[\gamma' z + c'\overline{z} + \gamma\big(az + b\overline{z} + Aw + B\overline{w} + g\big) \\ &\qquad + c\big(\overline{az} + \overline{b}z + \overline{A}\overline{w} + \overline{B}w + \overline{g}\big)\big]\big\} \\ &= \operatorname{Re}\big\{(\gamma\overline{z} + \overline{c}z)\big[\gamma' z + c'\overline{z} + (\gamma a + c\overline{b})z + (\gamma b + c\overline{a})\overline{z} + \gamma\big(Aw + B\overline{w_j} + g\big) \\ &\qquad + c\big(\overline{A}\overline{w} + \overline{B}w + \overline{g}\big)\big]\big\} \end{aligned}$$

for almost all $t \in \mathcal{K}$. Short computation gives $(\gamma a + c\overline{b})c = (\gamma b + c\overline{a})\gamma$, and from this we get

$$\begin{aligned} UU' \le{}& \operatorname{Re}\{(\gamma\overline{z} + \overline{c}z)(\gamma' z + c'\overline{z})\} + \operatorname{Re}\Big\{(\gamma\overline{z} + \overline{c}z)(\gamma a + c\overline{b})\Big(z + \frac{c}{\gamma}\overline{z}\Big)\Big\} \\ &+ \operatorname{Re}\big\{(\gamma\overline{z} + \overline{c}z)\big(\gamma(Aw + B\overline{w}) + c(\overline{A}\overline{w} + \overline{B}w)\big)\big\} \\ &+ \operatorname{Re}\big\{(\gamma\overline{z} + \overline{c}z)(\gamma g + c\overline{g})\big\}\,. \end{aligned}$$

Consequently,

$$UU' \le U^2 \operatorname{Re}\Big(a + \frac{c}{\gamma}\overline{b}\Big) + U\big(|\gamma| + |c|\big)|Aw + B\overline{w}| + U|\gamma g + c\overline{g}| + U^2 \operatorname{Re}\frac{\gamma' z + c'\overline{z}}{\gamma z + c\overline{z}}$$

for almost all $t \in \mathcal{K}$. Applying Lemma 1 to the last term, we obtain

$$\operatorname{Re}\frac{\gamma' z + c'\overline{z}}{\gamma z + c\overline{z}} \le \vartheta\,.$$

Using this inequality together with (7), the assumption (ii) and the relation $\operatorname{Re}\left(a+\frac{c}{\gamma}\overline{b}\right)=\operatorname{Re}a$, we obtain

$$\begin{aligned}UU' &\le U^2(\operatorname{Re}a+\vartheta+\kappa_0)+U\big(\kappa_1|\gamma(\tau)w+c(\tau)\overline{w}|\big)\\ &\quad+U(|\gamma|+|c|)\Big(\frac{|A||w|+|B||\overline{w}|}{|\gamma(\tau)|-|c(\tau)|}(|\gamma(\tau)|-|c(\tau)|)\Big)\\ &\le U^2(\operatorname{Re}a+\vartheta+\kappa_0)+U\Big[\kappa_1+(|A|+|B|)\frac{|\gamma|+|c|}{|\gamma(\tau)|-|c(\tau)|}\Big]|\gamma(\tau)w+c(\tau)\overline{w}|\\ &\le U^2(\operatorname{Re}a+\vartheta+\kappa_0)+U\psi|\gamma(\tau)w+c(\tau)\overline{w}|\end{aligned}$$

for almost all $t\in\mathcal{K}$. Consequently,

$$U'\le U(\operatorname{Re}a+\vartheta+\kappa_0)+\psi|\gamma(\tau)w+c(\tau)\overline{w}| \tag{13}$$

for almost all $t\in\mathcal{K}$.

Recalling that $U'(t)=0$ for almost all $t\in\mathcal{M}$, we can see that the inequality (13) is valid for almost all $t\ge t_1$ for which $z(t)$ is defined.

From (12) and (13) we have

$$V'\le U(\operatorname{Re}a+\vartheta+\kappa_0+\beta)+(\psi-\tau'\beta)\,|\gamma(\tau)w+c(\tau)\overline{w}|+\beta'\int_{\tau(t)}^{t}U(s)\,ds\,.$$

As the functions $\tau(t)$ and $\beta(t)$ fulfill the condition (6), we obtain

$$V'(t)\le U(t)\theta(t)+\beta'(t)\int_{\tau(t)}^{t}U(s)\,ds\,, \tag{14}$$

and from the assumption (iv) we get

$$V'(t)-\Lambda(t)V(t)\le 0 \tag{15}$$

for almost all $t\ge t_1$ for which the solution $z(t)$ exists.

