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ALMOST COMPLEX PROJECTIVE STRUCTURES AND THEIR MORPHISMS

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Abstract. We discuss almost complex projective geometry and the relations to a distinguished class of curves. We present the geometry from the viewpoint of the theory of parabolic geometries and we shall specify the classical generalizations of the concept of the planarity of curves to this case. In particular, we show that the natural class of J-planar curves coincides with the class of all geodesics of the so called Weyl connections and preserving of this class turns out to be the necessary and sufficient condition on diffeomorphisms to become homomorphisms or anti-homomorphisms of almost complex projective geometries.

1. Almost complex projective structures

An almost complex structure [8] is a smooth manifold equipped with a smooth linear complex structure on each tangent space. The existence of this structure is a necessary, but not sufficient, condition for a manifold to be a complex manifold. That is, every complex manifold is an almost complex manifold, but not vice-versa. The sufficient condition is vanishing of the Nijenhuis tensor

$$N(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]\,. \tag{1}$$

Definition 1.1. Let M be a smooth manifold of dimension $2n$. An *almost complex structure* on M is a smooth trace-free affinor J in $\Gamma(T^\star M \otimes TM)$ satisfying $J^2 = -\operatorname{id}_{TM}$.

For better understanding, we describe an almost complex structure at each tangent space in a fixed basis, i.e. with the help of real matrices:

$$\mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \mathbb{J} & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \mathbb{J} \end{pmatrix},$$

where J represents the multiplication by $i = \sqrt{-1}$ on each tangent space. We can equivalently define an almost complex structure (M,J) as a reduction of the linear

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frame bundle P^1M to the structure group preserving this affinor J, i.e. for our choice of affinor J the group is

$$\left\{ \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \dots & A_{n,n} \end{pmatrix} \middle| A_{i,j} = \begin{pmatrix} a_{i,j} & -b_{i,j} \\ b_{i,j} & a_{i,j} \end{pmatrix}, \det(A) \neq 0 \right\} \cong GL(n, \mathbb{C})\,.$$

Let us finally remark that the classical theory states that an almost complex structure is integrable if and only if the Nijenhuis tensor (1) vanishes for all smooth vector fields X and Y on M [8]. These G-structures are of infinite type but each choice of ∇ defines a geometry of finite type (with morphisms given by the affine maps) [5]. Hence, one can define an almost complex version of projective structures, and this will be done below.

Two connections are called projectively equivalent if they share the geodesics as unparametrized curves. Let us remind that in both smooth and holomorphic settings this means

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi \tag{2}$$

for as smooth or holomorphic one form Υ.

In the realm of almost complex structures this leads to our definition:

Definition 1.2. Let M be a smooth manifold of dimension $2n$, let J be an almost complex structure and let ∇ be a linear connection preserving J. A couple $(J, [\nabla])$, where $[\nabla]$ is the class of connections obtainned from ∇ by the transformation given by all smooth one-forms Υ

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta - \Upsilon(J\xi)J\eta + \Upsilon(\xi)\eta - \Upsilon(J\eta)J\xi + \Upsilon(\eta)\xi \tag{3}$$

is *an almost complex projective structure* on M. An almost complex projective structure is called *normal* if its torsion coincides with the Nijenhuis tensor.

We actually understand the real group $GL(n, \mathbb{C})$ as the subgroup of the $GL(2n, \mathbb{R})$ and the one form Υ defines a real linear map from the tangent space to the reals in any point. Hence one can view this as a complex valued linear function to $\mathbb{C}$ by complex linear extension. This is the way how to see that (3) is special case of (2) in the holomorphic case.

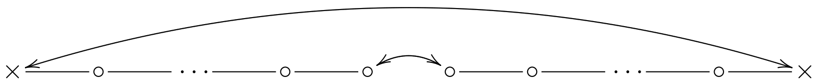
2. The homogeneous model

The homogeneous model (Klein geometry) is a homogeneous space $M \cong G/P$ together with a transitive action on M by a Lie group G, which acts as the symmetry group of the geometry [2]. In our example, the Lie group G is $SL(n+1, \mathbb{C})$ and P is the usual parabolic subgroup of all matrices of the form

$$\begin{pmatrix} c & W \\ 0 & C \end{pmatrix}, \quad \text{where} \quad C \in GL(n, \mathbb{C}),\ c \in GL(1, \mathbb{C}),\ c \cdot \det(C) = 1$$

and C has positive real determinant.

