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EXAMPLES OF HOMOTOPY LIE ALGEBRAS

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Abstract. We look at two examples of homotopy Lie algebras (also known as L_∞ algebras) in detail from two points of view. We will exhibit the algebraic point of view in which the generalized Jacobi expressions are verified by using degree arguments and combinatorics. A second approach using the nilpotency of Grassmann-odd differential operators Δ to verify the homotopy Lie data is shown to produce the same results.

1. Introduction

Homotopy Lie algebras, or L_∞ algebras, have been a topic of great interest to both mathematical physicists and to algebraists. By considering two different points of view, one can hope to gain a deeper understanding of these structures. In this note, we provide notations and definitions used by both communities, and hopefully illuminate both perspectives. On one hand, the second author and his collaborators [4, 5] have algebraicly constructed two concrete finite dimensional examples of homotopy Lie algebras from first principles. On the other hand, the first author and his collaborators have developed a generalization of the Batalin-Vilkovisky formalism [1] in which a nilpotent, Grassmann-odd, differential operator Δ may be used to identify L_∞ structures, cf. Lemma in Sec. 2.3 of Ref. [3], and Theorem 3.6 in Ref. [2]. This method is here applied to rederive the two examples of the second author and his collaborators.

2. Homotopy Lie algebras

We begin by recalling the definition of an L_∞ algebra [7], [6].

Definition 1. An L_∞ algebra structure on a $\mathbb{Z}$ graded vector space V is a collection of graded skew symmetric linear maps $l_n : V^{\otimes n} \to V$ of degree $2-n$ that satisfy generalized Jacobi identities

$$(2.1)\qquad \sum_{i+j=n+1} \sum_{\sigma} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} \times l_j\big(l_i(v_{\sigma(1)}, \dots, v_{\sigma(i)}), v_{\sigma(i+1)}, \dots, v_{\sigma(n)}\big) = 0\,,$$

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where $(-1)^\sigma$ is the sign of the permutation, $e(\sigma)$ is the Koszul sign which is equal to -1 raised to the product of the degrees of the permuted elements, and σ is taken over all $(i, n-i)$ unshuffles.

This is the cochain complex point of view; for chain complexes, require the maps l_n to have degree $n-2$.

2.1. **Desuspension.**

We will require an equivalent way to describe homotopy Lie algebra data that will be compatible with the operator approach.

Definition 2. Let $S^c(W)$ be the cofree cocommutative coassociative coalgebra on the graded vector space W. Then an L_∞ algebra structure on W is a coderivation $D\colon S^c(W) \to S^c(W)$ of degree $+1$ such that $D^2 = 0$.

Given an L_∞ algebra structure (V, l_i) as in Definition 1, we may desuspend V to obtain the graded vector space $W =\downarrow V$, where $W_n = V_{n+1}$ and $\downarrow$ is the desuspension operator. Define $D\colon S^c(W) \to S^c(W)$ by $D = \hat{l}_0 + \hat{l}_1 + \hat{l}_2 + \ldots$, where each $\hat{l}_n$ is a degree $+1$ symmetric map given by

$$\hat{l}_n = (-1)^{\frac{n(n-1)}{2}} \downarrow \circ l_n \circ \uparrow^{\otimes n} \colon S^c(W) \to W\,, \tag{2.2}$$

and then extended to a coderivation in the usual fashion. We will demonstrate this construction explicitly in the examples.

The examples that we consider will be structures on relatively small graded vector spaces: $V = V_0 \oplus V_1$, where each V_i is finite dimensional. When we desuspend V, we will consider the graded vector space $W = W_{-1} \oplus W_0$.

We now describe the Δ operator approach.

3. The Δ operator approach

3.1. **Vector space W with two Fermions.**

To be concrete, we let $\dim(W_{-1}) = 2$. We use Greek indices $\alpha, \beta, \ldots \in \{1, 2\}$ for a Fermionic basis $\theta_\alpha \in W_{-1}$ with Grassmann parity $\varepsilon(\theta_\alpha) = 1$. On the other hand, it will be useful to allow W_0 in the beginning to have infinitely many dimensions, and only at the very end perform a consistent truncation to a finite dimensional subspace. We use roman indices $i, j, \ldots \in \{1, 2, \ldots\}$ for the infinitely many Bosonic/even variables $x_i \in W_0$ with Grassmann parity $\varepsilon(x_i) = 0$. Hence, we are given a (super) vector space

$$W := W_{-1} \oplus W_0\,, \quad W_{-1} := \mathrm{span}\langle \theta_1, \theta_2 \rangle\,, \quad W_0 := \mathrm{span}\langle x_1, x_2, \ldots, x_i, \ldots \rangle\,. \tag{3.1}$$

We will for simplicity here only consider one kind of grading, although it is easy to generalize to several $\mathbb{Z}_2$ and $\mathbb{Z}$ gradings. In Section 2 we introduced a $\mathbb{Z}$ grading, called the degree. From an operational point of view, only a $\mathbb{Z}_2$ grading, the so-called Grassmann parity ε, is needed. We shall start by only considering the $\mathbb{Z}_2$ grading ε, and only later implement the full $\mathbb{Z}$ grading. This will lead to "selection rules", i.e., further restrictions.