Notice that, with respect to (4),

$$V(t)\ge U(t)\ge\big(|\gamma(t)|-|c(t)|\big)|z(t)|\ge\mu|z(t)| \tag{16}$$

for all $t\ge t_1$ for which $z(t)$ is defined.

Suppose that the condition (9) holds, and choose arbitrary $0<\varepsilon$. Put

$$\Delta=\max_{s\in[T_1,t_1]}\big(|\gamma(s)|+|c(s)|\big)\,,\qquad L=\sup_{T\le t<\infty}\int_T^t\Lambda(s)\,ds$$

and

$$\delta=\mu\varepsilon\Delta^{-1}\big(1+\beta(t_1)(t_1-T_1)\big)^{-1}\exp\Big\{\int_T^{t_1}\Lambda(s)ds-L\Big\}\,,$$

where $\mu>0$, $T_1\ge t_0$ and $T\ge T_1$ are the numbers from the condition (i).

If the initial function $z_0(t)$ of the solution $z(t)$ satisfies $\max\limits_{s\in[T_1,t_1]}|z_0(s)|<\delta$, then the multiplication of (15) by $\exp\{-\int_{t_1}^t\Lambda(s)\,ds\}$ and the integration over $[t_1,t]$ yield

$$V(t)\exp\Big\{-\int_{t_1}^{t}\Lambda(s)\,ds\Big\}-V(t_1)\le 0 \tag{17}$$

for all $t \geq t_1$ for which $z(t)$ is defined. From (16) and (17) we gain

$$\begin{aligned}\mu|z(t)| \leq V(t) \leq V(t_1)\exp\Big\{\int_{t_1}^{t}\Lambda(s)ds\Big\} &\leq \Big[(|\gamma(t_1)|+|c(t_1)|)|z(t_1)| \\ &\quad + \beta(t_1)\max_{s\in[T_1,t_1]}|z(s)|\int_{T_1}^{t_1}\big(|\gamma(s)|+|c(s)|\big)\,ds\Big]\exp\Big\{\int_{t_1}^{t}\Lambda(s)\,ds\Big\} \\ &\leq \Big[\Delta\max_{s\in[T_1,t_1]}|z_0(s)|+\beta(t_1)\max_{s\in[T_1,t_1]}|z_0(s)|\Delta(t_1-T_1)\Big]\exp\Big\{\int_{t_1}^{t}\Lambda(s)\,ds\Big\},\end{aligned}$$

i.e.,

$$\mu|z(t)| \leq \Delta\max_{s\in[T_1,t_1]}|z_0(s)|\big[1+\beta(t_1)(t_1-T_1)\big]\exp\Big\{L-\int_{T}^{t_1}\Lambda(s)ds\Big\} < \mu\varepsilon\,.$$

Thus we have $|z(t)| < \varepsilon$ for all $t \geq t_1$ and we conclude that the trivial solution of the equation (1) is stable.

Now suppose that the condition (10) is valid. Then, in view of the first part of Theorem 1, for $K > 0$ there is a $\rho > 0$ such that $\max\limits_{s\in[T_1,t_1]}|z_0(s)| < \rho$ implies that the solution $z(t)$ of (1) exists for all $t \geq t_1$ and satisfies $|z(t)| < K$, where K is arbitrary real constant. Hence, from this and (16), we have

$$|z(t)| \leq \mu^{-1}V(t) \leq \mu^{-1}V(t_1)\exp\Big\{\int_{t_1}^{t}\Lambda(s)\,ds\Big\}$$

for all $t \geq t_1$. This inequality with the condition (10) give

$$\lim_{t\to\infty} z(t) = 0\,,$$

which completes the proof. □

Remark 1. Since

$$\vartheta = \frac{\operatorname{Re}(\gamma\gamma' - \bar{c}c') + |\gamma c' - \gamma' c|}{\gamma^2 - |c|^2} \leq \frac{(|\gamma'|+|c'|)(|\gamma|+|c|)}{\gamma^2-|c|^2} = \frac{|\gamma'|+|c'|}{|\gamma|-|c|}\,,$$

it follows from (4) that we can replace the function ϑ in (8) by $\frac{1}{\mu}(|\gamma'|+|c'|)$.