The Lie algebra of P is a parabolic subalgebra of the real form $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ of the complex algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n+1, \mathbb{C}) \oplus \mathfrak{sl}(n+1, \mathbb{C})$ corresponding to the diagram:



The Maurer-Cartan form on G provides the homogeneous model by the structure of $|1|$-graded parabolic geometries of type (G, P). In matrix form, we can illustrate the grading from our example as:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \,,$$

where

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ A_{2,1} & 0 & \dots & 0 \\ \vdots & & & \\ A_{m,1} & 0 & \cdots & 0 \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} A_{1,1} & 0 & \dots & 0 \\ 0 & A_{2,2} & \dots & A_{2,m} \\ 0 & \vdots & \dots & \vdots \\ 0 & A_{m,2} & \dots & A_{m,m} \end{pmatrix},$$

$$\mathfrak{g}_1 = \begin{pmatrix} 0 & A_{1,2} & \dots & A_{1,m} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

3. Parabolic geometries and Weyl connections

The $|1|$-graded parabolic geometries are normally completely given by certain classical G-structures on the underlying manifolds. In our case, however the semisimple Lie algebra belong the the series of exceptions and only the choice of an appropriate class of connections defines the Cartan geometry completely [1, 2].

The normalization of the Cartan geometries is based on cohomological interpretation of the curvature. In our case, the appropriate cohomology is computed by the Künneth formula from the classical Kostant's formulae and the computation provides three irreducible components of the curvature. Torsion type harmonic curvature components consists of maps which are type $(0, 2)$, i.e. conjungate linear in booth arguments [7]. Hence for normal structures the torsion has to be a type $(0, 2)$, but is well known that for a linear connection which preserves an almost complex structure the $(0, 2)$-component of the torsion is a non-zero multiple of the Nijenhuis tensor. Therefore, a normal almost complex projective structure $(M, J, [\nabla])$ has to have this minimal torsion.

The general theory provides for each normal almost complex projective structure $(M, J, [\nabla])$ the construction of the unique principal bundle $\mathcal{G} \to M$ with structure group P, equipped by the normal Cartan connection ω. There is the distinguished class of the so called *Weyl connections* corresponding to all choices of reductions of the parabolic structure group to its reductive subgroup G_0. All Weyl connections are parametrized just by smooth one-forms and they all share the torsion of the Cartan connection.

The transformation formulae for the Weyl connections are generally given by the Lie bracket in the algebra in question.

$$\hat{\nabla}_X Y = \nabla_X Y + [[X, \Upsilon], Y] \tag{4}$$

where we use the frame forms $X, Y\colon \mathcal{G} \to \mathfrak{g}_{-1}$ of vector fields, and similarly for $\Upsilon\colon \mathcal{G} \to \mathfrak{g}_1$. Consequently, $[\Upsilon, X]$ is a frame form of an affinor valued in $\mathfrak{g}_0$ and the bracket with Y expresses the action of such an affinor on the vector field. According to the general theory, this transformation rule works for all covariant derivatives ∇ of Weyl connections. In the proof of Theorem 4.3 we will see that for an almost complex projective geometry, the rule (4) coincides with (3).