3.2. **Algebra.**
For an operational point of view, we use the fact the cocommutative coalgebra $S^c(W)$ has the same underlying vector space as the (super) symmetric algebra $\mathcal{A} := \mathrm{Sym}^\bullet(W)$, where

$$x_i \otimes x_j = x_j \otimes x_i \,, \qquad x_i \otimes \theta_\alpha = \theta_\alpha \otimes x_i \,, \qquad \theta_\alpha \otimes \theta_\beta = -\theta_\beta \otimes \theta_\alpha \,, \tag{3.2}$$

or

$$z \otimes w = (-1)^{\varepsilon(z)\varepsilon(w)} w \otimes z \tag{3.3}$$

for short, where z, $w \in W$.

3.3. **Bracket hierarchy $\Phi^\bullet \equiv \hat{l}_\bullet$.**
The family of maps $\hat{l}_\bullet$ on W will be denoted by $\Phi^\bullet$ to conform with notation used in Ref. [3] and Ref. [2]. We shall not always write (super) symmetric tensor symbol $\otimes$ explicitly. The sign convention is as follows:

$$\begin{aligned}
\varepsilon\left(\Phi^n(z_1 \otimes z_2 \otimes \cdots \otimes z_n)\right) &= 1 + \varepsilon(z_1) + \varepsilon(z_2) + \ldots + \varepsilon(z_n) \,, \\
\Phi^n(\cdots \otimes z_k \otimes z_{k+1} \otimes \ldots) &= (-1)^{\varepsilon(z_k)\varepsilon(z_{k+1})} \Phi^n(\cdots \otimes z_{k+1} \otimes z_k \otimes \ldots) \,, \\
\Phi^n(\lambda z_1 \otimes z_2 \otimes \cdots \otimes z_n) &= (-1)^{\varepsilon(\lambda)} \lambda \Phi^n(z_1 \otimes z_2 \otimes \cdots \otimes z_n) \,, \\
\Phi^n(z_1 \otimes \cdots \otimes z_n \lambda) &= \Phi^n(z_1 \otimes \cdots \otimes z_n) \lambda \,, \\
z_k \lambda &= (-1)^{\varepsilon(z_k)\varepsilon(\lambda)} \lambda z_k \quad .
\end{aligned} \tag{3.4}$$

Here λ is a super number. We shall use multi-index notation

$$\begin{gathered}
m = (m_1, m_2, \ldots, m_i, \ldots) \,, \qquad |m| = \sum_{i=1}^{\infty} m_i \,, \qquad m! = \prod_{i=1}^{\infty} m_i! \,, \\
x^{\otimes m} = x_1^{\otimes m_1} \otimes x_2^{\otimes m_2} \otimes \cdots \otimes x_i^{\otimes m_i} \otimes \ldots \,.
\end{gathered} \tag{3.5}$$

The most general bracket hierarchy $\Phi^\bullet$ on W is

$$\Phi^{|m|}(x^{\otimes m}) = c_m^\alpha \theta_\alpha \,, \tag{3.6}$$

$$\Phi^{|m|+1}(\theta_\alpha \otimes x^{\otimes m}) = b_{\alpha m}^i x_i \,, \tag{3.7}$$

$$\Phi^{|m|+2}(\theta_\alpha \otimes \theta_\beta \otimes x^{\otimes m}) = \epsilon_{\alpha\beta} a_m^\gamma \theta_\gamma \,, \tag{3.8}$$

where a_m^α, $b_{\alpha m}^i$ and c_m^γ are coefficients, and where

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha} \,, \qquad \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta_\gamma^\alpha \,, \qquad \epsilon^{12} = 1 = \epsilon_{21} \,. \tag{3.9}$$

3.4. **The Δ operator.**
Define generating functions

$$f^\alpha(p) := \sum_m a_m^\alpha \frac{p^m}{m!} \,, \qquad g_\alpha^i(p) := \sum_m b_{\alpha m}^i \frac{p^m}{m!} \,, \qquad h^\alpha(p) := \sum_m c_m^\alpha \frac{p^m}{m!} \,. \tag{3.10}$$

Define Δ operator

$$\Delta := \Delta_2 + \Delta_1 + \Delta_0 \,, \tag{3.11}$$

$$\Delta_2 := \frac{1}{2}\theta_\gamma f^\gamma(\frac{\partial}{\partial x})\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta}\frac{\partial}{\partial\theta_\alpha} \,, \tag{3.12}$$

$$\Delta_1 := x_i g^i_\alpha(\frac{\partial}{\partial x})\frac{\partial}{\partial\theta_\alpha} \,, \tag{3.13}$$

$$\Delta_0 := \theta_\alpha h^\alpha(\frac{\partial}{\partial x}) \,. \tag{3.14}$$

We will from now on not always write the $\frac{\partial}{\partial x}$ dependence explicitly in the formula for Δ.

