ARCHIVUM MATHEMATICUM (BRNO)
Tomus 46 (2010), 25–38

FREE ALGEBRAS IN VARIETIES

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Abstract. We define varieties of algebras for an arbitrary endofunctor on a cocomplete category using pairs of natural transformations. This approach is proved to be equivalent to one of equational classes defined by equation arrows. Free algebras in the varieties are investigated and their existence is proved under the assumptions of accessibility.

In universal algebra we deal with varieties – classes of algebras satisfying a certain collection of identities (pairs of terms of the corresponding language). This concept was generalized by Adámek and Porst in [3]. They worked with algebras for an endofunctor on a cocomplete category and used the free-algebra construction developed by Adámek in [1] (a chain of term-functors) to define equation arrows as certain regular epimorphisms. Using them they defined equational categories as analogues to varieties. These categories are later studied in [5].

We focus on another approach to varieties of algebras for a functor. We also use the free-algebra construction and define a natural term as a natural transformation with codomain in a term-functor. A pair of natural terms with a common domain will be called a natural identity and will be satisfied on an algebra if both of its natural transformations have the same term-evaluation on this algebra. Natural identities induce classes of algebras, which are proved to be precisely the classes defined by means of equation arrows. We present several examples of such classes and show that, in some cases, this approach essentially simplifies the presentation.

In the second chapter we investigate free algebras in a variety. Induction by natural identities allows us to make a restriction on identities with domains preserving the colimits of some small chains. Such identities will be called accessible. These cases still cover most of the usual examples and we prove that such varieties have free algebras. The proof uses a conversion of variety to a category of algebras for a diagram of monads used by Kelly in [7] to define algebraic colimit of monads. His theorem proving the existence of free objects of this category yields the existence of free algebras in the variety induced by accessible identities.

Notational convention. The constant functor mapping the objects to object X will be denoted by C_X. The initial object in a cocomplete category will be denoted by 0. For functors, we omit the brackets and the composition mark $\circ$ when possible. The class of objects and morphisms of a category will be denoted by Ob and Mor,

2000 *Mathematics Subject Classification*: primary 08C05; secondary 18C05, 18C20.
Key words and phrases: cocomplete category, free algebra, variety, natural transformation.
Received April 7, 2009, revised September 2009. Editor J. Rosický.

respectively. The concrete isomorphism between two concrete categories over $\mathcal{C}$ (i.e., the isomorphism preserving the forgetful functor) will be denoted by $\cong_{\mathcal{C}}$.

1. Classes of algebras

1.1. Algebras and equational classes.

Definition 1.1. Let F be an endofunctor on a category $\mathcal{C}$. By $\mathbf{Alg}\, F$ we denote the category of F-algebras – its objects are pairs (A, α), where $\alpha\colon FA \to A$ is a morphism in $\mathcal{C}$. The morphism $\phi_F\colon (A,\alpha) \to (B,\beta)$ of F-algebras is a morphism $\phi\colon A \to B$ such that $\phi \circ \alpha = \beta \circ F\phi$. The subscript $_F$ in the notation of morphism is usually omitted.

Remark 1.1. There is a forgetful functor $\mathcal{Z}_F : \mathbf{Alg}\, F \to \mathcal{C}$ assigning to an algebra (A, α) its underlying object A.

From now on, $\mathcal{C}$ will denote a cocomplete category. Let us recall the *free-algebra construction* (introduced in [1], generalized in [3]). We will show the definition in the functorial form.

Definition 1.2. We will use transfinite induction to define *term functors* $F_n\colon C \to \mathcal{C}$, for $n \in \mathcal{O}rd$ and natural transformations $w_{m,n} : F_m \to F_n$, for $m \leq n$:

Initial step: $F_0 = \mathrm{Id}_{\mathcal{C}}$, $w_{0,0} = \mathrm{id}$

Isolated step: Let $F_{n+1} = FF_n + \mathrm{Id}_{\mathcal{C}}$, the transformations $w_{0,n+1} = \iota_{n+1}$ and $q_n\colon FF_n \to F_{n+1}$ are the canonical injections of $\mathrm{Id}_{\mathcal{C}}$ and FF_n, respectively, into the coproduct and $w_{m+1,n+1} = [Fw_{m,n}, \mathrm{id}_{\mathrm{Id}_{\mathcal{C}}}]$ for $m \leq n$ is defined by

$$\begin{tikzcd}
FF_m \arrow[r, "Fw_{m,n}"] \arrow[d, "q_m"'] & FF_n \arrow[d, "q_n"] \\
F_{m+1} \arrow[r, "w_{m+1,n+1}"] & FF_n + \mathrm{Id}_{\mathcal{C}} = F_{n+1} \\
\mathrm{Id}_{\mathcal{C}} \arrow[u, "\iota_{m+1}"] \arrow[ur, "\iota_{n+1}"']
\end{tikzcd}$$

If m is a limit ordinal, then we define $w_{m,m+1}$ as the unique factorization of $\{w_{k,m+1} \mid k < m\}$ over the colimit cocone $\{w_{k,m} \mid k < m\}$.

Limit step: $F_n = \operatorname*{colim}_{m<n} F_m$ and $w_{m,n}$ is the corresponding component of the colimit cocone.

The construction gives rise to the transformation $y_n\colon F \to F_n$ defined by

$$y_n = w_{1,n} \circ q_0$$

for every ordinal $n > 0$.

To distinguish the transformations for different functors we put the name of the functor in the superscript: $w^F_{m,n}$, q^F_n, ι^F_n, y^F_n.

