ARCHIVUM MATHEMATICUM (BRNO)
Tomus 45 (2009), 241–254

CONSTANT JACOBI OSCULATING RANK OF $\mathbf{U(3)/(U(1)\times U(1)\times U(1))}$

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Dedicated to Salud Bartoll

Abstract. In this paper we obtain an interesting relation between the covariant derivatives of the Jacobi operator valid for all geodesic on the flag manifold $M^6 = U(3)/(U(1)\times U(1)\times U(1))$. As a consequence, an explicit expression of the Jacobi operator independent of the geodesic can be obtained on such a manifold. Moreover, we show the way to calculate the Jacobi vector fields on this manifold by a new formula valid on every g.o. space.

1. Introduction

It is well-known that locally Riemannian symmetric spaces (i.e. the spaces whose local geodesic symmetries are isometries) are completely determined by the condition $\nabla\mathcal{R} = 0$, where $\mathcal{R}$ denotes the curvature tensor. Moreover, D'Atri spaces (i.e. the spaces whose local geodesic symmetries are volume-preserving) generalize locally symmetric spaces and they are also determined by an infinite series of curvature conditions called odd Ledger's conditions. The Ledger conditions are derived from the so-called Ledger recurrence formula. This formula is derived using calculations involving the Jacobi operator (see [8]). However, between locally symmetric spaces and D'Atri spaces there are many kinds of Riemannian manifolds namely, symmetric-like manifolds for which such a natural characterization is not known. Therefore a natural question appears: Can one determine the symmetric-like manifolds by relations that involve only the Jacobi operator and its covariant derivatives?

The first steps to answer this question has been done on normal homogeneous spaces, naturally reductive spaces and g.o. spaces. It is well-known (cf. [7, Chapter X, Sections 2, 3]) that a Riemannian homogeneous space $(M = G/H, g)$ with its origin $p = \{H\}$ and with an $\mathrm{ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is *naturally reductive* (with respect to this decomposition) if and only if for any vector $X \in \mathfrak{m}\backslash\{0\}$, the curve $\gamma(t) = \tau(\exp tX)(p)$ is a geodesic with respect to the Riemannian connection. Here, exp and $\tau(h)$ denote the Lie exponential map of

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2000 *Mathematics Subject Classification*: primary 53C21; secondary 53C25, 53C30, 53C20.

Key words and phrases: naturally reductive space, g.o. space, Jacobi operator, Jacobi osculating rank.

The author's work has been supported by D.G.I. (Spain) and FEDER Project MTM 2007-65852, the network MTM2008-01013-E/ and by DFG Sonderforschungsbereich 647.

G and the left transformation of G/H induced by $h \in G$ respectively. Moreover, if there exits a bi-invariant metric on $\mathfrak{g}$ whose restriction to $\mathfrak{m}$ is the metric g (using the canonical identification $\mathfrak{m} \cong T_0M$) the naturally reductive space is called *normal homogeneous*. Now, natural reductivity is still a special case of a more general property: Each geodesic of $(M = G/H, g)$, is an orbit of a one-parameter group of isometries $\{\exp\ tZ\}$, $Z \in \mathfrak{g}$. Riemannian homogeneous spaces with this property are called *g.o. spaces*. Note that every normal space is naturally reductive, and every naturally reductive space is a g.o. space while the converse in both cases is false. For example, $M^6 = \frac{U(3)}{U(1)\times U(1)\times U(1)}$ is a naturally reductive space that it is not normal and the Kaplan's example [6] is a g.o. space which is not naturally reductive.

A. M. Naveira and A. Tarrío [10] found the relation independent of the geodesic satisfied on the normal manifold $V_1 = Sp(2)/SU(2)$ while they developed a recursive method for solving the Jacobi equation on such a manifold. This method is based on the fact that the Jacobi operator has constant osculating rank on every fix geodesic over naturally reductive spaces. Moreover, in [9] the authors obtain de relation independent of the geodesic satisfied on the Wilking manifold $V_3 = (SO(3)\times SU(3))/\dot{U}(2)$, endowed with a particular bi-invariant metric which made it normal space. Recently, in [4] (see also Chapter 3 of [2]) the authors extended on g.o. spaces the study made on naturally reductive spaces in [10]. Moreover, they found the relation independent of the geodesic satisfied on Kaplan's example. Finally, they also settling the concept of constant and non-constant Jacobi osculating rank of a g.o. space. After that, we can say that V_1 and V_3 have Jacobi osculating rank $\mathtt{r} = 2$ and for the Kaplan's example we have $\mathtt{r} = 4$. Moreover, in the previous examples the Jacobi osculating rank is constant because the coefficients of the relation are independent of the geodesic.

In this paper, we will treat more deeply the notions of naturally reductive space and Jacobi operator along a geodesic. In the third section, we will remind the concepts of Jacobi osculating rank and constant Jacobi osculating rank of a given g.o. space stablished in [4]. Moreover, we will propose a non-recursive formula for solving the Jacobi equation. In the last section, we will study the manifold M^6. It will become the first known example of a non-symmetric and, even non-normal, naturally reductive space where there exists a relation independent of the geodesic between the covariant derivatives of the Jacobi operator. This will leads up to conclude, on such an example, that the Jacobi osculating rank is 4 and it is constant. Moreover, as application, we will get the explicit expression of the Jacobi operator valid for all geodesics. Finally, we will show how to use the non-recursive method to obtain the Jacobi fields along an arbitrary geodesic.

The author would like to thank professors S. Bartoll, A. M. Naveira and A. H. Rocamora for their useful conversations.

2. Preliminaries

Let $M = G/H$ be a homogeneous space, where G is a connected Lie group and, unless otherwise stated, acts effectively on M. We say that G/H is reductive if the Lie algebra $\mathfrak{g}$ of G may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{h}$ of H and an $\mathrm{ad}(H)$-invariant subspace $\mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$, we denote the projections of a given $Z\in\mathfrak{g}$ onto $\mathfrak{h}$ and $\mathfrak{m}$ by $Z_{\mathfrak{h}}$ and $Z_{\mathfrak{m}}$, respectively. Moreover, there is a natural identification of $\mathfrak{m}\subset\mathfrak{g}=T_eG$ with the tangent space T_pM via the projection $\pi\colon G\to G/H=M$.

Definition 1. A reductive homogeneous space $M=G/H$ with a G-invariant metric g is said to be *naturally reductive* if it satisfies the condition

$$\langle [X,Y]_{\mathfrak{m}},Z\rangle+\langle Y,[X,Z]_{\mathfrak{m}}\rangle=0 \quad\text{for}\quad X,Y,Z\in\mathfrak{m}\,. \tag{1}$$

Here, $\langle\ ,\ \rangle$ denotes the induced inner product on $\mathfrak{m}$ from the metric g. Since g is G- invariant, $\langle\ ,\ \rangle$ is $\mathrm{Ad}(H)$-invariant.