We use following Corollary 1 to find an important example which shows, in connection with the article [1], that it is worth to consider the condition (2').

Corollary 1. *Let $a(t) \equiv a \in \mathbb{C}$, $b(t) \equiv b \in \mathbb{C}$, $|\operatorname{Im} a| > |b|$. Suppose that $\lim\limits_{t\to\infty}\tau(t) = \infty$, $\tau(t) \leq t$ for $t \geq T_1$, where $T_1 \geq t_0$. Let $\rho_0, \rho_1\colon [T_1,\infty) \to \mathbb{R}$ be such that*

$$|g(t,z,w)| \leq \rho_0(t)|z| + \rho_1(t)|w| \tag{18}$$

for $t \geq T_1$, $|z| < R$, $|w| < R$ and $\rho_0 \in L_{\mathrm{loc}}([T_1,\infty),\mathbb{R})$.
Let $\beta \in AC_{\mathrm{loc}}([T_1,\infty),\mathbb{R}_+)$ satisfy

$$\tau'(t)\beta(t) \geq \left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{\frac{1}{2}}\big(\rho_1(t)+|A(t)|+|B(t)|\big) \quad a.e.\ on\ [T_1,\infty)\,.$$

If

$$\limsup_{t\to\infty}\int^t \max\Big(\operatorname{Re} a+\Big(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\Big)^{\frac12}\rho_0(s)+\beta(s),\frac{\beta'(s)}{\beta(s)}\Big)\,ds<\infty\,, \tag{19}$$

then the trivial solution of equation (1) *is stable. If*

$$\lim_{t\to\infty}\int^t \max\Big(\operatorname{Re} a+\Big(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\Big)^{\frac12}\rho_0(s)+\beta(s),\frac{\beta'(s)}{\beta(s)}\Big)\,ds=-\infty\,, \tag{20}$$

then the trivial solution of (1) *is asymptotically stable.*

Proof. Choose $T\ge T_1$ such that $\tau(t)\ge T_1$ for $t\ge T$. Denote $z=z(t)$ and $w=z(\tau(t))$ again. Since $a,b\in\mathbb{C}$ are constants, then also γ and c are constants and we have $\vartheta(t)\equiv 0$. Using the condition (18) we get

$$\begin{aligned}|\gamma g(t,z,w)+c\overline{g}(t,z,w)| &\le (|\gamma|+|c|)\big(\rho_0(t)|z|+\rho_1(t)|w|\big)\\ &=\frac{|\gamma|+|c|}{|\gamma|-|c|}(|\gamma|-|c|)\big(\rho_0(t)|z|+\rho_1(t)|w|\big)\\ &\le\frac{|\gamma|+|c|}{|\gamma|-|c|}\big(\rho_0(t)|\gamma z+c\overline{z}|+\rho_1(t)|\gamma w+c\overline{w}|\big)\end{aligned}$$

and it follows that the condition (ii) holds with

$$\kappa_0(t)=\frac{|\gamma|+|c|}{|\gamma|-|c|}\rho_0(t)\,,\qquad \kappa_1(t)=\frac{|\gamma|+|c|}{|\gamma|-|c|}\rho_1(t)$$

and $\lambda(t)\equiv 0$.

Since

$$\frac{|\gamma|+|c|}{|\gamma|-|c|}=\frac{|\operatorname{Im} a|+\sqrt{|\operatorname{Im} a|^2-|b|^2}+|b|}{|\operatorname{Im} a|+\sqrt{|\operatorname{Im} a|^2-|b|^2}-|b|}=\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{\frac12}\,,$$

in view of (8) we obtain

$$\begin{aligned}\psi(t)&=\left(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\right)^{\frac12}\big(\rho_1(t)+|A(t)|+|B(t)|\big)\,,\\ \theta(t)&=\operatorname{Re} a+\frac{|\gamma|+|c|}{|\gamma|-|c|}\rho_0(t)+\beta(t)=\operatorname{Re} a+\Big(\frac{|\operatorname{Im} a|+|b|}{|\operatorname{Im} a|-|b|}\Big)^{\frac12}\rho_0(t)+\beta(t)\,.\end{aligned}$$

However, $\beta(t)$ is positive on $[T,\infty)$. Hence we may choose $\Lambda(t)=\max\big(\theta(t),\frac{\beta'(t)}{\beta(t)}\big)$ and the assertion follows from Theorem 1. □

Now we are able to give an example mentioned on the page 225 in connection with the article [1].

