ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 13–23

A NEW CHARACTERIZATION OF MATHIEU GROUPS

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Abstract. Let G be a finite group and $\operatorname{nse}(G)$ the set of numbers of elements with the same order in G. In this paper, we prove that a finite group G is isomorphic to M, where M is one of the Mathieu groups, if and only if the following hold:

(1) $|G| = |M|$,

(2) $\operatorname{nse}(G) = \operatorname{nse}(M)$.

1. Introduction

It is well known that the conjugacy class sizes play an important role in determining the structure of a finite group. The connection between the conjugacy class sizes and the structures of finite groups has been studied extensively (see [3], [5], [8], for example).

Analogically, let $m_i(G) := |\{g \in G \mid \text{ the order of } g \text{ is } i\}|$ (m_i for short), be the number of elements of order i, and $\operatorname{nse}(G) := \{m_i(G) \mid i \in \pi_e(G)\}$, the set of sizes of elements with the same order. We now consider the influence of the set $\operatorname{nse}(G)$ and $|G|$ on G.

For the set $\operatorname{nse}(G)$, the most important problem is related to the Thompson's problem.

Let G be a finite group and $M_t(G) = \{g \in G \mid g^t = 1\}$. Two finite groups G_1 and G_2 are of the same order type if and only if $|M_t(G_1)| = |M_t(G_2)|$, where $t = 1, 2, \ldots$. In 1987, J. G. Thompson put forward the following problem:

Thompson's problem. Suppose G_1 and G_2 are of the same order type. If G_1 is solvable, is G_2 necessarily solvable?

Professor W. J. Shi made the above problem public in 1989 (see [10]). Unfortunately, no one can solve it or even give a counterexample till now.

We found that the set $\operatorname{nse}(G)$ plays an important role in determining structure of a finite group, too. Surely, the set $\{|M_t(G)| \mid t = 1, 2, \ldots\}$ can determine the set $\operatorname{nse}(G)$. However, if the set $\operatorname{nse}(G)$ is known, what can we say about $|M_t(G)|$?

2000 *Mathematics Subject Classification*: primary 20D60; secondary 20D06.

Key words and phrases: finite group, solvable group, order of element.

Project supported by the science and technology plan project of Henan province (Grant No. 94300510100).

Received August 5, 2008, revised September 2009. Editor J. Trlifaj.

Main theorem. A group G is isomorphic to M, where M is a Mathieu group, if and only if the following hold:

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M)$.

In this paper, $n_p(G)$ always denotes the number of Sylow p-subgroups of G, that is, $n_p(G) = |\,\mathrm{Syl}_p(G)|$, $\pi(G)$ the set of all prime divisors of $|G|$. And $\varphi(x)$ denotes the Euler function of x. We always use $|G|$ for order of finite group G, $\mid$ to denote division relationship and $||$ denote that the prime upon the left is in its highest possible power divides the argument upon the right. All further unexplained notation is standard (see [10]).

2. Lemmas

Lemma 2.1 ([6])**.** *Suppose G is a finite solvable group with $|G| = mn$, where $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}, (m, n) = 1$, $p_1, \dots, p_r$ are distinct primes. Let $\pi = \{p_1, \dots, p_r\}$ and let h_m be the number of π-Hall subgroups of G. Suppose that $h_m = q_1^{\beta_1} \dots q_s^{\beta_s}$, where $q_1, \dots, q_s$ are distinct primes. Then following conditions are true for all i:*

(1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ *for some p_j.*

(2) *The order of some chief factor of G is divided by $q_i^{\beta_i}$.*

A finite group G is called a K_n-group, if $|\pi(G)| = n$.

Lemma 2.2 ([12])**.** *Let G be a simple K_4-group, then G is isomorphic to one of the following groups:*

1) A_7, A_8, A_9, A_{10};

2) M_{11}, M_{12}, J_2;

3) (a) $L_2(r)$, *where r is a prime and satisfies*

$$r^2 - 1 = 2^a \cdot 3^b \cdot u^c$$

with $a \geq 1, b \geq 1, c \geq 1, u > 3, u$ is prime;

(b) $L_2(2^m)$, *where m satisfies:*

$$\begin{cases} 2^m - 1 = u\,; \\ 2^m + 1 = 3t^b\,. \end{cases}$$

with $m \geq 1, u, t$ primes, $t > 3$, $b \geq 1$.

(c) $L_2(3^m)$, *where m satisfies:*

$$\begin{cases} 3^m + 1 = 4t\,; \\ 3^m - 1 = 2u^c\,. \end{cases}$$

or

$$\begin{cases} 3^m + 1 = 4t^b\,; \\ 3^m - 1 = 2u\,. \end{cases}$$

with $m \geq 1, u, t$ odd primes, $b \geq 1$, $c \geq 1$.