Lemma 3.1. *Let $(M, J, [\nabla])$ be a normal almost complex projective structure on a smooth manifold M. The Weyl connections of the uniquely determined normal parabolic geometry of type (G, P), where $G = SL(n+1, \mathbb{C})$ and P is the usual parabolic subgroup, coincide exactly with the distinguish class $[\nabla]$.*

Proof. We already know that the torsion of normal almost complex projective structure is exactly the Nijenhuis tensor from definition and we have computed that the torsion of any Weyl connection is the Nijenhuis tensor, too. In particular, any connection from $[\nabla]$ is Weyl connection. From the general theory we know that having chosen Weyl connection ∇, we obtain any parabolic geometry and following computation shows that it is independent on projective class $[\nabla]$. Consider $\tilde{\nabla}$ from projective class $[\nabla]$, we shall prove that the connection $\tilde{\nabla}_\xi \eta + [[\eta, \Upsilon], \xi]$ belongs to $[\nabla]$, for any one form Υ. This is the consequence of the computation of the Lie bracket in $\mathfrak{g}$ of the corresponding elements $\eta, \xi \in \mathfrak{g}_{-1}$, $\Upsilon(u) \in \mathfrak{g}_1$:

$$\left[\left[\begin{pmatrix} 0 & \dots & 0 \\ \eta_{2,1} & \dots & 0 \\ \vdots & \dots & \vdots \\ \eta_{m,1} & \dots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Upsilon_{1,2} & \dots & \Upsilon_{1,m} \\ \vdots & \dots & \dots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & \dots & 0 \\ \xi_{2,1} & \dots & 0 \\ \vdots & \dots & \vdots \\ \xi_{m,1} & \dots & 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} -(\Upsilon_{1,2}\eta_{2,1} + \dots + \Upsilon_{1,m}\eta_{m,1}) & 0 & \dots & 0 \\ 0 & \eta_{2,1}\Upsilon_{1,2} & \dots & \eta_{2,1}\Upsilon_{1,m} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \eta_{m,1}\Upsilon_{1,m} & \dots & \eta_{m,1}\Upsilon_{1,m} \end{pmatrix}, \begin{pmatrix} 0 & \dots & 0 \\ \xi_{2,1} & \dots & 0 \\ \vdots & \dots & \vdots \\ \xi_{m,1} & \dots & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 \\ (\eta_{2,1}\Upsilon_{1,2}\xi_{2,1} + \dots + \eta_{2,1}\Upsilon_{1,m}\xi_{m,1}) + (\xi_{2,1}\Upsilon_{1,2}\eta_{2,1} + \dots + \xi_{2,1}\Upsilon_{1,m}\eta_{m,1}) & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ (\eta_{m,1}\Upsilon_{1,2}\xi_{2,1} + \dots + \eta_{m,1}\Upsilon_{1,m}\xi_{m,1}) + (\xi_{m,1}\Upsilon_{1,2}\eta_{2,1} + \dots + \xi_{m,1}\Upsilon_{1,m}\eta_{m,1}) & 0 & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \dots & 0 \\ \eta_{2,1}(\Upsilon_{1,2}\xi_{2,1} + \dots + \Upsilon_{1,m}\xi_{m,1}) + \xi_{2,1}(\Upsilon_{1,2}\eta_{2,1} + \dots + \Upsilon_{1,m}\eta_{m,1}) & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \eta_{m,1}(\Upsilon_{1,2}\xi_{2,1} + \dots + \Upsilon_{1,m}\xi_{m,1}) + \xi_{m,1}(\Upsilon_{1,2}\eta_{2,1} + \dots + \Upsilon_{1,m}\eta_{m,1}) & 0 & \dots & 0 \end{pmatrix}$$

Now, we start to looking at the constant part of rows

$$\Upsilon_{1,2}\xi_{2,1} + \dots + \Upsilon_{1,m}\xi_{m,1} \tag{5}$$

substituting for the $\Upsilon_{i,j} := \begin{pmatrix} \Upsilon^1_{i,j} & -\Upsilon^2_{i,j} \\ \Upsilon^2_{i,j} & \Upsilon^1_{i,j} \end{pmatrix}$ now and for the $\xi_{i,j} := \begin{pmatrix} \xi^1_{i,j} & -\xi^2_{i,j} \\ \xi^2_{i,j} & \xi^1_{i,j} \end{pmatrix}$ later step we see that