Example 1. Consider equation (1), where $a(t)\equiv-\sqrt5+2i$, $b(t)\equiv 1$, $A(t)\equiv 0$, $B(t)\equiv 0$,

$$g(t,z,w)=\frac{2}{\sqrt3}\,\mathrm{e}^{it}\,z+\frac{1}{2t}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t}\,w\,.$$

Assume that $t_0 = 1$, $R = \infty$ and $\tau(t) = \ln(t)$. Put $T_1 = t_0$, $T = \mathrm{e}^{T_1} = \mathrm{e}$. Then $\rho_0(t) \equiv \frac{2}{\sqrt{3}}$, $\rho_1(t) = \frac{1}{2t}(\sqrt{15} - \sqrt{14})\,\mathrm{e}^{-t}$. We have

$$\max\Big(\frac{|a|-|b|}{|a|}\operatorname{Re} a + \Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac12}\rho_0(t) + \beta(t), \frac{\beta'(t)}{\beta(t)}\Big)$$
$$= \max\Big(-\frac{2}{3}\sqrt{5} + \sqrt{2}\frac{2}{\sqrt{3}} + \beta(t), \frac{\beta'(t)}{\beta(t)}\Big) \geq \frac{2}{3}(\sqrt{6} - \sqrt{5}) > 0$$

for

$$\tau'(t)\beta(t) = \frac{1}{t}\beta(t) \geq \Big(\frac{|a|+|b|}{|a|-|b|}\Big)^{\frac12}\big(\rho_1(t) + |A(t)| + |B(t)|\big) = \frac{1}{t\sqrt{2}}(\sqrt{15} - \sqrt{14})\,\mathrm{e}^{-t}\,,$$

hence we cannot apply Corollary 2.5 from the paper [1].

On the other hand, if we use

$$\tau'(t)\beta(t) = \frac{1}{t}\beta(t) = \frac{\sqrt{3}}{2t}(\sqrt{15}-\sqrt{14})\,\mathrm{e}^{-t} \geq \Big(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\Big)^{\frac12}\big(\rho_1(t)+|A(t)|+|B(t)|\big)\,,$$

we have

$$\max\Big(\operatorname{Re} a + \Big(\frac{|\operatorname{Im} a| + |b|}{|\operatorname{Im} a| - |b|}\Big)^{\frac12}\rho_0(t) + \beta(t), \frac{\beta'(t)}{\beta(t)}\Big)$$
$$= \max\Big(-\sqrt{5} + 2 + \frac{\sqrt{3}}{2}(\sqrt{15} - \sqrt{14})\,\mathrm{e}^{-t}, -1\Big)$$
$$\leq -\sqrt{5} + 2 + \frac{\sqrt{3}}{2}(\sqrt{15} - \sqrt{14}) < -\frac{12}{100} < 0\,,$$

thus Corollary 1 guarantees the stability and also asymptotic stability of the trivial solution of the considered equation.

The following corollary is similar to Corollary 2 in [5]. However, the conditions are more general, therefore the proof is included.

Corollary 2. *Assume that the conditions* (i), (ii) *and* (iii) *are valid with* $\lambda(t) \equiv 0$. *If* $\beta(t)$ *is monotone and bounded on* $[T, \infty)$ *and if*

$$\limsup_{t\to\infty} \int^t [\theta(s)]_+ ds < \infty,$$

where $[\theta(t)]_+ = \max\{\theta(t), 0\}$, *then the trivial solution of* (1) *is stable.*

Proof. Suppose firstly that β is non-increasing on $[T, \infty)$. Then $\beta' \leq 0$ a.e. on $[T, \infty)$.

If $\beta(T_2) = 0$ for some $T_2 \geq T$, then $\beta(t) \equiv 0$ on $[T_2, \infty)$. Consequently, Λ has to satisfy only the inequality $\theta(t) \leq \Lambda(t)$ a.e. on $[T_2, \infty)$, so we may choose $\Lambda(t) = \theta(t)$ on $[T_2, \infty)$. It follows that $\Lambda(t) = \theta(t) \leq \max\{\theta(t), 0\} = [\theta(t)]_+$.