(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8{}^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.

Corollary 2.3 ([12])**.** *Let G be a simple group of order $2^a \cdot 3^b \cdot 5 \cdot p^c$, where p is a prime, $p \neq 2, 3, 5$ and $abc \neq 0$. Then G is isomorphic to one of the following groups: A_7, A_8, A_9; M_{11}, M_{12}; $L_2(q)$, $q = 11, 16, 19, 31, 81$; $L_3(4)$, $L_4(3)$, $S_6(2)$, $U_4(3)$ or $U_5(2)$. In particular, if $p = 11$, then $G \cong M_{11}$, M_{12} or $L_2(11)$. If $p = 7$, then $G \cong A_7$, A_8, A_9, $L_3(4)$, $S_6(2)$ or $U_4(3)$.*

Lemma 2.4. *Let G be a simple K_4-group and $\pi(G) \subseteq \{2, 3, 5, 7, 11, 23\}$. Then G is isomorphic to one of the following simple groups:*
(1) A_7, A_8, A_9, A_{10};
(2) M_{11}, M_{12}, J_2;
(3) $L_2(11)$, $L_2(23)$;
(4) $L_2(49)$, $L_3(4)$, $S_4(7)$, $S_6(2)$, $O_8^+(2)$, $U_3(5)$, $U_4(3)$, $U_5(2)$.

Proof. If G is isomorphic to one of the groups of 1), 2), or 3)(d) in Lemma 2.2, we can easily get (1), (2) and (4) by [4].

Suppose now that G is isomorphic to one of the groups of (a), (b) or (c) in 3) of Lemma 2.2.

(I) If G is isomorphic to $L_2(r)$ in Lemma 2.2, then $r \in \{5, 7, 11, 23\}$.
If $r = 5$ or 7, then $|\pi(r^2 - 1)| = 2$, a contradiction.
If $r = 11$, then $r^2 - 1 = 2^3 \cdot 3 \cdot 5$. Thus $G \cong L_2(11)$.
If $r = 23$, then $r^2 - 1 = 2^4 \cdot 3 \cdot 11$. Thus $G \cong L_2(23)$.

(II) If G is isomorphic to $L_2(2^m)$ in Lemma 2.2, then $u \in \{3, 5, 7, 11, 23\}$.
If $u = 3$, then $m = 2$ and $3t^b = 5$, a contradiction.
If $u = 5$, then $2^m - 1 = 5$, a contradiction.
If $u = 7$, then $m = 3$ and $3t^b = 9$, thus $t = 3, b = 1$, this contradicts $t > 3$.
If $u = 11$, then $2^m - 1 = 11$, a contradiction.
If $u = 23$, then $2^m - 1 = 23$, a contradiction.

Similarly, we can prove that G is not isomorphic to $L_2(3^m)$ in Lemma 2.2. □

Lemma 2.5 ([2])**.** *Let α_i be a positive integer ($i = 1, \ldots, 5$), p a prime and $p \notin \{2, 3, 5, 7\}$. If G is a simple group and $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot p^{\alpha_5}$, then G is isomorphic to one of the following simple groups: A_{11}, A_{12}, M_{22}, HS, McL, He, $L_2(q)$ ($q = 2^6, 5^3, 7^4, 29, 41, 71, 251, 449, 4801$), $L_3(3^2)$, $L_4(2^2)$, $L_4(7)$, $L_5(2)$, $L_6(2)$, $O_5(7^2)$, $O_7(3)$, $O_9(2)$, $S_6(3)$, $O_8^+(3)$, $G_2(2^2)$, $G_2(5)$, $U_3(19)$, $U_4(5)$, $U_4(7)$, $U_5(3)$, $U_6(2)$, ${}^2D_4(2)$. In particular, if $p = 11$, then G is isomorphic to one of the following simple groups: A_{11}, A_{12}, M_{22}, HS, McL, $U_6(2)$.*

Lemma 2.6 ([1])**.** *Let α_i be a positive integer ($i = 1, \ldots, 6$), $p > 11$ a prime. If G is a simple group and $|G| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot 11^{\alpha_5} \cdot p^{\alpha_6}$, then G is isomorphic to one of the following simple groups: A_{13}, A_{14}, A_{15}, A_{16}, M_{23}, M_{24}, J_1, Suz, Co_2,Co_3, $M(22)$, F_3, $L_2(769)$, $L_2(881)$, $L_3(11)$, $L_6(3)$, $U_7(2)$, ${}^2D_5(2)$. In particular, if $p = 23$, then G is isomorphic to one of the following simple groups: M_{23}, M_{24}, Co_2, Co_3.*

Lemma 2.7 ([7]). *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $U_3(3)$, $L_3(3)$, $U_4(2)$.*