Moreover, we know by [7, Chapter X, Sections 2, 3] that, on a naturally reductive space, the Riemannian connection is given by $(\nabla_XY)_0=\frac{1}{2}[X,Y]_{\mathfrak{m}}$ and the curvature tensor R of the Riemannian connection satisfies

$$\begin{aligned}\big(R(X,Y)Z\big)_0=&-\frac{1}{4}[X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}}+\frac{1}{4}[Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}}\\&+\frac{1}{2}[[X,Y]_{\mathfrak{m}},Z]_{\mathfrak{m}}+[[X,Y]_{\mathfrak{h}},Z]\end{aligned}\tag{2}$$

for all X, Y, $Z\in\mathfrak{m}$.

A useful technique to describe the curvature along a geodesic in a Riemannian manifold is the use of the Jacobi operator. Let M be an n-dimensional, connected, real analytic Riemannian manifold and let γ or γ_x be a geodesic through $p=\gamma(0)\in M$ whose tangent vector at p is $x\in T_pM$. The Jacobi operator along the geodesic γ is a symmetric linear operator defined by $\mathcal{J}(X):=R(\dot\gamma,X)\dot\gamma$, where X denotes a differentiable vector field along γ.

In particular, if $M=G/H$ is a naturally reductive space, the Jacobi operator can be determined using only the information provided by the Lie brackets. In fact, by (2) we get $\mathcal{J}_0(X_0)=\big(R(\dot\gamma,X)\dot\gamma\big)_0=-[[X_0,x]_{\mathfrak{h}},x]-\dfrac{1}{4}\,[[X_0,x]_{\mathfrak{m}},x]_{\mathfrak{m}}$.

The covariant derivative of the Jacobi operator $\mathcal{J}$, is the self-adjoint tensor field defined by $\mathcal{J}^{1)}(X)=(\nabla_{\dot\gamma}R)(\dot\gamma,X)\dot\gamma$. The successive higher derivatives are defined in a natural manner. Accordingly A. M. Naveira - A. Tarrío [10], these can be determined at $p\in M$ by

$$\mathcal{J}_0^{k)}(X_0)=\tag{3}$$
$$(-1)^{k-1}\frac{1}{2^k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\Big[[\ \overset{k+2)}{\cdots}\ [X_0,x]_{\mathfrak{m}},x]_{\mathfrak{m}}\ \overset{i+1)}{\cdots},x]_{\mathfrak{h}},x]_{\mathfrak{m}}\dots,x\Big]_{\mathfrak{m}}.$$

Finally, it is known on naturally reductive spaces that, on the one hand, there exists a connection $\overline{\nabla}$, namely canonical connection, such that the curvature B and the torsion T of $\overline{\nabla}$ are parallel. The Jacobi equation written in terms of $\overline{\nabla}$ is the differential equation $\overline{\nabla}^2_{\dot\gamma}Y-T(\dot\gamma,\overline{\nabla}^1_{\dot\gamma}Y)+B(\dot\gamma,Y)\dot\gamma=0$ which has constant coefficients. On the other hand, it is well known [5] that the canonical connection and the Riemannian connection, ∇, have the same geodesics and, thus, the same

Jacobi fields. Therefore, the Jacobi equation can be rewritten in terms of ∇ as $\nabla^2_{\dot\gamma} Y + R(\dot\gamma, Y)\dot\gamma = 0$ where R is the curvature associated to the Riemannian connection. For simplicity of notation, we write $Y'' + \mathcal{J}Y = 0$ for the Jacobi equation throughout the paper. Here, Y denotes the Jacobi tensor field along the geodesic γ (i.e., the solution of the Jacobi equation).

3. Constant Jacobi osculating rank of a g.o. space. Resolution of the Jacobi equation

K. Tsukada [11] pointed out on naturally reductive spaces and, the first author and A. M. Naveira [4] showed on g.o. spaces that, in certain sense, the curvature operator can be seen as a curve in the space $\Re(\mathfrak{m})$ of curvature tensors on $\mathfrak{m}$. For a unit vector $x \in \mathfrak{m}$, the tensor $R_x(t)$ obtained by parallel translation of the curvature tensor along γ_x given by $R_x(t) = \mathrm{e}^{t\nabla_x} \cdot R_x(0)$ is a curve in $\Re(\mathfrak{m})$. Here, the dot indicates the action of $\mathrm{e}^{t\nabla_x}$ on the space $\Re(\mathfrak{m})$.

More explicitly, let $U(t)$, $V(t)$, $W(t)$ be vector fields along the geodesic γ_x such that $U(0) = u$, $V(0) = v$, $W(0) = w$ are unit vectors in $\mathfrak{m}$. By (1), ∇_x is a skew-symmetric linear endomorphism of $(\mathfrak{m}, \langle\ ,\ \rangle)$. Therefore, e^{∇_x} is a linear isometry of $(\mathfrak{m}, \langle\ ,\ \rangle)$ and $\mathrm{e}^{t\nabla_x}$ is a 1-parameter subgroup of the group of linear isometries of $\Re(\mathfrak{m})$. Moreover, the *curve* $R_x(t)(u,v)w = \mathrm{e}^{\nabla_{tx}} R_x(0)\left(\mathrm{e}^{\nabla_{-tx}} u, \mathrm{e}^{\nabla_{-tx}} v\right) \mathrm{e}^{\nabla_{-tx}} w$ has *constant osculating rank* r_γ in the classical sense. Hence, there is a number $r_\gamma \in \mathbb{N}$ for every fixed but arbitrary geodesic $\gamma_x(t)$ and there exist smooth functions $a_1, \ldots, a_{r_\gamma} : I \to M$ such that

$$R_x(t) = R_x(0) + a_1(t) R_x^{1)}(0) + \cdots + a_r(t) R_x^{r)}(0) \quad \text{for all} \quad t \in I\,. \tag{4}$$

Here, $R_x^{k)}(0)(u,v)w = \left(\nabla_x^k R\right)(u,v)w$ for $0 \le k \le r_\gamma$. In addition, by the classical constant osculating rank definition there are constants $\alpha_1, \ldots, \alpha_{r_\gamma}$ depending of the fixed geodesic such that

$$\alpha_1 \mathcal{R}_x^{1)}(t) + \cdots + \alpha_{r_\gamma} \mathcal{R}_x^{r_\gamma)}(t) + \mathcal{R}_x^{r_\gamma+1)}(t) = 0\,. \tag{5}$$

It is obvious on every g.o. space that for every fixed but arbitrary geodesic $\gamma_x(t)$ the associated Jacobi operator, $\mathcal{J}_x(t)$, has also constant osculating rank in the classical sense. To shorten notation, we will write throughout the paper $\mathcal{J}_t$ instead of $\mathcal{J}_x(t)$ and $\mathcal{J}_0$ instead of $\mathcal{J}_x(0)$. Therefore, we can rewrite (4) and (5) substituting the curvature operator by the Jacobi operator. In particular at $t = 0$, we will always have constants $\alpha_1, \ldots, \alpha_{r_\gamma}$ depending of the fixed geodesic, such that

$$\alpha_1 \mathcal{J}_0^{1)} + \cdots + \alpha_{r_\gamma} \mathcal{J}_0^{r_\gamma)} + \mathcal{J}_0^{r_\gamma+1)} = 0\,. \tag{6}$$

Now, we put $\mathtt{r} = \max\{r_\gamma : \text{for all } \gamma \text{ geodesic of a given g.o. space}\}$. In [4] the authors proved that there exits constants $\beta_1, \ldots, \beta_{\mathtt{r}}$, on every geodesic of a given g.o. space such that

$$\beta_1 \mathcal{J}_0^{1)} + \cdots + \beta_{\mathtt{r}} \mathcal{J}_0^{\mathtt{r})} + \mathcal{J}_0^{\mathtt{r}+1)} = 0\,. \tag{7}$$

From now on, we say that $\mathtt{r}\in\mathbb{N}$ is the **Jacobi osculating rank** of a given g.o. space. Of course, $\mathtt{r}\leq n^2$ and the constants of each relation still depend of the corresponding geodesic.