On the other way, if $\beta(t) > 0$ on $[T, \infty)$, we may put $\Lambda(t) = \max\big\{\theta(t), \frac{\beta'(t)}{\beta(t)}\big\}$. Then

$$\Lambda(t) = \max\Big\{\theta(t), \frac{\beta'(t)}{\beta(t)}\Big\} \leq \max\{\theta(t), 0\} = [\theta(t)]_+\,.$$

In both cases, Λ satisfies the condition (iv) and the inequality $\Lambda(t) \le [\theta(t)]_+$ on $[T_2, \infty)$, hence

$$\limsup_{t\to\infty} \int^t \Lambda(s)\,ds \le \limsup_{t\to\infty} \int^t [\theta(s)]_+ ds < \infty\,.$$

Now assume that β is non-decreasing on $[T, \infty)$. Then $\beta' \ge 0$ a.e. on $[T, \infty)$.

If $\beta(t) \equiv 0$ on $[T, \infty)$, we may treat it as above.

Otherwise, there is some $T_3 \ge T$ such that $\beta(t) > 0$ on $[T_3, \infty)$ and we may choose $\Lambda(t) = \max\bigl\{\theta(t), \frac{\beta'(t)}{\beta(t)}\bigr\}$ on $[T_3, \infty)$. Clearly Λ satisfies the condition (iv) on $[T_3, \infty)$. Since $\beta' \ge 0$ a.e. on $[T, \infty)$, it follows that $\frac{\beta'}{\beta} \ge 0$ a.e. on $[T_3, \infty)$. Hence

$$\Lambda(t) = \max\bigl\{\theta(t), \frac{\beta'(t)}{\beta(t)}\bigr\} \le \bigl\{[\theta(t)]_+, \frac{\beta'(t)}{\beta(t)}\bigr\} \le [\theta(t)]_+ + \frac{\beta'(t)}{\beta(t)}$$

and then

$$\begin{aligned}\limsup_{t\to\infty} \int^t \Lambda(s)\,ds &\le \limsup_{t\to\infty} \int^t [\theta(s)]_+ ds + \limsup_{t\to\infty} \int^t \frac{\beta'(t)}{\beta(t)}\,ds \\ &\le \limsup_{t\to\infty} \int^t [\theta(s)]_+ ds + \limsup_{t\to\infty} \bigl(\ln(\beta(t))\bigr) - \ln(\beta(T_3)) < \infty\end{aligned}$$

since β is bounded on $[T, \infty)$.

The statement follows from Theorem 1. $\square$

Remark 2. If the function β does not satisfy the assumptions of Corollary 2, we can try to find function $\beta^*(t) \ge \beta(t)$ which is monotone and bounded on $[T, \infty)$ such that all conditions stated in Corollary 2 become true.

However, in some cases it is not possible to find such function β^* while the trivial solution of the equation (1) is stable, as we can see in Example 2.

Example 2. Let us study the equation (1) where $a(t) \equiv -4-3i$, $b(t) \equiv 1$, $A(t) \equiv 0$, $B(t) \equiv 0$,

$$g(t, z, w) = \frac{1}{\sqrt{2}}(1 - \sin t)\,\mathrm{e}^{it}\, z + \frac{1}{t\sqrt{2}}(2 + \sin t)w\,.$$

Assume that $t_0 = 1$, $R = \infty$ and $\tau(t) = \ln(t)$. Put $T_1 = t_0$, $T = \mathrm{e}^{T_1} = \mathrm{e}$. As g satisfies the inequality (18) from Corollary refc1 with $\rho_0(t) = \frac{1}{\sqrt{2}}(1 - \sin t)$ and $\rho_1(t) = \frac{1}{t\sqrt{2}}(2 + \sin t)$, the condition (ii) is valid for $\kappa_0(t) = 1 - \sin t$, $\kappa_1(t) = \frac{1}{t}(2 + \sin t)$ and $\lambda(t) \equiv 0$. Moreover, $\vartheta(t) \equiv 0$.