$$\begin{pmatrix} \Upsilon^1_{1,2} & -\Upsilon^2_{1,2} \\ \Upsilon^2_{1,2} & \Upsilon^1_{1,2} \end{pmatrix} \xi_{2,1} + \dots + \begin{pmatrix} \Upsilon^1_{1,m} & -\Upsilon^2_{1,m} \\ \Upsilon^2_{1,m} & \Upsilon^1_{1,m} \end{pmatrix} \xi_{m,1}$$

$$= \begin{pmatrix} \Upsilon^1_{1,2} & 0 \\ 0 & \Upsilon^1_{1,2} \end{pmatrix} \xi_{2,1} + \begin{pmatrix} 0 & -\Upsilon^2_{1,2} \\ \Upsilon^2_{1,2} & 0 \end{pmatrix} \xi_{2,1} + \dots + \begin{pmatrix} \Upsilon^1_{1,m} & 0 \\ 0 & \Upsilon^1_{1,m} \end{pmatrix} \xi_{m,1} + \begin{pmatrix} 0 & -\Upsilon^2_{1,m} \\ \Upsilon^2_{1,m} & 0 \end{pmatrix} \xi_{m,1}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Upsilon^1_{1,2} \cdot \xi_{2,1} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Upsilon^2_{1,2} \cdot \xi_{2,1} + \dots + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Upsilon^1_{1,m} \cdot \xi_{m,1} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Upsilon^2_{1,m} \cdot \xi_{m,1}$$

$$= \Upsilon^1_{1,2} \begin{pmatrix} \xi^1_{2,1} & -\xi^2_{2,1} \\ \xi^2_{2,1} & \xi^1_{2,1} \end{pmatrix} + J\Upsilon^2_{1,2} \begin{pmatrix} \xi^1_{2,1} & -\xi^2_{2,1} \\ \xi^2_{2,1} & \xi^1_{2,1} \end{pmatrix} + \dots + \begin{pmatrix} \xi^1_{m,1} & -\xi^2_{m,1} \\ \xi^2_{m,1} & \xi^1_{m,1} \end{pmatrix} + J\Upsilon^2_{1,m} \begin{pmatrix} \xi^1_{m,1} & -\xi^2_{m,1} \\ \xi^2_{m,1} & \xi^1_{m,1} \end{pmatrix}$$

$$= E\Upsilon^1_{1,2}\xi^1_{2,1} + J\Upsilon^1_{1,2}\xi^2_{2,1} + J\Upsilon^2_{1,2}\xi^1_{2,1} + J^2\Upsilon^2_{1,2}\xi^2_{2,1} + \dots$$

$$= (\Upsilon^1_{1,2}\xi^1_{2,1} - \Upsilon^2_{1,2}\xi^2_{2,1})E + (\Upsilon^2_{1,2}\xi^1_{2,1} + \Upsilon^1_{1,2}\xi^2_{2,1})J + \dots$$

And finally,

$$\begin{pmatrix} 0 & 0 & \cdots \\ \big((\Upsilon^1_{1,2}\xi^1_{2,1} - \Upsilon^2_{1,2}\xi^2_{2,1})E + (\Upsilon^2_{1,2}\xi^1_{2,1} + \Upsilon^1_{1,2}\xi^2_{2,1})J + \cdots\big)\eta_{2,1} + (\cdots)\xi_{2,1} & 0 & \cdots \\ \vdots & 0 & \cdots \\ \big((\Upsilon^1_{1,2}\xi^1_{2,1} - \Upsilon^2_{1,2}\xi^2_{2,1})E + (\Upsilon^2_{1,2}\xi^1_{m,1} + \Upsilon^1_{1,2}\xi^2_{m,1})J + \cdots\big)\eta_{2,1} + (\cdots)\xi_{2,1} & 0 & \cdots \end{pmatrix}$$

$$(6)\quad \big((\Upsilon^1_{1,2}\xi^1_{2,1} - \Upsilon^2_{1,2}\xi^2_{2,1})E + (\Upsilon^2_{1,2}\xi^1_{2,1} + \Upsilon^1_{1,2}\xi^2_{2,1})J + \cdots\big) \begin{pmatrix} \eta_{2,1} \\ \vdots \\ \eta_{m,1} \end{pmatrix} + (\cdots) \begin{pmatrix} \xi_{2,1} \\ \vdots \\ \xi_{m,1} \end{pmatrix}$$