Proposition 1 ([4]). *Let (M,g) be a g.o. space with Jacobi osculating rank $\mathtt{r}$. Let $\gamma_x(t) : I\to M$ be a geodesic and $\beta_1,\dots,\beta_{\mathtt{r}}$ constants depending of γ such that $\beta_1\mathcal{J}_0^{1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_0^{\mathtt{r})}+\mathcal{J}_0^{\mathtt{r}+1)}=0$. Then,*

$$\beta_1\mathcal{J}_t^{1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_t^{\mathtt{r})}+\mathcal{J}_t^{\mathtt{r}+1)}=0 \quad \textit{for all} \quad t\in I\,. \tag{8}$$

Moreover, there are smooth functions $a_1,\dots,a_{\mathtt{r}}\colon I\to\mathbb{R}$ that determine the Jacobi operator along $\gamma_x(t)$. More explicitly, we obtain that

$$\mathcal{J}_t=\mathcal{J}_0+a_1(t)\mathcal{J}_0^{1)}+\cdots+a_{\mathtt{r}}(t)\mathcal{J}_0^{\mathtt{r})}\,. \tag{9}$$

Finally, note that if the constants β_i, $i=1,\dots,\mathtt{r}$, of (8) does *not depend* of the geodesic then we can find an expression of type (8) valid *for all* geodesic of a given g.o. space. Moreover, we obtain an expression of type (9) for the Jacobi operator valid *for all* geodesic of a given g.o. space. When this happen, we say that the given g.o. space has **constant Jacobi osculating rank**.

Remark 1. Although a g.o. space has constant Jacobi osculating rank, the operator $\mathcal{J}_x(t)$ with respect to every fixed geodesic $\gamma_x(t)$ has still its own (as a curve) constant osculating rank r_γ that could be less than or equal to the Jacobi osculating rank $\boldsymbol{r}$ of the g.o. space. (See Example 1 of Section 4 or [3]).

In the remainder of this section, we present a non-recursive method for solving the Jacobi equation on g.o. spaces. This method is based in the recursive method proposed in [10].

Let $\{E_i\}$, $i=1,\dots,n$, be a orthonormal basis of T_pM. We denote by $\{Q_i\}$, $i=1,\dots,n$, the orthonormal frame field obtained by parallel translation of the basis $\{E_i\}$ along a geodesic γ such that $\gamma(0)=p$ and $\dot\gamma(0)=x$. Moreover, let Q_t be the column vector function corresponding to $\{Q_i\}$. Then all covariant derivatives of Q_t are zero. Recall that a vector field Y_t along a geodesic γ of a manifold M is called *Jacobi vector field* if it satisfies the *Jacobi equation* along γ: $Y_t''+\mathcal{J}_tY_t=0$ with initial conditions $Y_0=0$, $Y_0'=Q_0$. Let $Y_t=A_tQ_t$, where A_t is a real analytic matrix function, be the Jacobi vector field along the geodesic γ. Then the Jacobi equation can be rewritten as $A_t^{2)}+A_t\mathcal{J}_t=0$ with initial values $A_0=0$, $A_0^{1)}=I$ where we consider the covariant differentiation with respect to $\dot\gamma$ and I is the identity transformation of T_pM. Therefore, to obtain the expression of the Jacobi fields it is enough to know the development in Taylor's series of A_t, due to $A_t=\sum_{k=0}^{\infty}\frac{t^k}{k!}A_0^{k)}$. It is clear that $A_0^{0)}=0$ and $A_0^{1)}=I$. Now, we firstly calculate $A_t^{k)}$ to obtain the expression of $A_0^{k)}$ for $k\geq 2$. If we successively differentiate the identity $A_t^{2)} = -\mathcal{J}_t A_t$, we get using Leibniz's formula that

$$\begin{aligned}
A_t^{n+2)} = &- \sum_{k=0}^{n} \binom{n}{k} \mathcal{J}_t^{n-k)} A_t^{k)} = -\Big[\mathcal{J}_t^{n)} A_t + n\mathcal{J}_t^{n-1)} A_t^{1)} \\
&- \sum_{k_1=2}^{n} \binom{n}{k_1} \mathcal{J}_t^{n-k_1)} \Big[\mathcal{J}_t^{k_1-2)} A_t + (k_1-2)\mathcal{J}_t^{k_1-3)} A_t^{1)} \\
&- \sum_{k_2=2}^{k_1-2} \binom{k_1-2}{k_2} \mathcal{J}_t^{k_1-2-k_2)} \Big[\mathcal{J}_t^{k_2-2)} A_t + (k_2-2)\mathcal{J}_t^{k_2-3)} A_t^{1)} \\
&\vdots \\
&- \sum_{k_j=2}^{k_{j-1}-2} \binom{k_{j-1}-2}{k_j} \mathcal{J}_t^{k_{j-1}-2-k_j)} \Big[\mathcal{J}_t^{k_j-2)} A_t + (k_j-2)\mathcal{J}_t^{k_j-3)} A_t^{1)}\Big] \dots \Big]\Big]\Big]
\end{aligned}$$

for all $n \geq 0$. Thus, if $t = 0$ and we use the initial conditions, we conclude that

$$\begin{aligned}
A_0^{n+2)} = &- \sum_{k=0}^{n} \binom{n}{k} \mathcal{J}_0^{n-k)} A_0^{k)} = -\Big[n\mathcal{J}_0^{n-1)} - \sum_{k_1=3}^{n} \binom{n}{k_1} \mathcal{J}_0^{n-k_1)} \Big[(k_1-2)\mathcal{J}_0^{k_1-3)} \\
&- \sum_{k_2=3}^{k_1-2} \binom{k_1-2}{k_2} \mathcal{J}_0^{k_1-2-k_2)} \Big[(k_2-2)\mathcal{J}_0^{k_2-3)} \\
&\vdots \\
&- \sum_{k_j=3}^{k_{j-1}-2} \binom{k_{j-1}-2}{k_j} \mathcal{J}_0^{k_{j-1}-2-k_j)} \Big[(k_j-2)\mathcal{J}_0^{k_j-3)}\Big] \dots \Big]\Big]\Big]
\end{aligned} \tag{10}$$

for all $n \geq 0$ where $j = \frac{n-1}{2}$ if n is odd and $j = \frac{n-2}{2}$ if n is even.