Lemma 2.8. *Let G be a finite group, $P \in \mathrm{Syl}_p(G)$, where $p \in \pi(G)$. Suppose that G has a normal series $K \trianglelefteq L \trianglelefteq G$. If $P \leq L$ and $p \nmid |K|$, then the following hold:*

(1) $N_{G/K}(PK/K) = N_G(P)K/K$;

(2) $|G\colon N_G(P)| = |L\colon N_L(P)|$, *that is,* $n_p(G) = n_p(L)$;

(3) $|L/K\colon N_{L/K}(PK/K)|t = |G\colon N_G(P)| = |L\colon N_L(P)|$, *that is,* $n_p(L/K)$ $t = n_p(G) = n_p(L)$, *for some positive integer t. And* $|N_K(P)|t = |K|$.

Proof. (1) follows from [7].

(2) By Frattini argument, $G = N_G(P)L$. Hence $|G\colon N_G(P)| = |N_G(P)L\colon N_G(P)|$ $= |L\colon N_L(P)|$.

(3) By (1), we have $|L/K\colon N_{L/K}(PK/K)| = |L/K\colon N_L(P)K/K| = |L\colon N_L(P)K| \big| |L\colon N_L(P)|$, then $|L/K\colon N_{L/K}(PK/K)|t = |G\colon N_G(P)| = |L\colon N_L(P)|$ for some positive integer t.

As $|L/K\colon N_{L/K}(PK/K)|t = |L\colon N_L(P)|$, then $|N_L(P)|t = |N_L(P)K|$. Hence $|N_K(P)|t = |K|$. □

Lemma 2.9 ([9]). *Each simple K_5-group is isomorphic to one of the following simple groups:*

(a) $L_2(q)$ *where q satisfies* $|\pi(q^2-1)| = 4$;

(b) $L_3(q)$ *where q satisfies* $|\pi(q^2-1)(q^3-1)| = 4$;

(c) $U_3(q)$ *where* $|\pi(q^2-1)(q^3+1)| = 4$;

(d) $O_5(q)$ *where* $|\pi(q^4-1)| = 4$;

(e) $S_z(2^{2^m+1})$ *where* $|\pi((2^{2^m+1}-1)(2^{2^{4m+2}}+1))| = 4$;

(f) $R(q)$ *where q is an odd power of 3 and* $|\pi(q^2-1)| = 3$;

(h) *one of the 30 other simple groups: A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, $O_9(2)$, $PSp_6(3)$, $PSp_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, ${O_8}^+(3)$, ${O_8}^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$, $G_2(9)$.*

Lemma 2.10. *Let G be a simple K_5-group and $|G|\big|2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$. Then $G \cong M_{22}$.*

Proof. Assume G is isomorphic to $L_2(q)$ in Lemma 2.9. Then 11 or $23\big|\ |L_2(q)|$.

(1) If $11\big||L_2(q)|$, we claim $q \neq 11$, otherwise if $q = 11$, then $|\pi(q^2-1)| = 3$, which contradicts $|\pi(q^2-1)| = 4$.

If $q = 2^m$, then $11\big|2^{2m}-1$. So we have $5\big|m$ and $31\big||L_2(q)|$, a contradiction.

If $q = 3, 3^2, 5, 7$ or 23, then $|\pi(q^2-1)| < 4$, a contradiction.

If $q = 3^3$, then $13\big|q^2-1\big||G|$, a contradiction.

(2) If $23\big||L_2(q)|$, we also get a contradiction similar to (1).

Hence G is not isomorphic to $L_2(q)$.

Similarly, G is not isomorphic to $L_3(q)$, $U_3(q)$, $O_5(q)$, $Sz(2^{2m+1})$ and $R(q)$.

In (h) of Lemma 2.9, we see that $M_{22} \cong G$. □

3. Proof of the main theorem

We shall present a separate proof for each of the Mathieu groups.

Theorem 3.1. *Let G be a group. Then $G \cong M_{11}$ if and only if the following hold:*
(1) $|G| = |M|$,
(2) $\text{nse}(G) = \text{nse}(M_{11}) = \{1, 165, 440, 990, 1584, 1320, 1980, 1440\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, let H be a $\{2, 5, 11\}$-Hall subgroup of G. All $\{2, 5, 11\}$-Hall subgroups in G are conjugate, so the number of $\{2, 5, 11\}$-Hall subgroups of G is: $|G\colon N_G(H)|\big|3^2$.

Now we calculate the number of elements of order 11 in G. By Sylow theorem we have that $n_{11}(H) = 1$ in H. So the number m of elements with order 11 in G is: $10 \le m \le 90$ and $10\big|m$, but $m \notin \text{nse}(G)$.

Thus G is unsolvable. Since $p\big|\big||G|$, where $p \in \{5, 11\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G\,,$$

such that L/K is a simple K_3-group or a simple K_4-group.