Finally the last formula (6) is exactly $\Upsilon(\xi)\eta - \Upsilon(J\xi)J\eta + \Upsilon(\eta)\xi - \Upsilon(J\eta)J\xi$, i.e. the connection $\tilde{\nabla}$ belongs to $[\nabla]$. □

4. J-planar curves

One of the basic ideas in differential geometry is the idea of geodetics, i.e. parameterized curves $c\colon \mathbb{R} \to M$ whose tangent vector is parallel, with respect to a linear connection ∇, along the curve itself, i.e. the condition for any suitable parametrization reads $\nabla_{\dot c}\dot c \in \langle \dot c\rangle$. We follow the classical planarity concept [6] with respect to an almost complex structure J.

Definition 4.1. Let (M, J) be an almost complex structure. A smooth curve $c : \mathbb{R} \to M$ is called J-planar with respect to a linear connection ∇ if $\nabla_{\dot c}\dot c \in \langle \dot c, J(\dot c)\rangle$, where $\dot c$ means the tangent velocity field along the curve c and the brackets indicate the linear hulls of the two vectors in the individual tangent spaces.

It is easy to see that geodetics are J-planar curves, but we have to be careful about dimension of M. Let M have dimension two. Consider any curve $c : \mathbb{R} \to M$. Clearly, the curve c satisfies the J-planar identity trivially, because of dimension of $\langle \dot c, J(\dot c)\rangle \cong \mathbb{R}^2$. Hence any curve c is J-planar for an almost complex structure of dimension 2. Let us first prove the fact that any geodesic of ∇ is J-planar for $\bar{\nabla}$, implies that $\bar{\nabla}$ lies in the almost complex projective equivalence class of ∇.

Lemma 4.2. *Let M be a smooth manifold of dimension $2n$, where $n > 1$ and $(J, [\nabla])$ be an almost complex projective structure on M. Let $\bar{\nabla}$ be a linear connection on M, such that ∇ and $\bar{\nabla}$ preserve J and they have the same torsion. If any geodesic* *of ∇ is J-planar for $\bar{\nabla}$, then $\bar{\nabla}$ lies in the almost complex projective equivalence class of ∇.*

Proof. First, let us consider the difference tensor $P(X,Y) = \bar{\nabla}_X(Y) - \nabla_X(Y)$ and one can see that its value in each tangent space is symmetric in each tangent space because both connections share the same torsion. Since both ∇ and $\bar{\nabla}$ preserve J, the difference tensor P is complex linear in the second variable. By symmetry it is thus complex bilinear.

Consider $X = \dot{c}$ and the deformation $P(X,X) := \bar{\nabla}_X(X) - \nabla_X(X)$ equals $\Upsilon_1(X)(X) + \Upsilon_2(X)J(X)$ because c is geodetics with respect to ∇ and J-planar with respect to $\bar{\nabla}$. In this case we shall verify

$$\begin{aligned} P(X,X) &= \Upsilon_1(X)(X) + \Upsilon_2(X)J(X) \\ -P(X,Y) &= P(JX,JX) = \Upsilon_1(JX)(JX) - \Upsilon_2(JX)(JX) \\ \Upsilon_1(JX)(JX) - \Upsilon_2(JX)(X) &= -\Upsilon_1(X)(X) - \Upsilon_2(X)(X) \\ \Upsilon_2(X) &= -\Upsilon_1(JX) \end{aligned}$$

i.e. $P(X,X) = \Upsilon_1(X)(X) - \Upsilon_1(JX)J(X)$ and one shall compute

$$\begin{aligned} P(X,Y) &= \frac{1}{2}\big(P(X+Y,X+Y) - P(X,X) - P(Y,Y)\big) \\ &= \frac{1}{2}\big(\Upsilon_1(X+Y)(X+Y) - \Upsilon_1(JX+JY)J(X+Y)\big) \\ &\quad - \Upsilon_1(X)(X) + \Upsilon_1(JX)J(X) - \Upsilon_1(Y)(Y) + \Upsilon_1(JY)J(Y) \end{aligned}$$

by polarization. It is clear by construction that $\Upsilon_i(tX) = t\Upsilon_i(X)$ for $t \in \mathbb{R}$ and that $P(sX,tY) = stP(X,Y)$ for any $s,t \in \mathbb{R}$.