Finally, it is clear by the previous part of this section that for every fixed geodesic of a g.o. space the curve $\mathcal{J}_t$ has constant osculating rank r_γ. Hence, we can always assume that a relation of type (6) is satisfied and we only need $\mathcal{J}_0, \mathcal{J}_0^{1)}, \dots, \mathcal{J}_0^{r_\gamma)}$ to obtain $A_0^{k)}$. In this situation, we need to fix the geodesic. Thus, we determine the Jacobi vector field along that specific geodesic.

On the other hand, due to every g.o. space has Jacobi osculating rank $\mathtt{r}$, we can always assume that a relation of type (7) is satisfied and we only need $\mathcal{J}_0$, $\mathcal{J}_0^{1)}$, …, $\mathcal{J}_0^{\mathtt{r})}$ to determine the coefficients $A_0^{k)}$. In this situation, we also need to fix the geodesic. Thus, we also determine the Jacobi vector field along that specific geodesic. Therefore, the previous formulation (using formula (6)) is better. Nevertheless, if the g.o. space has constant Jacobi osculating rank, with this second possibility (using formula (7)) we will obtain a solution of the Jacobi equation valid for all geodesic of the given g.o. space.

In the next section, we will calculate the constant Jacobi osculating rank of an specific naturally reductive space. Moreover, we will calculate the first Jacobi vector field working on a geodesic which has $r_\gamma = \mathtt{r}$.

4. On the naturally reductive space $\frac{U(3)}{U(1)\times U(1)\times U(1)}$

Let $M^6 = G/H$ be a homogeneous space where $G = U(3)$ and $H = U(1)\times U(1)\times U(1)$. It is well-known that the correspondent Lie algebras satisfy $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ and they have the following dimensions: $\dim(\mathfrak{g}) = 9$, $\dim(\mathfrak{h}) = 3$ and $\dim(\mathfrak{m}) = 6$. Let $\langle A, B\rangle = \operatorname{Re}\big(\operatorname{Tr}(A\bar{B}^t)\big) = -\operatorname{Re}\big(\operatorname{Tr}(AB)\big)$ be the inner product on $\mathfrak{u}(3) = \{X \in \mathfrak{gl}(3,\mathbb{C}) : \bar{X}^t = -X\}$. This inner product is an $\operatorname{ad}\big(U(3)\big)$-invariant inner product on $\mathfrak{u}(3)$. Let us denote by $\{E_i\}$, $i = 1,\dots,9$, the orthonormal basis of $\mathfrak{g} = \mathfrak{u}(3)$. In particular, the $\mathfrak{m}$'s basis is given by

$$E_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad E_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}, \quad E_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},$$

$$E_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \mathrm{i} & 0\\ \mathrm{i} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad E_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -\mathrm{i}\\ 0 & 0 & 0\\ -\mathrm{i} & 0 & 0\end{pmatrix}, \quad E_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & \mathrm{i}\\ 0 & \mathrm{i} & 0\end{pmatrix},$$

and the $\mathfrak{h}$'s basis is given by

$$E_7 = \begin{pmatrix} \mathrm{i} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad E_8 = \begin{pmatrix} 0 & 0 & 0\\ 0 & \mathrm{i} & 0\\ 0 & 0 & 0\end{pmatrix}, \quad E_9 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \mathrm{i}\end{pmatrix}.$$

Moreover, the structure of the Lie algebra $\mathfrak{g}$ can be obtained by a straightforward computation.

Lemma 2. *The multiplication table associate to the Lie algebra $\mathfrak{g}$ is*

$$\begin{aligned}
&[E_1, E_2] = -\frac{1}{\sqrt{2}}E_3, && [E_1, E_3] = \frac{1}{\sqrt{2}}E_2, && [E_1, E_4] = E_7 - E_8,\\
&[E_1, E_5] = \frac{1}{\sqrt{2}}E_6, && [E_1, E_6] = -\frac{1}{\sqrt{2}}E_5, && [E_1, E_7] = -E_4,\\
&[E_1, E_8] = E_4, && [E_1, E_9] = 0, && [E_2, E_3] = -\frac{1}{\sqrt{2}}E_1,\\
&[E_2, E_4] = -\frac{1}{\sqrt{2}}E_6, && [E_2, E_5] = -E_7 + E_9, && [E_2, E_6] = \frac{1}{\sqrt{2}}E_4,\\
&[E_2, E_7] = E_5, && [E_2, E_8] = 0, && [E_2, E_9] = -E_5,\\
&[E_3, E_4] = \frac{1}{\sqrt{2}}E_5, && [E_3, E_5] = \frac{-1}{\sqrt{2}}E_4, && [E_3, E_6] = E_8 - E_9,\\
&[E_3, E_7] = 0, && [E_3, E_8] = -E_6, && [E_3, E_9] = E_6,\\
&[E_4, E_5] = \frac{1}{\sqrt{2}}E_3, && [E_4, E_6] = \frac{-1}{\sqrt{2}}E_2, && [E_4, E_7] = E_1,\\
&[E_4, E_8] = -E_1, && [E_4, E_9] = 0, && [E_5, E_6] = \frac{1}{\sqrt{2}}E_1,\\
&[E_5, E_7] = -E_2, && [E_5, E_9] = E_2, && [E_6, E_8] = E_3,
\end{aligned}$$

$$[E_6, E_9] = -E_3, \quad [E_5, E_8] = [E_6, E_7] = [E_7, E_8] = [E_7, E_9] = [E_8, E_9] = 0.$$

Now, it is easy to conclude from Lemma 2 that $\langle E_7\rangle$, $\langle E_8\rangle$ and $\langle E_9\rangle$ are commuting and isomorphic to $\mathfrak{u}(1)$. Therefore, the group generated by $\mathfrak{h}$ is analytically isomorphic to $\mathfrak{u}(1)\times\mathfrak{u}(1)\times\mathfrak{u}(1)$. Furthermore, $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ is an ad(H)$-$invariant decomposition (in particular the reductive condition $[\mathfrak{m},\mathfrak{h}]\subset\mathfrak{m}$ is satisfied). Moreover, $\langle\ ,\ \rangle$ is an ad(H)-invariant inner product on $\mathfrak{m}$ such that the relation (1) is satisfied. Hence, under the above inner product on $\mathfrak{m}$, M is a naturally reductive space.

In the remainder of this section, our purpose is to obtain an expression of type (9). It means that we want to determine explicitly the Jacobi operator along an arbitrary geodesic γ with initial vector $x=\dot\gamma(0)$. From now on, we consider that

$$x=\sum_{i=1}^{6}x_iE_i \quad\text{with}\quad |x|^2=\sum_{i=1}^{6}(x_i)^2=1\,. \tag{11}$$

There is no loss of generality in assuming that $x\in\mathfrak{m}$ is a unit vector. Nevertheless, it will be sometimes convenient to ignore it. Furthermore, we denote by $\{Q_i\}$, $i=1,\dots,6$, the orthonormal frame field obtained by parallel translation of the basis $\{E_i\}$, $i=1,\dots,6$, along the geodesic γ. More precisely, to determine explicitly the Jacobi operator along an arbitrary geodesic, firstly, we have to calculate which is the Jacobi osculating rank $\mathtt{r}$ of M^6. For that, we must obtain the relation of type (7) that it is satisfied on M^6. Let us start with the following technical lemma whose proof is direct from Lemma 2 and (11).