3.5. Koszul bracket hierarchy $\Phi^\bullet_\Delta$.

Define Koszul brackets hierarchy $\Phi^\bullet_\Delta$ as

$$\Phi^n_\Delta(z_1 \otimes \cdots \otimes z_n) := \underbrace{[[\ldots[\Delta, L_{z_1}], \ldots], L_{z_n}]}_{n \text{ commutators}} 1 \,, \tag{3.15}$$

$$\Phi^0_\Delta := \Delta(1) \equiv \theta_\alpha c^\alpha_0 \,, \tag{3.16}$$

where

$$L_z(w) := zw \tag{3.17}$$

is the left multiplication operator with algebra element z.

It is easy to check that the $\Phi^\bullet_\Delta$ Koszul brackets hierarchy (3.15)-(3.16) reproduces the original $\Phi^\bullet$ bracket hierarchy (3.6)-(3.8):

$$\Phi^\bullet_\Delta = \Phi^\bullet \,. \tag{3.18}$$

3.6. L_∞ structure and nilpotency conditions.

A consequence of Lemma in Section 2.3 of [3], or alternatively Theorem 3.6 in [2], is that $\Phi^\bullet_\Delta$ forms a homotopy Lie algebra if and only if Δ is nilpotent (of order two), i.e., Δ squares to zero,

$$\Delta^2 \equiv \frac{1}{2}[\Delta, \Delta] = 0 \,. \tag{3.19}$$

We calculate:

$$[\Delta_2, \Delta_2] = 0 \,, \tag{3.20}$$

$$[\Delta_2, \Delta_1] = \frac{1}{2}x_i g^i_\gamma f^\gamma \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta}\frac{\partial}{\partial\theta_\alpha} \,, \tag{3.21}$$

$$[\Delta_1, \Delta_1] = 2x_i g^i_{\alpha,j} g^j_\beta \frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} = x_i g^i_{\alpha,j}\epsilon^{\alpha\beta} g^j_\beta \epsilon_{\gamma\delta}\frac{\partial}{\partial\theta^\delta}\frac{\partial}{\partial\theta^\gamma} \,, \tag{3.22}$$

$$[\Delta_2, \Delta_0] = \theta_\gamma f^\gamma h^\alpha \epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta} \,, \tag{3.23}$$

$$[\Delta_1, \Delta_0] = \theta_\alpha h^\alpha{}_{,i} g^i_\beta \frac{\partial}{\partial\theta^\beta} + x_i g^i_\alpha h^\alpha \,, \tag{3.24}$$

$$[\Delta_0, \Delta_0] = 0 \,. \tag{3.25}$$

For instance, eq. (3.21) is proved as follows. Write shorthand $\Delta_2 = \theta_\gamma D^\gamma$, where

$$D^\gamma := \frac{1}{2} f^\gamma(\frac{\partial}{\partial x})\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta}\frac{\partial}{\partial\theta_\alpha}\,. \tag{3.26}$$

Then

$$\begin{aligned}
[\Delta_2, \Delta_1] &= \theta_\gamma[D^\gamma, \Delta_1] + [\theta_\gamma, \Delta_1]D^\gamma \\
&= \theta_\gamma[D^\gamma, \Delta_1] + [\Delta_1, \theta_\gamma]D^\gamma \\
&= \theta_\gamma[D^\gamma, x_i g^i_\delta \frac{\partial}{\partial\theta_\delta}] + [x_i g^i_\alpha \frac{\partial}{\partial\theta_\alpha}, \theta_\gamma]D^\gamma \\
&= \theta_\gamma[D^\gamma, x_i]g^i_\delta \frac{\partial}{\partial\theta_\delta} + x_i g^i_\alpha [\frac{\partial}{\partial\theta_\alpha}, \theta_\gamma]D^\gamma \\
&= \frac{1}{2}\theta_\gamma f^\gamma{}_{,i}\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta_\beta}\frac{\partial}{\partial\theta_\alpha}g^i_\delta\frac{\partial}{\partial\theta_\delta} + x_i g^i_\gamma D^\gamma\,. \qquad (3.27)
\end{aligned}$$

Note that the first term on the right–hand side of (3.27) must vanish because it contains three Fermionic derivatives, but there are only two different Fermions. The second term yields the result (3.21).