Assuming that X and Y are complex linearly independent we compare the coefficients of X in the expansions of $P(sX,tY) = stP(X,Y)$ as above to get

$$s\Upsilon_1(sX+tY) - s\Upsilon_1(sX) = st\big(\Upsilon_1(X+Y) - \Upsilon_1(X)\big)\,.$$

Dividing by s and, putting $t = 1$ and taking the limit $s \to 0$, we conclude that $\Upsilon_1(X+Y) = \Upsilon_1(X) + \Upsilon_1(Y)$.

We have proved that the form Υ_1 is linear in X and

$$(X,Y) \to \Upsilon_1(X)Y - \Upsilon_1(JX)JY + \Upsilon_1(Y)X - \Upsilon_1(JY)JX$$

is a symmetric complex bilinear map which agrees with $P(X,Y)$ if both arguments coincide, it always agrees with P by polarization and $\bar{\nabla}$ lies in the almost complex projective equivalence class of ∇. $\square$

Next, we observe that there is a nice link between J-planar curves and connections from the class defining an almost complex projective structure.

Theorem 4.3. *Let $(M,J,[\nabla])$ be a smooth normal almost complex projective structure on a manifold M. A curve c is J-planar with respect to at least one Weyl connection $\bar{\nabla}$ on M if and only if there is a parametrization of c which is a geodesic* *trajectory of some Weyl connection ∇. Moreover, this happens if and only if c is J-planar with respect to all Weyl connections.*

Proof. For a Weyl connection ∇ and a trajectory $c : \mathbb{R} \to M$, the defining equation for J-planarity reads $\nabla_{\dot c}\dot c \in \langle \dot c, J(\dot c)\rangle$. Now, we will prove that the formula (4) for the change of the Weyl connections implies

$$\hat\nabla_{\dot c}\dot c = \nabla_{\dot c}\dot c + [[\dot c, \Upsilon], \dot c] \in \langle \dot c, J(\dot c)\rangle$$

for any holomorphic one form Υ. Indeed, this is the consequence of equation $\nabla_{\dot c}\dot c \in \langle \dot c, J(\dot c)\rangle$ and the computation of the Lie bracket in $\mathfrak{g}$ of the corresponding elements $\dot c \in \mathfrak{g}_{-1}$, $\Upsilon(u) \in \mathfrak{g}_1$, but one can see in (5) that $\Upsilon_{1,2}\dot c_{2,1} + \cdots + \Upsilon_{1,m}\dot c_{m,1} \in \mathbb{C}$ and we are done. Clearly, if $c\colon \mathbb{R} \to M$ is a geodetics with respect to a connection ∇ then c is a J-planar with respect to connection $\hat\nabla$ because of identity $\hat\nabla_{\dot c}\dot c = \nabla_{\dot c}\dot c + [[\dot c, \Upsilon], \dot c] \in \langle \dot c, J(\dot c)\rangle$ from above. Thus, the geodetics c are J-planar with respect to connection $\hat\nabla$. i.e. with respect to all Weyl connections. On the other hand, let us suppose that $c\colon \mathbb{R} \to M$ is J-planar with respect to $\bar\nabla$, i.e. $\bar\nabla_{\dot c}\dot c = a(\dot c)\dot c + b(\dot c)J(\dot c)$ for some functions $a(\dot c)$ and $b(\dot c)$ along the curve. We have to find one form Υ such that the formula for the transformed connection kills all the necessary terms along the curve c. Since there are many such forms Υ such that:

$$0 = \Upsilon_{1,2}\dot c_{2,1} + \cdots + \Upsilon_{1,m}\dot c_{m,1}\,,$$

there is a Weyl connection ∇ such that $\nabla_{\dot c}\dot c \in \langle \dot c\rangle$. Thus, the J-planar curves with respect to one Weyl connection ∇ are J-planar with respect to any Weyl connection $\hat\nabla$. □