Lemma 3. *The operators $[E_k,x]_{\mathfrak{m}}$ and $[E_k,x]_{\mathfrak{h}}$, $k=1,\dots,6$ are given by*

$$\begin{aligned}
&[E_1,x]_{\mathfrak{m}}=\frac{1}{\sqrt2}(x_3E_2-x_2E_3-x_6E_5+x_5E_6)\,, && [E_1,x]_{\mathfrak{h}}=x_4(E_7-E_8)\,.\\
&[E_2,x]_{\mathfrak{m}}=\frac{1}{\sqrt2}(-x_3E_1+x_1E_3+x_6E_4-x_4E_6)\,, && [E_2,x]_{\mathfrak{h}}=x_5(-E_7+E_9)\,.\\
&[E_3,x]_{\mathfrak{m}}=\frac{1}{\sqrt2}(x_2E_1-x_1E_2-x_5E_4+x_4E_5)\,, && [E_3,x]_{\mathfrak{h}}=x_6(E_8-E_9)\,.\\
&[E_4,x]_{\mathfrak{m}}=\frac{1}{\sqrt2}(-x_6E_2+x_5E_3-x_3E_5+x_2E_6)\,, && [E_4,x]_{\mathfrak{h}}=x_1(-E_7+E_8)\,.\\
&[E_5,x]_{\mathfrak{m}}=\frac{1}{\sqrt2}(x_6E_1-x_4E_3+x_3E_4-x_1E_6)\,, && [E_5,x]_{\mathfrak{h}}=x_2(E_7-E_9)\,.\\
&[E_6,x]_{\mathfrak{m}}=\frac{1}{\sqrt2}(-x_5E_1+x_4E_2-x_2E_4+x_1E_5)\,, && [E_6,x]_{\mathfrak{h}}=x_3(-E_8+E_9)\,.
\end{aligned} \tag{12}$$

In general, the n^{th} covariant derivative of the Jacobi operator at the origin of a manifold is given by the matrix $\mathcal{J}_0^{n)}=\big(\mathcal{J}_{ij}^{n}(0)\big)$ where $\mathcal{J}_{ij}^{n}(0)=\langle\mathcal{J}_t^{n)}(Q_i),Q_j\rangle(0)=\langle\mathcal{J}_0^{n)}(E_i),E_j\rangle$. Here, on M^6 we determine $\mathcal{J}_0^{n)}(E_i)$, $i=1,\dots,6$, $n=0,1,\dots,5$ by a lengthy but elementary recurrence calculation using (3) and (12). Afterwards, we easily obtain $\mathcal{J}_0^{n)}$, $n=0,1,\dots,5$. Here, we only write the expression of $\mathcal{J}_0^{0)}$, $\mathcal{J}_0^{1)}$

and the elements $\mathcal{J}_{11}^{n)}(0)$, $n=3,5$, for the illustration of the next result's proof. However, the full expressions for the matrices $\mathcal{J}_0^{n)}$, $n=2,3,4,5$ can be seen in [1].

The explicit expression of $\mathcal{J}_0^{0)}=\left(\mathcal{J}_{ij}^0(0)\right)=\left(\mathcal{J}_{ij}^0\right)$, $i,j=1,\dots,6$, is given by

$$
\begin{aligned}
&\mathcal{J}_{11}^0=\frac{1}{8}\left(x_2^2+x_3^2+16x_4^2+x_5^2+x_6^2\right), && \mathcal{J}_{12}^0=\frac{-1}{8}\left(x_1x_2+9x_4x_5\right),\\
&\mathcal{J}_{14}^0=-2x_1x_4, && \mathcal{J}_{13}^0=\frac{-1}{8}\left(x_1x_3+9x_4x_6\right),\\
&\mathcal{J}_{15}^0=\frac{1}{8}\left(9x_2x_4-x_1x_5\right), && \mathcal{J}_{16}^0=\frac{1}{8}\left(9x_3x_4-x_1x_6\right),\\
&\mathcal{J}_{22}^0=\frac{1}{8}\left(x_1^2+x_3^2+x_4^2+16x_5^2+x_6^2\right), && \mathcal{J}_{23}^0=\frac{-1}{8}\left(x_2x_3+9x_5x_6\right),\\
&\mathcal{J}_{25}^0=-2x_2x_5, && \mathcal{J}_{24}^0=\frac{1}{8}\left(9x_1x_5-x_2x_4\right),\\
&\mathcal{J}_{26}^0=\frac{1}{8}\left(9x_3x_5-x_2x_6\right), && \mathcal{J}_{33}^0=\frac{1}{8}(x_1^2+x_2^2+x_4^2+x_5^2+16x_6^2),\\
&\mathcal{J}_{34}^0=\frac{1}{8}\left(9x_1x_6-x_3x_4\right), && \mathcal{J}_{35}^0=\frac{1}{8}\left(9x_2x_6-x_3x_5\right),\\
&\mathcal{J}_{36}^0=-2x_3x_6, && \mathcal{J}_{44}^0=\frac{1}{8}\left(16x_1^2+x_2^2+x_3^2+x_5^2+x_6^2\right),\\
&\mathcal{J}_{55}^0=\frac{1}{8}\left(x_1^2+16x_2^2+x_3^2+x_4^2+x_6^2\right), && \mathcal{J}_{45}^0=\frac{-1}{8}\left(x_4x_5+9x_1x_2\right),\\
&\mathcal{J}_{46}^0=\frac{-1}{8}\left(x_4x_6+9x_1x_3\right), && \mathcal{J}_{56}^0=\frac{-1}{8}\left(x_5x_6+9x_2x_3\right),\\
&\mathcal{J}_{66}^0=\frac{1}{8}(x_1^2+x_2^2+16x_3^2+x_4^2+x_5^2).
\end{aligned}
\tag{13}
$$