Altogether, the nilpotency condition $\Delta^2 = 0$ read

$$g^i_\gamma f^\gamma + g^i_{\alpha,j}\epsilon^{\alpha\beta}g^j_\beta = 0\,, \tag{3.28}$$

$$f^\alpha h^\gamma \epsilon_{\gamma\beta} + h^\alpha{}_{,i}g^i_\beta = 0\,, \tag{3.29}$$

$$g^i_\alpha h^\alpha = 0\,. \tag{3.30}$$

3.7. **Special cases.**
Let us now discuss special cases. Let us assume $h^\alpha \equiv 0$. Then the two last nilpotency conditions are satisfied, and only the first of the three nilpotency conditions (3.28)–(3.30) remains.

Notice that we can explain $h^\alpha \equiv 0$ as a "selection rule" from the degree $\mathbb{Z}$ grading, where $x_i \in W_0$ have degree 0; $\theta_\alpha \in W_{-1}$ have degree -1; the brackets $\Phi^\bullet_\Delta$ have degree $+1$; and the Δ operator has degree $+1$. Then $c^\alpha_m \equiv 0$, $h^\alpha \equiv 0$, and $\Delta_0 \equiv 0$.

Let us assume only one Bosonic/even variable $x \equiv x_1$, i.e., $0 = x_2 = x_3 = \ldots$. Then the first nilpotency condition (3.28) reads:

$$g_\gamma f^\gamma + W(g_1, g_2) = 0\,, \tag{3.31}$$

where

$$W(g_1, g_2) := g'_\alpha \epsilon^{\alpha\beta} g_\beta \equiv g'_1 g_2 - g_1 g'_2 \tag{3.32}$$

is the Wronskian.

Let us assume that g_1 is given with $g_1(p = 0) \equiv b_{\alpha=1,m=0} \neq 0$. Then we can interpret the inverse $1/g_1$ as a formal power series.

If there is also given g_2, then we can e.g., choose

$$f^1 = -\frac{W(g_1, g_2)}{g_1}\,, \qquad f^2 = 0\,. \tag{3.33}$$

Or if there instead is also given f^1, then we can e.g., choose

$$\frac{g_2}{g_1} = \int dp \frac{f^1}{g_1}\,, \qquad f^2 = 0\,. \tag{3.34}$$

4. First example

4.1. **Algebra approach.**

The following L_∞ algebra was studied in [5]. Let $V = V_0 \oplus V_1$ be the graded vector space where V_0 has basis $\langle v_1, v_2\rangle$ and V_1 has basis $\langle w\rangle$. Define $l_n\colon V^{\otimes n} \to V$ by

$$\begin{aligned} l_1(v_1) = l_1(v_2) = w\,, \qquad l_2(v_1 \otimes v_2) = v_1\,, \qquad l_2(v_1 \otimes w) = w\,,\\ l_n\left(v_2 \otimes w^{\otimes n-1}\right) = C_n w \quad \text{for} \quad n \geq 3\,, \end{aligned} \tag{4.1}$$

and all other sectors are zero, and where $C_n = (-1)^{\frac{(n-2)(n-3)}{2}}(n-3)!\,$.

To verify the L_∞ relations (2.1), the summands in the L_∞ relation can be calculated as follows. The first summand reads

$$l_1 \circ l_n\left(v_1 \otimes v_2 \otimes w^{\otimes n-2}\right) = 0\,. \tag{4.2}$$

The next summand reads

$$\begin{aligned} &l_2 \circ l_{n-1}\left(v_1 \otimes v_2 \otimes w^{\otimes n-2}\right)\\ &\quad = (-1)^{n-1} l_2\left(l_{n-1}(v_2 \otimes w^{\otimes n-2}) \otimes v_1\right)\\ &\quad = (-1)^{n-1} C_{n-1} l_2(w \otimes v_1) = (-1)^n C_{n-1} w\,. \end{aligned} \tag{4.3}$$

For all $3 \leq k \leq n-3$ we have

$$l_k \circ l_{n-k+1}\left(v_1 \otimes v_2 \otimes w^{\otimes n-2}\right) = 0\,, \tag{4.4}$$

because each summand in this expansion contains the term $l_k(v_1 \otimes w^{\otimes k-1}) = 0$. The second-last summand reads

$$\begin{aligned} &l_{n-1} \circ l_2\left(v_1 \otimes v_2 \otimes w^{\otimes n-2}\right)\\ &\quad = l_{n-1}\left(l_2(v_1 \otimes v_2 \otimes w^{\otimes n-2}\right) - (n-2) l_{n-1}\left(l_2(v_1 \otimes w) \otimes v_2 \otimes w^{\otimes n-3}\right)\\ &\quad = l_{n-1}(v_1 \otimes w^{\otimes n-2}) - (n-2) l_{n-1}\left(w \otimes v_2 \otimes w^{\otimes n-3}\right)\\ &\quad = 0 + (n-2) l_{n-1}(v_2 \otimes w^{\otimes n-2}) = (n-2) C_{n-2} w\,. \end{aligned} \tag{4.5}$$