Since $\kappa_1(t) \in AC_{\mathrm{loc}}([T, \infty), \mathbb{R}_+)$ and $\tau'(t) > 0$ on $[T, \infty)$, we may choose $\beta(t) = \bigl(\tau'(t)\bigr)^{-1}\kappa_1(t) = 2 + \sin t$, hence the minimal function β^* which satisfies the conditions of Corollary 2 is constant function $\beta^* \equiv 3$. For this β^* we have $\theta^*(t) = \operatorname{Re} a + \vartheta(t) + \kappa_0(t) + \beta^*(t) = -4 + 0 + 1 - \sin t + 3 = -\sin t$. It is not difficult to compute that $\limsup\limits_{t\to\infty} \int_T^t [\theta^*(s)]_+ ds = \infty$, hence Corollary 2 is not suitable to prove stability of the trivial solution of equation (1).

Nevertheless, for β we get $\theta(t) = \operatorname{Re} a + \vartheta(t) + \kappa_0(t) + \beta(t) = -4 + 0 + 1 - \sin t + 2 + \sin t = -1$ and since $\beta > 0$ on $[T, \infty)$, we may choose $\Lambda(t) = \max\bigl\{\theta(t), \frac{\beta'(t)}{\beta(t)}\bigr\} =$

$\frac{\cos t}{2+\sin t}$. Then $\limsup_{t\to\infty} \int_T^t \Lambda(s)\,ds \le \ln\frac{3}{2} < \infty$, hence, in view of Theorem 1, the trivial solution of equation (1) is stable.

Theorem 2. *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and*

$$V(t) = |\gamma(t)z(t) + c(t)\overline{z}(t)| + \beta(t)\int_{\tau(t)}^{t} |\gamma(s)z(s) + c(s)\overline{z}(s)|\,ds\,, \tag{21}$$

where $z(t)$ is any solution of (1)*. Then*

$$\mu|z(t)| \le V(s)\exp\Big(\int_s^t \Lambda(\xi)\,d\xi\Big) + \int_s^t \lambda(\xi)\exp\Big(\int_\xi^t \Lambda(\sigma)\,d\sigma\Big)\,d\xi \tag{22}$$

for $t \ge s \ge t_1$ where $t_1 \ge T$.

Proof. Following the proof of the Theorem 1, we have

$$\begin{aligned} V'(t) &\le |\gamma(t)z(t) + c(t)\overline{z}(t)|\theta(t) + \beta'(t)\int_{\tau(t)}^{t} |\gamma(s)z(s) + c(s)\overline{z}(s)|\,ds + \lambda(t) \\ &\le \Lambda(t)V(t) + \lambda(t) \end{aligned}$$

a.e. on $[t_1,\infty)$. Using this inequality, we get

$$V'(t) - \Lambda(t)V(t) \le \lambda(t) \tag{23}$$

a.e. on $[t_1,\infty)$. Multiplying (23) by $\exp\big(-\int_s^t \Lambda(\xi)d\xi\big)$ gives

$$\Big[V(t)\exp\Big(-\int_s^t \Lambda(\xi)\,d\xi\Big)\Big]' \le \lambda(t)\exp\Big(-\int_s^t \Lambda(\xi)\,d\xi\Big)$$

a.e. on $[t_1,\infty)$. Integration over $[s,t]$ yields

$$V(t)\exp\Big(-\int_s^t \Lambda(\xi)\,d\xi\Big) - V(s) \le \int_s^t \lambda(\xi)\exp\Big(-\int_s^\xi \Lambda(\sigma)d\sigma\Big)\,d\xi\,, \tag{24}$$

and multiplying (24) by $\exp\big(\int_s^t \Lambda(\xi)d\xi\big)$, we obtain

$$V(t) \le V(s)\exp\Big(\int_s^t \Lambda(\xi)\,d\xi\Big) + \int_s^t \lambda(\xi)\exp\Big(\int_\xi^t \Lambda(\sigma)d\sigma\Big)\,d\xi\,.$$

The statement now follows from (16). □

From Theorem 2 we obtain several consequences.

Corollary 3. *Let the conditions* (i), (ii), (iii) *and* (iv) *be fulfilled and*

$$\limsup_{t\to\infty} \int_s^t \lambda(\xi)\exp\Big(-\int_s^\xi \Lambda(\sigma)d\sigma\Big)\,d\xi < \infty$$

for some $s \ge T$.