The explicit expression of $\mathcal{J}_0^{1)}=\left(\mathcal{J}_{ij}^1(0)\right)=\left(\mathcal{J}_{ij}^1\right)$, $i,j=1,\dots,6$, is given by

$$
\begin{aligned}
&\mathcal{J}_{11}^1=\mathcal{J}_{22}^1=\mathcal{J}_{33}^1=\mathcal{J}_{44}^1=\mathcal{J}_{55}^1=\mathcal{J}_{66}^1=0, \qquad \mathcal{J}_{14}^1=\mathcal{J}_{25}^1=\mathcal{J}_{36}^1=0,\\
&\mathcal{J}_{12}^1=\frac{-3}{2\sqrt{2}}\left(x_3(x_4^2-x_5^2)+x_6(x_1x_4-x_2x_5)\right),\\
&\mathcal{J}_{13}^1=\frac{3}{2\sqrt{2}}\left(x_2(x_4^2-x_6^2)+x_5(x_1x_4-x_3x_6)\right),\\
&\mathcal{J}_{15}^1=\frac{-3}{2\sqrt{2}}\left(x_6(x_2^2-x_4^2)+x_3(x_1x_4+x_2x_5)\right),\\
&\mathcal{J}_{16}^1=\frac{3}{2\sqrt{2}}\left(x_5(x_3^2-x_4^2)+x_2(x_1x_4+x_3x_6)\right),\\
&\mathcal{J}_{23}^1=\frac{-3}{2\sqrt{2}}\left(x_1(x_5^2-x_6^2)+x_4(x_2x_5-x_3x_6)\right),\\
&\mathcal{J}_{24}^1=\frac{3}{2\sqrt{2}}\left(x_6(x_1^2-x_5^2)+x_3(x_1x_4+x_2x_5)\right),\\
&\mathcal{J}_{26}^1=\frac{-3}{2\sqrt{2}}\left(x_4(x_3^2-x_5^2)+x_1(x_2x_5+x_3x_6)\right),
\end{aligned}
\tag{14}
$$

$$\mathcal{J}_{34}^{1} = \frac{-3}{2\sqrt{2}}\left(x_5(x_1^2 - x_6^2) + x_2(x_1x_4 + x_3x_6)\right) ,$$
$$\mathcal{J}_{35}^{1} = \frac{3}{2\sqrt{2}}\left(x_4(x_2^2 - x_6^2) + x_1(x_2x_5 + x_3x_6)\right) ,$$
$$\mathcal{J}_{45}^{1} = \frac{-3}{2\sqrt{2}}\left(x_3(x_2^2 - x_1^2) + x_6(x_1x_4 - x_2x_5)\right) ,$$
$$\mathcal{J}_{46}^{1} = \frac{3}{2\sqrt{2}}\left(x_2(x_3^2 - x_1^2) + x_5(x_1x_4 - x_3x_6)\right) ,$$
$$\mathcal{J}_{56}^{1} = \frac{-3}{2\sqrt{2}}\left(x_1(x_3^2 - x_2^2) + x_4(x_2x_5 - x_3x_6)\right) .$$

Finally, the elements $\mathcal{J}_{11}^{n}(0)$ of $\mathcal{J}_0^{n)}$, $n = 3, 5$, are

$$(15)\qquad \begin{aligned} \mathcal{J}_{11}^{3}(0) &= \frac{9}{8\sqrt{2}}\, x_4(x_2x_6 + x_3x_5)(x_2^2 - x_3^2 + x_5^2 - x_6^2)\,, \\ \mathcal{J}_{11}^{5}(0) &= -\frac{45}{64\sqrt{2}}\, x_4(x_2x_6 + x_3x_5)(x_2^2 - x_3^2 + x_5^2 - x_6^2)\,. \end{aligned}$$

Theorem 4. *The naturally reductive homogeneous space* $\mathbf{M^6} = \frac{U(3)}{U(1)\times U(1)\times U(1)}$ ***has constant Jacobi osculating rank*** $\mathtt{r} = \mathbf{4}$*. In fact, the derivatives of the Jacobi operator at the origin satisfy the identity*

$$(16)\qquad \frac{1}{16}|x|^4\mathcal{J}_0^{1)} + \frac{5}{8}|x|^2\mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0\,.$$

Proof. It is a straightforward and patient computation similar to that one used for $Sp(2)/SU(2)$ in [10] or for Kaplan's example in [4]. Let us consider the linear homogeneous system of equations $\{A\mathcal{J}_{11}^1(0) + B\mathcal{J}_{11}^2(0) + C\mathcal{J}_{11}^3(0) + D\mathcal{J}_{11}^4(0) = 0, A\mathcal{J}_{22}^1(0) + B\mathcal{J}_{22}^2(0) + C\mathcal{J}_{22}^3(0) + D\mathcal{J}_{22}^4(0) = 0, A\mathcal{J}_{26}^1(0) + B\mathcal{J}_{26}^2(0) + C\mathcal{J}_{26}^3(0) + D\mathcal{J}_{26}^4(0) = 0, A\mathcal{J}_{66}^1(0) + B\mathcal{J}_{66}^2(0) + C\mathcal{J}_{66}^3(0) + D\mathcal{J}_{66}^4(0) = 0\}$. Using the information provided by (14) and the expression for $\mathcal{J}_0^{n)}$, $n = 2, 3, 4$ given by [1] to study the explicit expression of the determinant of the previous system, we conclude that $A = B = C = D = 0$ is the only possible solution valid for all geodesic in N. Therefore, $\mathcal{J}_0^{1)}$, $\mathcal{J}_0^{2)}$, $\mathcal{J}_0^{3)}$ and $\mathcal{J}_0^{4)}$ are, in general, linearly independent. On the other hand, it is clear from (14) and (15) that $\frac{1}{16}|x|^4\mathcal{J}_{11}^1(0) + \frac{5}{8}|x|^2\mathcal{J}_{11}^3(0) + \mathcal{J}_{11}^5(0) = 0$. Analogously using (14) and $\mathcal{J}_0^{n)}$, $n = 3, 5$ of [1], we easily check that $\frac{1}{16}|x|^4\mathcal{J}_{ij}^1(0) + \frac{5}{8}|x|^2\mathcal{J}_{ij}^3(0) + \mathcal{J}_{ij}^5(0) = 0, \quad i, j = 1, \dots, 6$. Therefore, we obtain (16) and we state that $\mathtt{r} = 4$. Moreover, this relation is valid for all geodesic in M^6 due to $|x|^2 = 1$ (see (11)). Thus, the Jacobi osculating rank of M^6 is constant. □

The next example illustrate the affirmation given in Remark 1.

Example 1. Note that although the constant Jacobi osculating rank of M^6 is $\mathtt{r} = 4$, every geodesic has still its own constant osculating rank r_γ. For example, if we consider an arbitrary geodesic γ_1 on M^6 with $x_1 = x_2 = x_3$, $x_4 = x_5 = x_6 = 0$ (using the notation (11)) we easily obtain from (14) that $r_{\gamma_1} = 0$ because $\mathcal{J}_0^{1)} = 0$. Moreover, let γ_2 be an arbitrary geodesic on M^6 such that $x_1 \neq 0$, $x_2 = x_1/\sqrt{2} = x_5$,

$x_3 = x_1$, $x_4 = x_6 = 0$. In this case, $r_{\gamma_2} = 2$ because it is easy to check using (14) and $\mathcal{J}_0^{2)}$ of [1] that the determinant of the linear homogeneous system of equations $\{A\mathcal{J}_{11}^1(0) + B\mathcal{J}_{11}^2(0) = 0, A\mathcal{J}_{26}^1(0) + B\mathcal{J}_{26}^2(0) = 0\}$ is $\frac{9}{16\sqrt{2}}x_1^7 \neq 0$ and $\frac{1}{2}|x|^2\mathcal{J}_0^{1)} + \mathcal{J}_0^{3)} = 0$. Finally, let γ_3 be an arbitrary geodesic on M^6 such that $x_3 = x_4 = x_6 = 0$. In this case, we obtain that $r_{\gamma_3} = 4$ and $\frac{1}{16}|x|^4\mathcal{J}_0^{1)} + \frac{5}{8}|x|^2\mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0$ following the same steps as in the proof of the previous theorem.