If L/K is a simple K_3-group. Since 5 or $11\big||L/K|\big|2^4 \cdot 3^2 \cdot 5 \cdot 11$, then $L/K \cong A_5$ or A_6.

(1) Assume $L/K \cong A_5$. If $P_5 \in \text{Syl}_5(G)$, then $P_5K/K \in \text{Syl}_5(L/K)$. Also $n_5(L/K)t = n_5(G)$ for some positive integer t by Lemma 2.8.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$ and $5 \nmid t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \text{nse}(G)$, $m = 1584$ and $t = 66$. By Lemma 2.8, $66|N_K(P_5)| = |K|$. As $|K|\big|2^2 \cdot 3 \cdot 11$, thus $n_{11}(K) = 1$ or 12. So we get that the number of elements of order 11 in G is 10 or 120. But $10, 120 \notin \text{nse}(G)$, a contradiction.

(2) Assume $L/K \cong A_6$. If $P_5 \in \text{Syl}_5(G)$, then $P_5K/K \in \text{Syl}_5(L/K)$. Also $n_5(L/K)t = n_5(G)$ for some positive integer t by Lemma 2.8.

By [4], $n_5(L/K) = n_5(A_6) = 36$. Hence $n_5(G) = 36t$ and $5 \nmid t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 36t$. Since $m \in nse(G)$, $m = 1584$ and $t = 44$. As $|G| = |G\colon L|\,|L\colon K|\,|K|$ and $44|N_K(P_5)| = |K|$ by Lemma 2.8. Thus $44\big||K|$, then $44 \cdot |A_6|\big||G|$, that is, $2^5 \cdot 3^2 \cdot 5 \cdot 11\big||G|$, which is a contradiction.

So L/K is a simple K_4-group and $\pi(L/K) = \pi(G) = \{2, 3, 5, 11\}$. By Corollary 2.3 $L/K \cong M_{11}, M_{12}$ or $L_2(11)$. But $|L/K|\big|2^4 \cdot 3^2 \cdot 5 \cdot 11$, thus $L/K \cong M_{11}$ or $L_2(11)$.

Assume $L/K \cong L_2(11)$. Let $P_{11} \in \text{Syl}_{11}(G)$, then $P_{11}K/K \in \text{Syl}_{11}(L/K)$. By Lemma 2.8, $n_{11}(L/K)t = n_{11}(G)$ for some positive integer t and $11 \nmid t$.

By [4], $n_{11}(L/K) = n_{11}(L_2(11)) = 12$. Hence $n_{11}(G) = 12t$.

Thus the number m of elements of order 11 in G is: $m = n_{11}(G) \cdot 10 = 120t$. Since $m \in \{1, 165, 440, 990, 1584, 1320, 1980, 1440\}$, $m = 1440$ and $t = 12$. Therefor $12|N_K(P_{11})| = |K|$ by Lemma 2.8. As $|K|\big|12$, so $N_K(P_{11}) = 1$ and $|K| = 12$. And then $K \cap N_G(P_{11}) = K \cap C_G(P_{11}) = 1$. So $K \rtimes P_{11}$ is a Frobenius group, which means that $|P_{11}|\big|\,|\text{Aut}(K)|$, a contradiction.

So $L/K \cong M_{11}$, and hence $|L/K| = |M_{11}|$. Thus $K = 1$ and $G = L \cong M_{11}$. $\square$

Theorem 3.2. *Let G be a group. Then $G \cong M_{12}$ if and only if the following hold:*
(1) $|G| = |M|$,
(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{12}) = \{1, 891, 4400, 5940, 9504, 23760, 9504, 17280\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, let H be a $\{2, 5, 11\}$-Hall subgroup of G. G is solvable, and therefore all the $\{2, 5, 11\}$-Hall subgroups of G are conjugate. Hence the number of $\{2, 5, 11\}$-Hall subgroups of G is:

$$|G\colon N_G(H)|\big|3^3\,.$$

We have $n_{11}(H) = 1$ or 320 by Sylow theorem. Let m be the number of elements of order 11 in G. If $n_{11}(H) = 1$, then $10 \le m \le 270$. But $m \notin \mathrm{nse}(G)$, a contradiction.

If $n_{11}(H) = 320$, then $3200 \le m \le 86400$ and $10\big|m$. Since $m \in \{1, 891, 4400, 5940, 9504, 23760, 9504, 17280\}$, $m = 4400, 5940, 23760$ or 17280. And we have $n_{11}(G) \cdot 10 = m$ in G, that is, $n_{11}(G) = 11k + 1 = 440, 594, 2376$ or 1728 for some positive integer k. If $n_{11}(G) = 11k+1 = 440, 594$ or 2376, then this equation has no solution in N. If $n_{11}(G) = 11k + 1 = 1728 = 2^6 \cdot 3^3$, then we have $2^6 \equiv 1 \pmod{11}$ and $3^3 \equiv 1 \pmod{11}$ by Lemma 2.1, a contradiction.