By the Lemma 4.2 and Theorem 4.3 above, we can formulate:

Corollary 4.4. *Let $(M, J, [\nabla])$ be a smooth normal almost complex projective structure on a manifold M. A curve c is J-planar with respect to at least one connection $\bar\nabla$ from the class $[\nabla]$ on M if and only if there is a parametrization of c which is a geodesic trajectory of some connection ∇ from the class $[\nabla]$. Moreover, this happens if and only if c is J-planar with respect to all connections from the class $[\nabla]$.*

5. Generalized planar curves and mappings

Various concepts of generalized geodesics have been studied for almost complex, almost product and almost quaternionic geometries. Various structures on manifolds are defined as smooth distribution in the vector bundle $T^\star M \otimes TM$ of all endomorphisms of the tangent bundle. We present an abstract and general definition of the so called A-structures, where A is the linear span of the given affinors, and recover the classical theory of planar curves in this general setup.

Definition 5.1. Let M be a smooth manifold of dimension n. Let A be a smooth vector ℓ-rank subbundle in $T^*M \otimes TM$ $(\ell < n)$, such that the identity affinor $E = \mathrm{id}_{TM}$ restricted to T_xM belongs to $A_x \subset T^*M \otimes TM$ at each point $x \in M$. We say that M is equipped by ℓ-dimensional *A-structure*.

For any tangent vector $X \in T_xM$, we shall write $A(X)$ for the vector subspace

$$A(X) = \{F(X) \mid F \in A_xM\} \subset T_xM\,,$$

and we call $A(X)$ the *A-hull of the vector X*.

Definition 5.2. Let (M, A) be a smooth manifold M equipped with an ℓ-dimensional A-structure. We say that A-structure has

- *generic rank ℓ* if for each $x \in M$ the subset of vectors $(X, Y) \in T_xM \oplus T_xM$, such that the A-hulls $A(X)$ and $A(Y)$ generate a vector subspace $A(X) \oplus A(Y)$ of dimension 2ℓ is open and dense;
- *weak generic rank ℓ* if for each $x \in M$ the subset of vectors

$$\mathcal{V} := \{X \in T_xM \mid \dim A(X) = \ell\}$$

  is open and dense in T_xM.

Lemma 5.3. *Let M be a smooth manifold,* $\dim M > 2$. *Every almost complex structure (M, J) on a manifold M, has weak generic rank 2.*

Proof. Let as assume that $X \notin \mathcal{V} := \{X \in T_xM \mid \dim A(X) = 2\}$, i.e. there is $F = aE + bJ$ which belongs to space $A = \langle E, J\rangle$, such that $F(X) = 0$. For $G := \frac{a}{a^2+b^2}E + \frac{-b}{a^2+b^2}J \in A$ we have $GF(X) = EX = 0$ and $X = 0$, finally. □

Theorem 5.4 ([3])**.** *Let $A \subset V^* \otimes V$ be a vector subspace of generic rank ℓ, and assume that $P(X, X) \in A(X)$ for same fixed symmetric tensor $P \in V^* \otimes V^* \otimes V$ and each vector $X \in V$. Then the induced mapping $P\colon V \to V^* \otimes V$ has values in A.*

Theorem 5.5 ([4])**.** *Let (M, A) be a smooth manifold of dimension n with ℓ-dimensional A-structure, such that $2\ell \leqq \dim M$. If A_x is an algebra (i.e. for all $f, g \in A_x$, $fg := f \circ g \in A_x$) for all $x \in M$ and A has weak generic rank ℓ, then the structure has generic rank ℓ.*

Corollary 5.6. *Each almost complex structure on a smooth manifold M has generic rank 2 because of Lemma 5.3 and Theorem 5.5 above.*

Definition 5.7. Let M be a smooth manifold equipped with an A-structure and a linear connection ∇.