The last summand reads

$$\begin{aligned} l_n \circ l_1(v_1 \otimes v_2 \otimes w^{\otimes n-2}) = l_n(w \otimes v_2 \otimes w^{\otimes n-2})\\ - l_n(w \otimes v_1 \otimes w^{\otimes n-2}) = -C_n w\,. \end{aligned} \tag{4.6}$$

Consequently, the nth Jacobi expression is satisfied if and only if

$$\sum_{p=1}^{n}(-1)^{p(n-p)} l_{n-p+1}\circ l_p\left(v_1\otimes v_2\otimes w^{\otimes n-2}\right)=0$$

$$\Leftrightarrow (-1)^{(n-1)1}(-1)^n C_{n-1}w+(-1)^{2(n-2)}(n-2)C_{n-1}w$$
$$+(-1)^{1(n-1)}(-1)C_n w=0$$

$$\Leftrightarrow (-1)C_{n-1}+(n-2)C_{n-1}+(-1)^n C_n=0$$

$$\Leftrightarrow C_n=(-1)^{n-1}(n-3)C_{n-1}\,. \tag{4.7}$$

One can check that C_n must equal $(-1)^{\frac{(n-2)(n-3)}{2}}(n-3)!$.

4.2. Desuspension.

We next desuspend V to obtain $W=W_{-1}\oplus W_0$ where W_{-1} has basis $\langle\theta_1,\theta_2\rangle$ and W_0 has basis $\langle x\rangle$ and then rewrite the L_∞ data in terms of degree $+1$ maps

$$\begin{gathered}\hat{l}_1(\theta_1)=\hat{l}_1(\theta_2)=x\,,\qquad \hat{l}_2(\theta_1\otimes\theta_2)=\theta_1\,,\\ \hat{l}_n\left(\theta_2\otimes x^{\otimes n-1}\right)=(-1)^n(n-3)!x\,,\end{gathered}\tag{4.8}$$

and all other sectors are zero. The $\hat{l}_n$'s will correspond to the Φ^n's in the next section.

4.3. Δ operator approach.

The algebra of [5] has only one Bosonic generator $x\equiv x_1$, and is given as

$$\Phi^2(\theta_1\otimes\theta_2)=\theta_1\,,\qquad \Phi^1(\theta_1)=x\,,\qquad \Phi^2(\theta_1\otimes x)=x\,,$$

$$\Phi^{m+1}\left(\theta_2\otimes x^{\otimes m}\right)=\begin{cases} x & \text{for}\quad m=0\,,\\ 0 & \text{for}\quad m=1\,,\\ -(-1)^m(m-2)!\;x & \text{for}\quad m\geq 2\,,\end{cases}\tag{4.9}$$

and all other sectors are zero. Thus the coefficients are

$$a^1_m=-\delta^0_m\,,\qquad a^2_m=0\,,\qquad b_{1m}=\delta^0_m+\delta^1_m\,,$$

$$b_{2m}=\begin{cases} 1 & \text{for}\quad m=0\,,\\ 0 & \text{for}\quad m=1\,,\\ -(-1)^m(m-2)! & \text{for}\quad m\geq 2\,.\end{cases}\tag{4.10}$$

The generating functions become

$$f^1(p)=-1\,,\qquad f^2(p)=0\,,\qquad g_1(p)=1+p\,,$$

$$g_2(p)=1-\sum_{m=2}^{\infty}\frac{(-p)^m}{m(m-1)}=(1+p)[1-\ln(1+p)]\,.\tag{4.11}$$

It is easy to check that the nilpotency condition (3.31) is satisfied. Alternatively, g_2 could have been predicted from eq. (3.34).

5. Second example

5.1. **Algebra approach.**

This next example was constructed by M. Daily [4]. Let $V = V_0 \oplus V_1$ with $\dim(V_1) \geq \dim(V_0)$. Denote the basis for V_0 by $\langle v_1, \ldots, v_i\rangle$ and the basis for V_1 by $\langle w_1, \ldots, w_j\rangle$. Define

$$l_1(v_i) = w_i\,, \qquad l_2(v_i \otimes v_j) = 0\,, \qquad l_2(v_i \otimes w_j) = w_i + w_j\,,$$

$$l_n(v_i \otimes v_j \otimes w\text{-terms}) = 0\,, \qquad l_n(v_i \otimes w\text{-terms}) = C_n w_i \quad \text{for} \quad n \geq 3\,, \tag{5.1}$$

and all other sectors are zero.