If $z(t)$ is any solution of (1)*, then*

$$z(t) = O\Big[\exp\Big(\int_s^t \Lambda(\xi)\,d\xi\Big)\Big]\,.$$

Proof. From the assumptions and (24) we can see that there are $K > 0$ and $S \geq s$ such that for $t \geq S$ we have

$$V(t)\exp\Big(-\int_s^t \Lambda(\xi)\,d\xi\Big) - V(s) \leq \int_s^t \lambda(\xi)\exp\Big(-\int_s^\xi \Lambda(\sigma)\,d\sigma\Big)\,d\xi \leq K < \infty\,.$$

Then

$$\mu|z(t)| \leq V(t) \leq (K + V(s))\exp\Big(\int_s^t \Lambda(\xi)\,d\xi\Big)\,.$$

□

Corollary 4. *Let the assumptions* (i), (ii), (iii) *and* (iv) *hold and let*

$$\limsup_{t\to\infty} \Lambda(t) < \infty \quad \textit{and} \quad \lambda(t) = O(\mathrm{e}^{\eta t})\,, \tag{25}$$

where $\eta > \limsup\limits_{t\to\infty} \Lambda(t)$. *If* $z(t)$ *is any solution of* (1), *then* $z(t) = O(\mathrm{e}^{\eta t})$.

Proof. The proof is identical to the proof of Corollary 6 in [4]. □

Remark 3. If $\lambda(t) \equiv 0$, we can take $L = 0$ in the proof of Corollary 4, and taking the inequalities from the proof of Corollary 6 in [4] into account, we obtain the following statement: there is an $\eta^* < \eta_0 < \eta$ such that $z(t) = o(\mathrm{e}^{\eta_0 t})$ holds for the solution $z(t)$ of (1).

Consider now a special case of equation (1) with $g(t, z, w) \equiv h(t)$:

$$z'(t) = a(t)z(t) + b(t)\overline{z}(t) + A(t)z\big(\tau(t)\big) + B(t)\overline{z}\big(\tau(t)\big) + h(t)\,, \tag{26}$$

where $h(t) \in L_{\mathrm{loc}}([t_0, \infty), \mathbb{C})$.

Corollary 5. *Let the assumption* (i) *be satisfied and suppose*

$$\limsup_{t\to\infty} \big(|\gamma(t)| + |c(t)|\big) < \infty\,. \tag{27}$$

Let $\tilde{\beta} \in AC_{\mathrm{loc}}([T, \infty), \mathbb{R}_+)$ *be such that*

$$\tau'(t)\tilde{\beta}(t) \geq \big(|A(t)| + |B(t)|\big)\frac{|\gamma(t)| + |c(t)|}{|\gamma(\tau(t))| - |c(\tau(t))|} \quad \textit{a.e. on} \;\; [T, \infty). \tag{28}$$

If h *is bounded,*

$$\limsup_{t\to\infty} \big[\operatorname{Re} a(t) + \vartheta(t) + \tilde{\beta}(t)\big] < 0 \tag{29}$$

and

$$\limsup_{t\to\infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} < 0\,, \tag{30}$$

then any solution of equation (26) *is bounded.*

If $h(t) = O(\mathrm{e}^{\eta t})$ *for any* $\eta > 0$,

$$\limsup_{t\to\infty} \big[\operatorname{Re} a(t) + \vartheta(t) + \tilde{\beta}(t)\big] \leq 0 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{\tilde{\beta}'(t)}{\tilde{\beta}(t)} \leq 0\,,$$

then any solution of (26) *satisfies* $z(t) = o(\mathrm{e}^{\eta t})$ *for any* $\eta > 0$.

Proof. Choose $R=\infty$, $\kappa_0(t)\equiv 0$, $\kappa_1(t)\equiv 0$, $\lambda(t)\equiv |h(t)|\sup\limits_{t\geq T}\bigl(|\gamma(t)|+|c(t)|\bigr)$ and $\beta(t)\equiv\tilde\beta(t)$, then $g(t,z,w)\equiv h(t)$ satisfies the condition (ii) and $\beta(t)$ satisfies (iii). Define $\theta(t)=\operatorname{Re}a(t)+\vartheta(t)+\tilde\beta(t)$ and $\Lambda(t)=\max\bigl(\theta(t),\frac{\tilde\beta'(t)}{\tilde\beta(t)}\bigr)$. The assumptions (29) and (30) give the estimate

$$\limsup_{t\to\infty}\Lambda(t)<0,$$

hence the first statement of Corollary 5 follows from Corollary 4.