- A smooth curve $c\colon \mathbb{R} \to M$ is said to be *A-planar* if $\nabla_{\dot c}\dot c \in A(\dot c)$.
- Let $\bar M$ be another manifold with a linear connection $\bar\nabla$ and B-structure. A diffeomorphism $f\colon M \to \bar M$ is called *(A, B)-planar* if each A-planar curve C on M is mapped onto the B-planar curve $f_\star C$ on M. In the special case, where A is the trivial structure given by $\langle E\rangle$, we talk about B-planar maps.

Theorem 5.8 ([3],[4])**.** *Let (M, A), (M', A') be smooth manifolds of dimension m equipped with A-structure and A'-structure of the same generic rank $\ell \le 2m$ and assume that the A-structure satisfies the property*

$$\forall X \in T_xM\,,\quad \forall F \in A\,,\quad \exists c_X \mid \dot c_X = X\,,\quad \nabla_{\dot c_X}\dot c_X = \beta(X)F(X)\,, \tag{7}$$

where $\beta(X) \neq 0$. If $f\colon M \to M'$ is an (A, A')-planar mapping, then f is a morphism of the A-structures, i.e. $f^*A' = A$.

An almost complex projective structure $(M, J, [\nabla])$ carries the A-structure with $A = \langle E, J\rangle$. Let us remind, that the A-planarity does not depend on the choice of the Weyl connection ∇ in the class in view of Theorem 4.3. Let us also note that if the diffeomorfismus f satisfies $f^*\bar{A} = A$, then one must have $f^*J = \pm J$, since J and $-J$ are the only elements in A whose square is minus the identity and likewise for $\bar{A}$.

Theorem 5.9. *Let $f\colon M \to M'$ be a diffeomorphism between two almost complex projective manifolds of dimension at least four. Then f is a homomorphism ($f^*J = J$) or an anti-homomorphism ($f^*J = -J$) of the almost complex projective structures if and only if it preserves the class of unparameterized geodesics of all Weyl connections on M and M'.*

Proof. By Lemma 5.3 an almost product complex structure has generic rank 2. In view of the theorem above, we only have to prove that an almost complex structure (M, J) has the property

$$\forall X \in T_xM\,, \quad \forall F \in \langle E, J\rangle\,, \quad \exists c_X \mid \dot{c}_X = X\,, \quad \nabla_{\dot{c}_X}\dot{c}_X = \gamma(X)F(X)\,.$$

Consider $F = \alpha E + \beta J \in \langle E, J\rangle$ and $X \in TM$. First we solve the system of equations:

$$\alpha = \Upsilon^1_{1,2}a^1_{1,2} - \Upsilon^2_{1,2}a^2_{1,2} + \cdots$$

$$\beta = \Upsilon^2_{1,2}a^1_{1,2} + \Upsilon^2_{1,2}a^1_{1,2} + \cdots$$

with respect to Υ. Second, we to define a new connection $\hat{\nabla}$, where: $\hat{\nabla}_{\dot{c}}\dot{c} = \nabla_{\dot{c}}\dot{c} + \big((\Upsilon^1_{1,2}a^1_{1,2} - \Upsilon^2_{1,2}a^2_{1,2} + \cdots)E + (\Upsilon^2_{1,2}a^1_{1,2} + \Upsilon^2_{1,2}a^1_{1,2} + \cdots)J\big)\dot{c} = (\alpha E + \beta J)\dot{c}$.

Now, if $f\colon M \to M'$ is an (A, A')-planar mapping, then f is a morphism of the A-structure, i.e. $f^*A' = A$ and therefore $f^*J = \pm J$.

Furthermore, Theorem 4.3 asserts that the J-planar curves are just the geodesics of the Weyl connections and Corollary 4.4 asserts that the J-planar curves are just the geodesics of the connections form the class $[\nabla]$ and this concludes the proof. □

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