We begin verification of the L_∞ algebra relations (2.1) with

$$\begin{aligned} &l_1 \circ l_2(v_i \otimes v_j) - l_2 \circ l_1(v_i \otimes v_j) \\ &\quad = l_1(0) - [l_2(l_1(v_i) \otimes v_j) - l_2(l_1(v_j) \otimes v_i)] \\ &\quad = 0 - l_2(w_i \otimes v_j) + l_2(w_j \otimes v_i) = w_j + w_i - (w_i + w_j) = 0\,. \end{aligned} \tag{5.2}$$

We next consider the generalized Jacobi expression evaluated on $v_i \otimes v_j \otimes w_k$. The first summand reads

$$l_1 \circ l_3(v_i \otimes v_j \otimes w_k) = 0\,. \tag{5.3}$$

The next summand reads

$$\begin{aligned} &l_2 \circ l_2(v_i \otimes v_j \otimes w_k) \\ &\quad = l_2(l_2(v_i \otimes v_j) \otimes w_k) - l_2(l_2(v_i \otimes w_k) \otimes v_j) + l_2(l_2(v_j \otimes w_k) \otimes v_i) \\ &\quad = 0 - l_2((w_i + w_k) \otimes v_j) + l_2((w_j + w_k) \otimes v_i) \\ &\quad = (w_j + w_i) + (w_j + w_k) - (w_i + w_j) - (w_i + w_k) \\ &\quad = -w_i + w_j\,. \end{aligned} \tag{5.4}$$

The last summand reads

$$\begin{aligned} l_3 \circ l_1(v_i \otimes v_j \otimes w_k) &= l_3(w_i \otimes v_j \otimes w_k) - l_3(w_j \otimes v_i \otimes w_k) \\ &= -C_3 w_j + C_3 w_i\,. \end{aligned} \tag{5.5}$$

Thus, the generalized Jacobi expression

$$\begin{aligned} &(l_1 \circ l_3 + l_2 \circ l_2 + l_3 \circ l_1)(v_i \otimes v_j \otimes w_k) = 0 \\ &\quad \Leftrightarrow w_i + w_j - C_3 w_j + C_3 w_i = 0 \Leftrightarrow C_3 = 1\,. \end{aligned} \tag{5.6}$$

For $n \geq 4$, we compute

$$\sum_{p=1}^{n} (-1)^{p(n-p)} l_{n-p+1} \circ l_p(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}})\,. \tag{5.7}$$

The first summand with $p = 1$ reads

$$
\begin{aligned}
l_n \circ l_1(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&= l_n(w_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&\quad - l_n(w_j \otimes v_i \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&= -C_n w_j + C_n w_i = C_n(w_i - w_j)\,. \qquad (5.8)
\end{aligned}
$$

The next summand with $p = 2$ reads

$$
\begin{aligned}
l_{n-1} \circ l_2(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&= -\sum_\alpha l_{n-1}(l_2(v_i \otimes w_{k_\alpha}) \otimes v_j \otimes w\text{-terms}) \\
&\quad + \sum_\alpha l_{n-1}(l_2(v_j \otimes w_{k_\alpha}) \otimes v_i \otimes w\text{-terms}) \\
&= -\sum_\alpha l_{n-1}((w_i + w_{k_\alpha}) \otimes v_j \otimes w\text{-terms}) \\
&\quad + \sum_\alpha l_{n-1}((w_j + w_{k_\alpha}) \otimes v_i \otimes w\text{-terms}) \\
&= 2(n-2)C_{n-1}(w_j - w_i)\,. \qquad (5.9)
\end{aligned}
$$

For $3 \le p \le n-2$, we have

$$
\begin{aligned}
l_{n-p+1} \circ l_p(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&= (-1)^{p-1}\binom{n-2}{p-1} l_{n-p+1}\left(l_p(v_i \otimes w - \text{terms}) \otimes v_j \otimes w\text{-terms}\right) \\
&\quad - (-1)^{p-1}\binom{n-2}{p-1} l_{n-p+1}\left(l_p(v_j \otimes w\text{-terms}) \otimes v_i \otimes w\text{-terms}\right) \\
&= (-1)^{p-1}\binom{n-2}{p-1} l_{n-p+1}\left(C_p w_i \otimes v_j \otimes w\text{-terms}\right) \\
&\quad - (-1)^{p-1}\binom{n-2}{p-1} l_{n-p+1}\left(C_p w_j \otimes v_i \otimes w\text{-terms}\right) \\
&= (-1)^p\binom{n-2}{p-1} C_{n-p+1}C_p w_j - (-1)^p\binom{n-2}{p-1} C_{n-p+1}C_p w_i \\
&= (-1)^{p+1}\binom{n-2}{p-1} C_{n-p+1}C_p(w_i - w_j)\,. \qquad (5.10)
\end{aligned}
$$

The second-last summand with $p = n-1$ reads

$$
\begin{aligned}
l_2 \circ l_{n-1}(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\
&= (-1)^{n-2} l_2(l_{n-1}(v_i \otimes w\text{-terms}) \otimes v_j) \\
&\quad - (-1)^{n-2} l_2(l_{n-1}(v_j \otimes w\text{-terms}) \otimes v_i) \\
&= (-1)^{n-2} l_2(C_{n-1}w_i \otimes v_j) - (-1)^{n-2} l_2(C_{n-1}w_j \otimes v_i) \\
&= (-1)^{n-1}C_{n-1}(w_j + w_i) - (-1)^{n-1}C_{n-1}(w_i + w_j) = 0\,. \qquad (5.11)
\end{aligned}
$$