Hence G is unsolvable. Since $p\big|\,|G|$, where $p \in \{5, 11\}$, G has a normal series as follows:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G\,,$$

such that L/K is a non-Abelian simple group. Since $|\pi(G)| = 4$, then $|\pi(L/K)| = 3$ or 4.

If $|\pi(L/K)| = 3$, then L/K is a simple K_3-group and $\pi(L/K) \subset \pi(G) = \{2, 3, 5, 11\}$. Hence G is isomorphic to one of the group: A_5, A_6 or $U_4(2)$ by Lemma 2.7.

(1) Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. By Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, then $m = 9504$ and $t = 396$. Therefore $396|N_K(P_5)| = |K|$ by Lemma 2.8. As $|K|\,\big|2^4 \cdot 3 \cdot 11$, and hence $n_{11}(K) = 1$, 12 or 144. So the number of elements of order 11 in G is: 10 or 120. But 10, $120, 1440 \notin \mathrm{nse}(G)$, a contradiction.

(2) Assume $L/K \cong A_6$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. By Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_6) = 36$. Hence $n_5(G) = 36t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 144t$. Since $m \in \mathrm{nse}(G)$, and hence $m = 9504$ and $t = 66$. By Lemma 2.8, $66|N_K(P_5)| = |K|$. As $|G| = |G\colon L|\,|L\colon K|\,|K|$, then $|K|\big|2^3 \cdot 3 \cdot 11$. So we have $n_{11}(K) = 1$ or 12. And then the number m of elements of order 11 in G is: $m = 10$ or 120. But 10, $120 \notin \mathrm{nse}(G)$, a contradiction.

(3) Assume $L/K \cong U_4(2)$, then $|U_4(2)| \,\big|\, |G|$, that is, $2^6 \cdot 3^4 \cdot 5 \big| \, |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$, a contradiction.

Hence L/K is a simple K_4- group and $\pi(L/K) = \{2, 3, 5, 11\}$, therefore $L/K \cong M_{11}, M_{12}$ or $L_2(11)$ by Corollary 2.3.

(1) Assume $L/K \cong M_{11}$, Let $P_{11} \in \mathrm{Syl}_{11}(G)$, then $P_{11}K/K \in \mathrm{Syl}_{11}(L/K)$. Also $n_{11}(L/K)t = n_{11}(G)$ for some positive integer t and $11 \nmid t$.

By [4], $n_{11}(L/K) = n_{11}(M_{11}) = 144$. Hence $n_{11}(G) = 144t$.

So the number m of elements of order 5 in G is: $m = n_{11}(G) \cdot 10 = 1440t$. Since $m \in \mathrm{nse}(G)$, $m = 17280$ and $t = 12$. By Lemma 2.8, $12|N_K(P_5)| = |K|$. As $|K| \big| 2^2 \cdot 3$, then $|K| = 2^2 \cdot 3$ and $N_K(P_{11}) = 1$. And $1 = N_K(P_{11}) \geq C_K(P_{11})$. So $K \rtimes P_{11}$ is a Frobenius group, and hence $|P_{11}| \,\big|\, |\mathrm{Aut}(K)|$, a contradiction.

(2) Assume $L/K \cong L_2(11)$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, then $P_{11}K/K \in \mathrm{Syl}_{11}(L/K)$. Also $n_{11}(L/K)t = n_{11}(G)$ for some positive integer t and $5 \nmid t$ by Lemma 2.8.

By [4], $n_{11}(L/K) = n_{11}(L_2(11)) = 12$. Hence $n_{11}(G) = 12t$ and $11 \nmid t$.

So the number m of elements of order 11 in G is: $m = n_{11}(G) \cdot 10 = 120t$. Since $m \in \mathrm{nse}(G)$, $m = 17280$ and $t = 144$. By Lemma 2.8, $144|N_K(P_{11})| = |K|$. As $|K| \,\big| 2^4 \cdot 3^2$, then $|K| = 2^4 \cdot 3^2$ and $N_K(P_{11}) = 1$. And $1 = N_K(P_{11}) \geq C_K(P_{11})$. So $K \rtimes P_{11}$ is a Frobenius group, and hence $|P_{11}| \,\big|\, |\mathrm{Aut}(K)|$, a contradiction.