The second statement follows from Corollary 4 as well, since

$$\limsup_{t\to\infty}\Lambda(t)\leq 0$$

and $z(t)=o(\mathrm{e}^{\eta t})$ for any $\eta>0$ if and only if $z(t)=O(\mathrm{e}^{\eta t})$ for any $\eta>0$. □

Remark 4. If $h(t)\equiv 0$ in Corollary 5, then, with respect to Corollary 4 and Remark 3, we gain the following assertion:

Suppose that assumptions (i) and (27) hold and for $\tilde\beta$ from Corollary 5 the inequality (28) is valid. If (29) and (30) are satisfied, then there is $\eta_0<0$ such that $z(t)=o(\mathrm{e}^{\eta_0 t})$ for any solution $z(t)$ of

$$z'(t)=a(t)z(t)+b(t)\overline{z}(t)+A(t)z\bigl(\tau(t)\bigr)+B(t)\overline{z}\bigl(\tau(t)\bigr)\,.$$

Theorem 3. *Let the assumptions* (i), (ii), (iii) *and* (iv) *be satisfied. Let* $\Lambda(t)\leq 0$ *a.e. on* $[T^*,\infty)$*, where* $T^*\in[T,\infty)$*. If*

$$\lim_{t\to\infty}\int^t\Lambda(s)\,ds=-\infty\quad\textit{and}\quad\lambda(t)=o\bigl(\Lambda(t)\bigr),\tag{31}$$

then any solution $z(t)$ *of equation* (1) *satisfies*

$$\lim_{t\to\infty}z(t)=0\,.$$

Proof. The proof is identical to the proof of Theorem 3 in [4]. □

Corollary 6. *Let the assumptions* (i) *and* (27) *hold and* $\tilde\beta\in AC_{\mathrm{loc}}([T,\infty),\mathbb{R}_+)$ *satisfy* (28)*. If the conditions* (29) *and* (30) *are fulfilled and* $h\in L_{\mathrm{loc}}\bigl([t_0,\infty),\mathbb{C}\bigr)$ *satisfies* $\lim\limits_{t\to\infty}h(t)=0$*, then*

$$\lim_{t\to\infty}z(t)=0$$

for any solution $z(t)$ *of equation* (26)*.*

Proof. Choose $R=\infty$, $\kappa_0(t)\equiv 0$, $\kappa_1(t)\equiv 0$, $\lambda(t)\equiv |h(t)|\sup\limits_{t\geq T}\bigl(|\gamma(t)|+|c(t)|\bigr)$ and $\beta(t)\equiv\tilde\beta(t)$ in the same way as in the proof of Corollary 5. Define $\theta(t)=\operatorname{Re}a(t)+\vartheta(t)+\tilde\beta(t)$ and $\Lambda(t)=\max\bigl(\theta(t),\frac{\tilde\beta'(t)}{\tilde\beta(t)}\bigr)$ again.

From (29) and (30) we have $\limsup\limits_{t\to\infty} \Lambda(t) < 0$, i.e. for $L < 0$, $L > \limsup\limits_{t\to\infty} \Lambda(t)$ there is $s \geq T$ such that $\Lambda(t) \leq L$ for all $t \geq s$. In particular, $\Lambda(t) \neq 0$ for $t \geq s$, hence

$$\lim_{t\to\infty} \frac{\lambda(t)}{\Lambda(t)} = \lim_{t\to\infty} \frac{|h(t)| \sup\limits_{t\geq T}\big(|\gamma(t)| + |c(t)|\big)}{\Lambda(t)} = 0\,,$$

which gives $\lambda(t) = o\big(\Lambda(t)\big)$.

Since $\Lambda(t) \leq L$ for all $t \geq s$, we get

$$\lim_{t\to\infty} \int_s^t \Lambda(\xi)\, d\xi \leq \lim_{t\to\infty} \int_s^t L d\tau = -\infty\,.$$

Thus (31) holds and we can apply Theorem 3 to the equation (26). $\square$

3. Conclusion

We tried to improve the results presented in [1] under the condition $\liminf\limits_{t\to\infty}\big(\,|\operatorname{Im} a(t)|\,|b(t)|\,\big) > 0$ instead of $\liminf\limits_{t\to\infty}\big(|a(t)| - |b(t)|\big) > 0$ considered in [1]. We obtained several results which are similar to the propositions in the related article but they can be more efficient, which is illustrated by an example.

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