The last summand with $p = n$ reads

$$l_1 \circ l_n(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) = 0\,. \tag{5.12}$$

We add together all of the above summands with $p = 1, 2, \ldots, n$ to obtain

$$\begin{aligned} &\sum_{p=1}^{n} (-1)^{p(n-p)} l_{n-p+1} \circ l_p(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) \\ &\qquad = (-1)^{n-1} C_n(w_i - w_j) - 2(n-2)C_{n-1}(w_i - w_j) \\ &\qquad\quad + \sum_{p=3}^{n-2} (-1)^{p(n-p)}(-1)^{p+1}\binom{n-2}{p-1} C_{n-p+1}C_p(w_i - w_j) + 0 + 0\,. \end{aligned} \tag{5.13}$$

So,

$$\begin{aligned} &\sum_{p=1}^{n} (\,-\,1)^{p(n-p)} l_{n-p+1} \circ l_p(v_i \otimes v_j \otimes w_{k_1} \otimes \cdots \otimes w_{k_{n-2}}) = 0 \\ &\quad \Leftrightarrow (-1)^{n-1} C_n - 2(n-2)C_{n-1} \\ &\quad + \sum_{p=3}^{n-2} (-1)^{pn+1}\binom{n-2}{p-1} C_{n-p+1}C_p = 0\,. \end{aligned} \tag{5.14}$$

One can then solve for

$$C_n = (-1)^n \Big[-2(n-2)C_{n-1} + \sum_{p=3}^{n-2} (-1)^{pn+1}\binom{n-2}{p-1} C_{n-p+1}C_p \Big] \tag{5.15}$$

with $C_3 = 1$.

5.2. **Desuspension.**

As before, we desuspend the vector space to obtain $W = W_{-1} \oplus W_0$ and convert the l_n's to degree $+1$ symmetric maps and end up with the homotopy Lie algebra structure given by

$$\begin{gathered} \hat{l}_1(\theta_i) = x_i\,, \qquad \hat{l}_2(\theta_i \otimes \theta_j) = 0\,, \qquad \hat{l}_2(\theta_i \otimes x_j) = x_i + x_j\,, \\ \hat{l}_n(\theta_i \otimes \theta_j \otimes x\text{-terms}) = 0\,, \\ \hat{l}_n(\theta_i \otimes x\text{-terms}) = (-1)^{\frac{n(n-1)}{2}}(-1)^{n-1} C_n x_i \quad \text{for} \quad n \geq 3\,, \end{gathered} \tag{5.16}$$

and all other sectors are zero. The last equation may be rewritten as

$$\hat{l}_n(\theta_i \otimes x\text{-terms}) = (2-n)^{n-2} x_i \quad \text{for} \quad n \geq 3\,. \tag{5.17}$$

5.3. **Δ operator approach.**

In the following we let $\dim(W_{-1}) = 2$, to conform with the theory developed in Section 3. Moreover, it is practical to let W_0 have infinitely many Bosonic generators

x_i. (It will be consistent to truncate the tail $0 = x_{N+1} = x_{N+2} = \ldots$ to reduce to only finitely many generators $x_1, \ldots, x_N$.) Then the second example is of the form

$$\begin{aligned} \Phi^1(\theta_\alpha) &= B_0 x_\alpha\,, \\ \Phi^2(\theta_\alpha \otimes x_i) &= B_1 x_\alpha + x_i\,, \\ \Phi^{|m|+1}(\theta_\alpha \otimes x^{\otimes m}) &= B_{|m|} x_\alpha \text{ for } |m| \geq 2\,, \end{aligned} \tag{5.18}$$

and all other sectors are zero, and where $B_0, B_1, B_2, \ldots$ are complex numbers with $B_0 \neq 0$. By scaling

$$\begin{gathered} x'_i = B_0 x_i\,, \quad \theta'_\alpha = \theta_\alpha\,, \quad \Phi' = \Phi\,, \\ B'_M = (B_0)^{M-1} B_M \quad \text{for} \quad M = 0, 1, 2, \ldots\,, \end{gathered} \tag{5.19}$$

(and by dropping the primes again afterwards) we will from now on always assume the initial condition

$$B_0 = 1\,. \tag{5.20}$$

We will below prove the following Proposition 3.

Proposition 3. *The $\Phi^\bullet$ bracket hierarchy* (5.18) *with initial condition* (5.20) *is a homotopy Lie algebra if and only if*

$$B_M = (1-M)^{M-1} \quad \textit{for} \quad M = 0, 1, 2, \ldots \tag{5.21}$$

(*with the convention that* $0^0 := 1$).