Now, due to $\mathtt{r} = 4$ and Proposition 1, there are four smooth functions $a_1, \ldots, a_4 \colon I \to \mathbb{R}$ that provide an expression of type (9) for the Jacobi operator valid for all geodesic of M^6. In the following, we will determine these functions.

Proposition 5. *The derivatives of the Jacobi operator at the origin on M^6 satisfy*

$$\mathcal{J}_0^{2i+5)} = A_i\mathcal{J}_0^{1)} + B_i\mathcal{J}_0^{3)}, \quad \mathcal{J}_0^{2i+6)} = A_i\mathcal{J}_0^{2)} + B_i\mathcal{J}_0^{4)} \quad \text{for} \quad i = 0, 1, \ldots \tag{17}$$

where $A_0 = -\frac{1}{16}|x|^4$, $B_0 = -\frac{5}{8}|x|^2$ and $A_i = A_0B_{i-1}$, $B_i = A_{i-1} + B_0B_{i-1}$ for $i = 1, 2, \ldots$ or, equivalently, for $i = 1, 2, \ldots$ we have

$$\begin{aligned} A_i &= \left[\frac{-1}{16}\left(\frac{-1}{8}\right)^{i-1} + \left(\frac{-1}{2}\right)^{i-1}\right]\frac{|x|^{2(i+2)}}{24}, \\ B_i &= \left[\left(\frac{-1}{8}\right)^{i} - 16\left(\frac{-1}{2}\right)^{i}\right]\frac{|x|^{2(i+1)}}{24}. \end{aligned} \tag{18}$$

Proof. We use induction method. For $i = 0$

$$\mathcal{J}_0^{5)} = -\frac{1}{16}|x|^4\mathcal{J}_0^{1)} - \frac{5}{8}|x|^2\mathcal{J}_0^{3)} = A_0\mathcal{J}_0^{1)} + B_0\mathcal{J}_0^{3)}.$$

For $i = k$, it holds

$$\mathcal{J}_0^{2k+5)} = A_k\mathcal{J}_0^{1)} + B_k\mathcal{J}_0^{3)}.$$

Then, for $i = k + 1$ we have

$$\begin{aligned} \mathcal{J}_0^{2(k+1)+5)} &= \left(\nabla_{\dot\gamma}^2(A_k(\nabla_{\dot\gamma}^1\mathcal{J}) + B_k(\nabla_{\dot\gamma}^3\mathcal{J})\right)(0) \\ &= A_k\mathcal{J}_0^{3)} + B_k\mathcal{J}_0^{5)} = A_k\mathcal{J}_0^{3)} + B_k(A_0\mathcal{J}_0^{1)} + B_0\mathcal{J}_0^{3)}) \\ &= A_0B_k\mathcal{J}_0^{1)} + (A_k + B_0B_k)\mathcal{J}_0^{3)} = A_{k+1}\mathcal{J}_0^{1)} + B_{k+1}\mathcal{J}_0^{3)}. \end{aligned}$$

Finally, to deduce the second formula of (17) from the first one, take the covariant derivative. Moreover, we have (18) due to the sequence $\{B_i\}$ is the solution of the second-order linear recurrence equation $x_i = A_0x_{i-2} + B_0x_{i-1}$ for $i = 2, 3, \ldots$. □

Theorem 6. *The Jacobi operator along an arbitrary geodesic γ of M^6 with $|x|^2 = 1$ can be written in the form*

$$\mathcal{J}_t = \mathcal{J}_0 + a_1(t)\mathcal{J}_0^{1)} + a_2(t)\mathcal{J}_0^{2)} + a_3(t)\mathcal{J}_0^{3)} + a_4(t)\mathcal{J}_0^{4)}$$

where

$$a_1(t) = \frac{8\sqrt{2}}{3}\sin\left(\frac{t}{2\sqrt{2}}\right) - \frac{\sqrt{2}}{3}\sin\left(\frac{t}{\sqrt{2}}\right), \quad a_2(t) = 10 - \frac{32}{3}\cos\left(\frac{t}{2\sqrt{2}}\right) + \frac{2}{3}\cos\left(\frac{t}{\sqrt{2}}\right),$$

$$a_3(t) = \frac{16\sqrt{2}}{3}\sin\left(\frac{t}{2\sqrt{2}}\right) - \frac{8\sqrt{2}}{3}\sin\left(\frac{t}{\sqrt{2}}\right), \; a_4(t) = 16 - \frac{64}{3}\cos\left(\frac{t}{2\sqrt{2}}\right) + \frac{16}{3}\cos\left(\frac{t}{\sqrt{2}}\right).$$

Proof. From Taylor's theorem and the previous technical proposition we have

$$\begin{aligned}\mathcal{J}_t &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathcal{J}_0^{k)} = \mathcal{J}_0 + t\mathcal{J}_0^{1)} + \frac{t^2}{2!}\mathcal{J}_0^{2)} + \frac{t^3}{3!}\mathcal{J}_0^{3)} + \frac{t^4}{4!}\mathcal{J}_0^{4)} + \sum_{i=0}^{\infty}\mathcal{J}_0^{2i+5)}\frac{t^{2i+5}}{(2i+5)!}\\ &+ \sum_{i=0}^{\infty}\mathcal{J}_0^{2i+6)}\frac{t^{2i+6}}{(2i+6)!} = \mathcal{J}_0 + t\mathcal{J}_0^{1)} + \frac{t^2}{2!}\mathcal{J}_0^{2)} + \frac{t^3}{3!}\mathcal{J}_0^{3)} + \frac{t^4}{4!}\mathcal{J}_0^{4)}\\ &+ \sum_{i=0}^{\infty}(A_i\mathcal{J}_0^{1)} + B_i\mathcal{J}_0^{3)})\frac{t^{2i+5}}{(2i+5)!} + \sum_{i=0}^{\infty}(A_i\mathcal{J}_0^{2)} + B_i\mathcal{J}_0^{4)})\frac{t^{2i+6}}{(2i+6)!}\,.\end{aligned}$$

Therefore,

$$\begin{aligned} a_1(t) &= t + \sum_{i=0}^{\infty}A_i\frac{t^{2i+5}}{(2i+5)!}\,, & a_2(t) &= \frac{t^2}{2!} + \sum_{i=0}^{\infty}A_i\frac{t^{2i+6}}{(2i+6)!}\,,\\ a_3(t) &= \tfrac{t^3}{3!} + \sum_{i=0}^{\infty}B_i\frac{t^{2i+5}}{(2i+5)!}\,, & a_4(t) &= \frac{t^4}{4!} + \sum_{i=0}^{\infty}B_i\frac{t^{2i+6}}{(2i+6)!}\,.\end{aligned}$$