So we get $L/K \cong M_{12}$, and hence $|L/K| = |M_{12}| = |G|$. Thus $K = 1$ and $G = L \cong M_{12}$. □

Theorem 3.3. *Let G be a group. Then $G \cong M_{22}$ if and only if the following hold:*

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{22}) = \{1, 1155, 12320, 41580, 88704, 36960, 126720, 55440, 80640\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, let H be a $\{3, 5, 7, 11\}$-Hall subgroup of G. G is solvable, and therefore all the $\{3, 5, 7, 11\}$-Hall subgroups of G are conjugate. Hence the number of $\{3, 5, 7, 11\}$-Hall subgroup of G is:

$$|G \colon N_G(H)| \,\big| 2^7 \,.$$

We have $n_{11}(H) = 1$ or 45 by Sylow theorem. Let m be the number of elements of order 11 in G.

If $n_{11}(H) = 1$, then $10 \leq m \leq 1280$ and $10 \big| \, m$. But $m \notin \mathrm{nse}(G)$, a contradiction.

If $n_{11}(H) = 45$, then $450 \leq m \leq 57600$ and $10 \big| \, m$. Since $m \in \mathrm{nse}(G)$, $m = 12320, 41580, 36960$ or 55440. And we have $n_{11}(G) \cdot 10 = m$, that is, $11k + 1 = 1232, 4158, 3696$ or 5544 in G for some positive integer k. But this equation has no solution in N.

Hence, G is unsolvable. Since $p \big| \, |G|$, where $p \in \{5, 7, 11\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G \,,$$

such that L/K is a non-Abelian simple group.

(1) L/K is not a simple K_3-group. Otherwise, $L/K \cong A_5, A_6, L_2(7)$ or $L_2(8)$ by Lemma 2.7 and [4].

Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also by Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \mid t$.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, then $m = 88704$ and $t = 3696$. By Lemma 2.8, $3696|N_K(P_5)| = |K|$.

As $|K| \,\big|\, 2^5 \cdot 3 \cdot 7 \cdot 11$, there must be $n_{11}(K) = 1, 56$ or 672. So the number m of elements of order 11 in G is: 10, 560 or 6720. But $m \notin \mathrm{nse}(G)$, a contradiction.

Similarly, L/K is not isomorphic to $A_6, L_2(7)$ or $L_2(8)$.

(2) L/K is not a simple K_4-group. Otherwise, by Corollary 2.3, we have $L/K \cong A_7$, A_8, M_{11}, $L_2(11)$ or $L_3(4)$.

Assume $L/K \cong A_7$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also $n_5(L/K)$ $t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_7) = 126$. Hence $n_5(G) = 126t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 504t$. Since $m \in nse(G)$, then $m = 88704$ and $t = 176$. By Lemma 2.8, $176|N_K(P_5)| = |K|$. As $|G| = |G\colon L|\,|L\colon K|\,|K|$, then $|K| \,\big|\, 2^4 \cdot 11$, that is, $|K| = 2^4 \cdot 11$. And then $n_{11}(K) = 1$. So the number m of elements of order 11 in G is: $m = 10$. But $10 \notin \mathrm{nse}(G)$, a contradiction.

Similarly, we can get that L/K is not isomorphic to A_8, M_{11}, $L_2(11)$ and $L_3(4)$.

Hence L/K is a simple K_5-group. By Lemma 2.10, $L/K \cong M_{22}$. So we have $|L/K| = |M_{22}| = |G|$. Thus $K = 1$ and $G = L \cong M_{22}$. $\square$

Theorem 3.4. *Let G be a group. Then $G \cong M_{23}$ if and only if the following hold:*

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{23}) = \{1, 3795, 56672, 318780, 680064, 850080, 1457280, 1275120, 1854720, 1360128, 887040\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we prove that G is unsolvable.

If G is solvable, then G contains a $\{3, 5, 7, 11, 23\}$-Hall subgroup. By Sylow theorem, $n_{23}(H) = 1$ or 231.

Moreover,

$$|G\colon N_G(H)| \,\big|\, 2^7 \,.$$

If $n_{23}(H) = 1$, then $22 \le m \le 2816$ and $22 \mid m$, but $m \notin \mathrm{nse}(G)$.

If $n_{23}(H) = 231$, then $5082 \le m \le 650496$ and $22 \mid m$, but $m \in \mathrm{nse}(G)$.

Hence $m = 56672$ or 318780. And we have $n_{23}(G) \cdot 22 = m$ in G, that is, $23k + 1 = 2576$ or 1440 for some positive integer k, but the equation has no solution in N.

Thus, G is unsolvable. Since $p\big|\,\big|\,|G|$, where $p \in \{5, 7, 11, 23\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G \,,$$

such that L/K is a non-Abelian simple group.

(1) L/K is not a simple K_3-group. Otherwise, L/K is isomorphic to A_5, A_6, $L_2(7)$, $L_2(8)$ or $U_3(3)$ by Lemma 2.7 and [4].

Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also by Lemma 2.8, $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 680064$ or 1360128 and $t = 28336$ or 56672, respectively.

If $m = 680064$ and $t = 28336$, then $28336|N_K(P_5)| = |K|$ by Lemma 2.8. As $|K| \mid 2^5 \cdot 3 \cdot 7 \cdot 11 \cdot 23$, we obtain $n_{23}(K) = 1$. So the number m of elements of order 23 in G is: $m = 22$. But $22 \in \mathrm{nse}(G)$, a contradiction.

If $m = 1360128$ and $t = 56672$. Similarly as above, we also get a contradiction.

Similarly, $L/K \ncong A_6, L_2(7), L_2(8)$ or $U_3(3)$.

(2) L/K is not a simple K_4-group. Otherwise, by Lemma 2.4 and [4], $L/K \cong A_7$, A_8, M_{11}, $L_3(4)$, $L_2(11)$ or $L_2(23)$.

If $L/K \cong A_7$, then $n_7(L/K) = 120$, $n_7(G) = 120t$ and $7 \nmid t$ for some positive integer t and $7 \nmid t$ by Lemma 2.8.

So the number m of elements of order 7 in G is: $m = n_7(G) \cdot 6 = 720t$. Since $m \in \mathrm{nse}(G)$, then $m = 1457280$ and $t = 2024$. So we have $2024|N_K(P_7)| = |K|$ by Lemma 2.8. As $|K| \mid 2^4 \cdot 11 \cdot 23$, there must be $n_{23}(K) = 1$. And then the number m of elements of order 23 in G is: $m = 22$. but $22 \notin nse(G)$, a contradiction.

Similar to the case in (1), we can get that $L/K \ncong A_7, A_8$ or $L_3(4)$.

(3) L/K is not a simple K_5-group. Otherwise, $L/K \cong M_{22}$ by Lemma 2.10. So we have $n_{11}(G) = n_{11}(L/K)t = 8064t$, where $11 \nmid t$, and the number m of elements of order 11 in G is: $m = n_{11}(G)10 = 80640t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 1854720$ or 887040 and $t = 23$ or 11, respectively.

Assume $m = 1854720$ and $t = 23$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, there is $23|N_K(P_{11})| = |K|$. As $|K| \mid 23$, then $|K| = 23$. So we have that the number of elements of order 23 in G is 22, but $22 \notin \mathrm{nse}(G)$, a contradiction.

Assume $m = 887040$ and $t = 11$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, then $11|N_K(P_{11})| = |K|$. We have $|K| \mid 23$, which is a contradiction.

So $\pi(L/K) = \{2, 3, 5, 7, 11, 23\}$ and $|L/K| \mid 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. By Lemma 2.6 and [4], $L/K \cong M_{23}$. Thus $|L/K| = |M_{23}| = |G|$. This implies $K = 1$ and $G = L \cong M_{23}$. □

Theorem 3.5. *Let G be a group. Then $G \cong M_{24}$ if and only if the following hold:*

(1) $|G| = |M|$,

(2) $\mathrm{nse}(G) = \mathrm{nse}(M_{24}) = \{1, 43263, 712448, 5100480, 4080384, 20401920, 11658240, 15301440, 12241152, 22256640, 40803840, 34974720, 32643072, 23316480, 21288960\}$.

Proof. The necessity is obvious. We only need to prove the sufficiency.

First, we will prove that G is unsolvable.

If G is solvable, then G contains a $\{3, 5, 7, 11, 23\}$-Hall subgroup. Moreover, the number of $\{3, 5, 7, 11\}$-Hall subgroups in G is $|G\colon N_G(H)| \mid 2^{10}$. By Sylow theorem, $n_{23}(H) = 1$ or 231.

If $n_{23}(H)$=1, then $10 \leq m \leq 40960$ and $22 \mid m$, but $m \notin \mathrm{nse}(G)$.

If $n_{23}(H)$=231, then $5082 \leq m \leq 5203968$ and $22 \mid m$. Since $m \in \mathrm{nse}(G)$, there must be $m = 712448, 5100480$ or 4080384.

So we have $n_{23}(G) \cdot 22 = m$, that is, $23k + 1 = 32384, 231840$ or 185472, but the equation has no solution in N.

Hence, G is unsolvable. Since $p\big|\,|G|$, where $p \in \{5, 7, 11\}$, G has a normal series:

$$1 \trianglelefteq K \trianglelefteq L \trianglelefteq G\,,$$

such that L/K is a non-Abelian simple group.

(1) L/K is not a simple K_3-group. Otherwise, L/K is isomorphic to: A_5, A_6, $L_2(7)$, $L_2(8)$ or $U_4(2)$ by Lemma 2.7 and [4].