Proof. The bracket coefficients are in this example

$$\begin{aligned} a^\alpha_m &= 0\,, \\ b^i_{\alpha m} &= \delta^i_\alpha B_{|m|} + \delta^0_{m_1}\delta^0_{m_2}\ldots\delta^0_{m_{i-1}}\delta^1_{m_i}\delta^0_{m_{i+1}}\ldots\,. \end{aligned} \tag{5.22}$$

The generating functions become

$$\begin{aligned} f^\alpha(p) &= 0\,, \\ g^i_\alpha(p) &= \delta^i_\alpha G(P) + p^i\,, \end{aligned} \tag{5.23}$$

where

$$G(P) = \sum_m B_{|m|}\frac{p^m}{m!} = \sum_{M=0}^\infty B_M \frac{P^M}{M!}\,, \tag{5.24}$$

and

$$P := \sum_{i=1}^\infty p^i\,. \tag{5.25}$$

The initial condition (5.20) becomes

$$G(P{=}0) = 1\,. \tag{5.26}$$

The nilpotency condition (3.28) reads

$$\begin{aligned} (\alpha \leftrightarrow \beta) = g^i_{\alpha,j} g^j_\beta &= (\delta^i_\alpha G'(P) + \delta^i_j)(\delta^j_\beta G(P) + p^j) \\ &= \delta^i_\alpha G'(P)(G(P)+P) + \delta^i_\beta G(P) + p^i\,. \end{aligned} \tag{5.27}$$

This is equivalent to the ODE

$$G'(P)(G(P)+P) = G(P) \tag{5.28}$$

$$\Leftrightarrow \frac{dP}{dG} = 1+\frac{P}{G} \Leftrightarrow \frac{d}{dG}\left[\frac{P}{G}\right] = \frac{1}{G} \Leftrightarrow \frac{P}{G} = \mathrm{Ln}(G) + \text{constant}\,. \tag{5.29}$$

We deduce from the initial condition (5.26) that the inverse function $P=P(G)$ is

$$P(G) = G\mathrm{Ln}(G) = -(1-G) + \sum_{n=2}^{\infty}\frac{(1-G)^n}{n(n-1)}\,. \tag{5.30}$$

Let us now recall the Lambert function $W = W(P)$, whose inverse function $P=P(W)$ is

$$P(W) = We^W\,. \tag{5.31}$$

(Hopefully, the reader will not be confused by the fact that we denote two different function $P=P(G)$ and $P=P(W)$ (and in fact also the "momentum" variable P itself) with the same symbol P. It should be clear from the context which is which.) Note that the Lambert function $W=W(P)$ has a zero in $P=0$

$$W(P{=}0) = 0\,. \tag{5.32}$$

By comparing eqs. (5.30) and (5.31) we deduce that the sought-for function $G=G(P)$ is just the exponential of the Lambert function

$$G(P) = e^{W(P)} = \frac{P}{W(P)}\,. \tag{5.33}$$

The Taylor expansion for the Lambert function $W=W(P)$ is

$$W(P) = \sum_{n=1}^{\infty}(-n)^{n-1}\frac{P^n}{n!}\,. \tag{5.34}$$

The Taylor coefficients with $n\geq 1$ follow from Lagrange's inversion formula, or simply by calculating

$$\begin{aligned} W^{(n)}(P{=}0) &= \frac{1}{n!}\oint_0 \frac{dP}{2\pi i}\frac{W'(P)}{P^n} = \frac{1}{n!}\oint_0\frac{dW}{2\pi i}\frac{1}{P(W)^n} \\ &= \frac{1}{n!}\oint_0\frac{dW}{2\pi i}\frac{e^{-nW}}{W^n} = \left.\frac{d^{n-1}}{dW^{n-1}}e^{-nW}\right|_{W=0} \\ &= (-n)^{n-1}\,. \end{aligned} \tag{5.35}$$

Similarly, the Taylor coefficients for the function $G=G(P)$ with $n\geq 1$ are

$$\begin{aligned} B_n = G^{(n)}(P{=}0) &= \frac{1}{n!}\oint_0\frac{dP}{2\pi i}\frac{G'(P)}{P^n} = \frac{1}{n!}\oint_0\frac{dP}{2\pi i}\frac{W'(P)e^{W(P)}}{P^n} \\ &= \frac{1}{n!}\oint_0\frac{dW}{2\pi i}\frac{e^W}{P(W)^n} = \frac{1}{n!}\oint_0\frac{dW}{2\pi i}\frac{e^{(1-n)W}}{W^n} = \left.\frac{d^{n-1}}{dW^{n-1}}e^{(1-n)W}\right|_{W=0} \\ &= (1-n)^{n-1}\,. \end{aligned} \tag{5.36}$$

The Taylor expansion for the function $G = G(P)$ is

$$G(P) = \sum_{n=0}^{\infty} (1-n)^{n-1} \frac{P^n}{n!} \,. \tag{5.37}$$

Both Taylor series (5.34) and (5.37) have radius of convergence equal to $1/e$, as may be seen by the ratio test. This completes the proof of Proposition 3. □

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