Using the ratio test, it is easy to verify the convergence of the series above. In fact, the sum of them establishes our claim that we also write as follows:

$$\begin{aligned}\mathcal{J}_t &= (\mathcal{J}_0 + 10\mathcal{J}_0^{2)} + 16\mathcal{J}_0^{4)}) - \sin\Big(\frac{t}{\sqrt{2}}\Big)\Big(\frac{\sqrt{2}}{3}(\mathcal{J}_0^{1)} + 8\mathcal{J}_0^{3)})\Big)\\ &+ \sin\Big(\frac{t}{2\sqrt{2}}\Big)\Big((\mathcal{J}_0^{1)} + 2\mathcal{J}_0^{3)})\frac{8\sqrt{2}}{3}\Big) + \cos\Big(\frac{t}{\sqrt{2}}\Big)\Big(\frac{2}{3}(\mathcal{J}_0^{2)} + 8\mathcal{J}_0^{4)})\Big)\\ &- \cos\Big(\frac{t}{2\sqrt{2}}\Big)\Big(\frac{32}{3}(\mathcal{J}_0^{2)} + 2\mathcal{J}_0^{4)})\Big)\,.\end{aligned} \tag{19}$$

□

In the following we will show how to use the non-recursive method for solving the Jacobi equation proposed in the third section. For that, we will calculate the first Jacobi vector field working on a geodesic in M^6 which has $r_\gamma = \mathtt{r} = 4$. Thus, we obtain the coefficients $A_0^{k)}$ given by (10) using only $\mathcal{J}_0^{n)}$, $n = 0, 1, 2, 3, 4$. The proof of the next proposition is a straightforward computation using (17) in (10).

Proposition 7. *The general expression with the reminder term $\mathcal{O}(t^{10})$ of each Jacobi vector field $(Y_t)_i$, $i = 1, \dots, 6$, along an arbitrary geodesic in the space M^6 is given by*

$$(Y_t)_i = \sum_{k=1}^{6}(A_t)_{ik}Q_k\,, \quad i = 1, \dots, 6 \tag{20}$$

where

$$
\begin{aligned}
(A_t)_{ik} &= I_{ik}t - \frac{1}{3!}\ (\mathcal{J}_0)_{ik}\ t^3 - \frac{1}{4!}[2(\mathcal{J}_0^{1)})_{ik}]t^4 - \frac{1}{5!}[3(\mathcal{J}_0^{2)})_{ik} - (\mathcal{J}_0\mathcal{J}_0)_{ik}]t^5 \\
&\quad - \frac{1}{6!}[4(\mathcal{J}_0^{3)})_{ik} - 4(\mathcal{J}_0^{1)}\mathcal{J}_0)_{ik} - 2(\mathcal{J}_0\mathcal{J}_0^{1)})_{ik}]\ t^6 - \frac{1}{7!}[5(\mathcal{J}_0^{4)})_{ik} \\
&\quad - 10(\mathcal{J}_0^{2)}\mathcal{J}_0)_{ik} - 10(\mathcal{J}_0^{1)}\mathcal{J}_0^{1)})_{ik} - 3(\mathcal{J}_0\mathcal{J}_0^{2)})_{ik} + (\mathcal{J}_0\mathcal{J}_0\mathcal{J}_0)_{ik}]\ t^7 \\
&\quad - \frac{1}{8!}\Big[-\frac{15}{4}|x|^2(\mathcal{J}_0^{3)})_{ik} - \frac{3}{8}|x|^4(\mathcal{J}_0^{1)})_{ik} - 20(\mathcal{J}_0^{3)}\mathcal{J}_0)_{ik} - 30(\mathcal{J}_0^{2)}\mathcal{J}_0^{1)})_{ik} \\
&\quad - 18(\mathcal{J}_0^{1)}\mathcal{J}_0^{2)})_{ik} - 4(\mathcal{J}_0\mathcal{J}_0^{3)})_{ik} + 6(\mathcal{J}_0^{1)}\mathcal{J}_0\mathcal{J}_0)_{ik} + 4(\mathcal{J}_0\mathcal{J}_0^{1)}\mathcal{J}_0)_{ik} \\
&\quad + 2(\mathcal{J}_0\mathcal{J}_0\mathcal{J}_0^{1)})_{ik}\Big]t^8 - \frac{1}{9!}\Big[-\frac{35}{8}|x|^2(\mathcal{J}_0^{4)})_{ik} - \frac{7}{16}|x|^4(\mathcal{J}_0^{2)})_{ik} - 35(\mathcal{J}_0^{4)}\mathcal{J}_0)_{ik} \\
&\quad - 70(\mathcal{J}_0^{3)}\mathcal{J}_0^{1)})_{ik} - 63(\mathcal{J}_0^{2)}\mathcal{J}_0^{2)})_{ik} + 28(\mathcal{J}_0^{1)}\mathcal{J}_0^{3)})_{ik} - 5(\mathcal{J}_0\mathcal{J}_0^{4)})_{ik} \\
&\quad + 21(\mathcal{J}_0^{2)}\mathcal{J}_0\mathcal{J}_0)_{ik} + 28(\mathcal{J}_0^{1)}\mathcal{J}_0^{1)}\mathcal{J}_0)_{ik} + 14(\mathcal{J}_0\mathcal{J}_0^{1)}\mathcal{J}_0)_{ik} + 10(\mathcal{J}_0\mathcal{J}_0^{2)}\mathcal{J}_0)_{ik} \\
&\quad + 10(\mathcal{J}_0\mathcal{J}_0^{1)}\mathcal{J}_0^{1)})_{ik} + 3(\mathcal{J}_0\mathcal{J}_0\mathcal{J}_0^{2)})_{ik} + (\mathcal{J}_0\mathcal{J}_0\mathcal{J}_0\mathcal{J}_0)_{ik}\Big]\ t^9 + \mathcal{O}(t^{10})\,.
\end{aligned}
$$

Example 2. Let γ_3 be the arbitrary geodesic on M^6 given in the Example 1. Then, we obtain the Jacobi field $(Y_t)_1$ along the geodesic γ_3 using Proposition 7.

$$
\begin{aligned}
(Y_t)_1 &= tQ_1 + \Big(-\frac{1}{3!8}t^3 + \frac{|x|^2}{5!8^2}t^5 - \frac{|x|^4}{7!8^3}t^7 + \frac{|x|^6}{9!8^4}t^9 + \mathcal{O}(t^{11})\Big) \\
&\quad \times \big((x_2^2 + x_5^2)Q_1 - x_1x_2Q_2 - x_1x_5Q_5\big) \\
&= tQ_1 + \Big(\frac{-t}{|x|^2} + \frac{2\sqrt{2}}{|x|^3}\sin(\frac{t|x|}{2\sqrt{2}})\Big)\big((x_2^2 + x_5^2)Q_1 - x_1x_2Q_2 - x_1x_5Q_5\big) \\
&\overset{(|x|=1)}{=} tx_1^2Q_1 + \Big(t - 2\sqrt{2}\sin\Big(\frac{t}{2\sqrt{2}}\Big)\Big)x_1(x_2Q_2 + x_5Q_5)\,.
\end{aligned}
$$

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