Assume $L/K \cong A_5$. If $P_5 \in \mathrm{Syl}_5(G)$, then $P_5K/K \in \mathrm{Syl}_5(L/K)$. Also $n_5(L/K)t = n_5(G)$ for some positive integer t and $5 \nmid t$ by Lemma 2.8.

By [4], $n_5(L/K) = n_5(A_5) = 6$. Hence $n_5(G) = 6t$.

So the number m of elements of order 5 in G is: $m = n_5(G) \cdot 4 = 24t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 4080384$, 12241152 or 32643072 and $t = 170016$, 510048 or 1360128, respectively.

Whenever $t = 170016$, 510048 or 1360128, we can get that $p\big|\,|K|$, where $p \in \{7, 11, 23\}$.

If K is solvable, we can get that the number m of elements of order 23 in G is: $m \notin \mathrm{nse}(G)$.

If K is unsolvable, similar to theorem 3.1, we also can get a contradiction.

Similarly, $L/K \ncong A_6, L_2(7), L_2(8)$ or $U_3(3)$.

(2) L/K is not a simple K_4-group. Otherwise, by Lemma 2.4 and [4], $L/K \cong A_7, A_8, M_{11}, M_{12}, L_2(11), L_2(23)$, or $L_3(4)$.

If $L/K \cong A_7$, then $n_7(L/K) = 120$, $n_7(G) = 120t$ and $7 \nmid t$ by Lemma 2.8.

So the number m of elements of order 7 in G is: $m = n_7(G) \cdot 6 = 720t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 11658240$, 34974720 or 23316480 and $t = 16192$, 48676 or 32384, respectively.

If $m = 11658240$ and $t = 16192$, then $16192|N_K(P_7)| = |K|$. As $|K| \mid 2^6 \cdot 3 \cdot 11 \cdot 23$, we obtain $n_{23}(K) = 1$ or 24. And then the number m of elements of order 23 in G is: $m = 22$ or 528. But 22, $528 \notin \mathrm{nse}(G)$, a contradiction.

If $m = 34974720$ and $t = 48676$, then $48676|N_K(P_7)| = |K|$. Now $|K| \mid 2^6 \cdot 3 \cdot 11 \cdot 23$, a contradiction.

If $m = 23316480$ and $t = 32384$, then $32384|N_K(P_7)| = |K|$. Now $|K| \mid 2^6 \cdot 3 \cdot 11 \cdot 23$, a contradiction.

Similarly as above, we also get that $L/K \ncong A_8$, A_9, M_{11}, M_{12}, $L_2(11)$, $L_2(23)$, or $L_3(4)$.

(3) L/K is not a simple K_5-group. Otherwise, $L/K \cong M_{22}$ by Lemma 2.10. So we have $n_{11}(G) = n_{11}(L/K)t = 8064t$, where $11 \nmid t$, and the number m of elements of order 11 in G is: $m = n_{11}(G)10 = 80640t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 22256640$ and $t = 276$. If $P_{11} \in \mathrm{Syl}_{11}(G)$, then $276|N_K(P_{11})| = |K|$. Now $|K| \mid 2^3 \cdot 3 \cdot 23$. Therefore $n_{23}(K) = 1$ or 24. So we have that the number of elements of order 23 in G is 22 or 528. But 22, $528 \notin \mathrm{nse}(G)$, a contradiction.

So $\pi(L/K) = \{2, 3, 5, 7, 11, 23\}$ and $|L/K| \mid 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. By Lemma 2.6 and [4], $L/K \cong M_{23}$ or M_{24}.

If $L/K \cong M_{23}$, we have $n_{23}(L/K)t = n_{23}(G)$ and $23 \nmid t$ by Lemma 2.8. Now, $n_{23}(L/K) = 40320$. Choose $P_{23} \in \mathrm{Syl}_{23}(G)$. Then $n_{23}(G) = 40320t$, where $23 \nmid t$. Hence the number m of elements of order 23 in G is: $m = n_{23}(G)22 = 887040t$. Since $m \in \mathrm{nse}(G)$, there must be $m = 21288960$ and $t = 24$. As $24|N_K(P_{23})| = |K|$ and $|K| \mid 2^3 \cdot 3$, we obtain $N_K(P_{23}) = 1$ and $|K| = 24$. And $1 = K \cap N_G(P_{23}) \geq K \cap C_G(P_{23})$. So $K \rtimes P_{23}$ is a Frobenius group, therefore $|P_{23}| \mid |\mathrm{Aut}(K)|$, a contradiction.

Hence $|L/K| = |M_{24}| = |G|$. We get $K = 1,\ G = L \cong M_{24}$.

□

Acknowledgement. The authors would like to thank the referee with deep gratitude for pointing out some mistakes in a previous version of the paper, especially his/her valuable suggestions on revising the paper, which make the proof of theorem read smoothly and more technically.

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