For every $m \leq n$, the construction yields the property:

$$w_{m,n} \circ q_m \quad = \quad q_n \circ Fw_{m,n}\,. \tag{1}$$

Remark 1.2. If we substitute Id_C for C_0 in initial step of construction, we get an equivalent concept.

As a consequence of the definition we obtain the following properties (see [3]).

Remark 1.3. Given an F-algebra (A, α), for every $n \in \mathcal{O}rd$, there is a morphism (a *term-evaluation* on (A, α))

$$\epsilon_{n,(A,\alpha)} \colon F_n A \to A$$

defined recursively by: $\epsilon_{0,(A,\alpha)} = \mathrm{id}_A$, $\epsilon_{n+1,(A,\alpha)} = [\alpha \circ F\epsilon_{n,(A,\alpha)}, \mathrm{id}_A]$,

$$\begin{tikzcd}
FF_n(A) \arrow[r, "F\epsilon_{n,(A,\alpha)}"] \arrow[d, "q_{n,A}"'] & F(A) \arrow[d, "\alpha"] \\
F_{n+1}(A) \arrow[r, "\epsilon_{n+1,(A,\alpha)}"] & A \\
A \arrow[u, "\iota_{n+1,A}"] \arrow[ur, "\mathrm{id}_A"'] &
\end{tikzcd}$$

and by $\epsilon_{l,(A,\alpha)} = \operatorname*{colim}_{m<l} \epsilon_{m,(A,\alpha)}$ for a limit ordinal l. Then, for every n, $m \leq n$, we have:

$$\epsilon_{n+1,(A,\alpha)} \circ q_{n,A} = \alpha \circ F\epsilon_{n,(A,\alpha)} \tag{2}$$

$$\epsilon_{n,(A,\alpha)} \circ \iota_{n,A} = \mathrm{id}_A \tag{3}$$

$$\epsilon_{m,(A,\alpha)} = \epsilon_{n,(A,\alpha)} \circ w_{m,n,A} \tag{4}$$

$$\epsilon_{n,(A,\alpha)} \circ y_{n,A} = \alpha \tag{5}$$

where the last property requires $n > 0$. We write the name of the functor in the superscript $\epsilon_{k,(A,\alpha)} = \epsilon^F_{k,(A,\alpha)}$, if necessary.

We recall here the notion of equational category of F-algebras introduced in [3].

Definition 1.3. Let X be an object of $\mathcal{C}$, $n \in \mathcal{O}rd$. An *equation arrow of arity n over X* is defined as a regular epimorphism $e : F_n X \to E$. The object X is called a *variable-object* of e.

We say that an F-algebra (A, α) *satisfies* an equation arrow $e \colon F_n X \to E$ if for every $f \colon X \to A$ there is a morphism $h \colon E \to A$ such that $\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e$.

For a class $\mathcal{E}$ of equation arrows, we define an *equational class of F-algebras induced by $\mathcal{E}$* as the class of all algebras satisfying all equations $e \in \mathcal{E}$. Viewed as a full subcategory of $\mathbf{Alg}\, F$, it is called an *equational category* and denoted by $\mathbf{Alg}\,(F, \mathcal{E})$. Equational category presentable in this way by a single equation arrow is called *single-based*.

As shown in [3], this approach generalizes the classical universal algebra on sets, since every identity uniquely determines the regular epimorphism on the set of all terms which is given by unifying the terms included in the identity.

1.2. **Naturally induced classes.**
Now we introduce a concept of algebras induced by natural transformations.

Definition 1.4. Let n be an ordinal and G be a $\mathcal{C}$-endofunctor. A natural transformation $\phi\colon G \to F_n$ is called a *natural term*, more precisely an *n-ary G-term*.

By *G-identity* we mean a pair of G-terms. Such pairs are called *natural identities*. Let ϕ and ψ be m-ary and n-ary G-terms, respectively. The functor G is called a *domain* and (m, n) is an *arity-couple* of identity (ϕ, ψ). If $m = n$, we say that (ϕ, ψ) has an *arity* of n.

We say that an F-algebra (A, α) *satisfies* the G-identity (ϕ, ψ), if

$$\epsilon_{m,(A,\alpha)} \circ \phi_A = \epsilon_{n,(A,\alpha)} \circ \psi_A \,.$$

Then we write

$$(A, \alpha) \models (\phi, \psi)\,.$$

For a class $\mathcal{I}$ of natural identities, we define a *naturally induced class of F-algebras* as the class of all algebras satisfying all identities $(\phi, \psi) \in \mathcal{I}$. The corresponding full subcategory of $\mathbf{Alg}\, F$ is denoted by $\mathbf{Alg}\,(F, \mathcal{I})$. If such a class is inducible by a single natural identity, we say that this class is *single-induced*.

Two natural identities are said to be algebraically equivalent iff they induce the same classes of F-algebras. Analogously, we define an algebraic equivalence of classes of natural identities. The algebraic equivalence relation will be denoted by $\approx$.

Remark 1.4.

(1) Arities of components of a natural identity can be arbitrarily raised. Clearly, for an identity (ϕ, ψ) of arity-couple (m_1, m_2), we have $(\phi, \psi) \approx (w_{m_1,n} \circ \phi, w_{m_2,n} \circ \psi)$ for every $n \geq \max\{m_1, m_2\}$. Hence, every natural identity is algebraically equivalent to the identity consisting of natural terms of the same arity.

(2) Every set $\mathcal{N} = \{(\phi_i, \psi_i) \mid i \in I\}$ of n-ary natural identities is algebraically equivalent to a singleton. Clearly, $\mathcal{N} \approx \{(\phi, \psi)\}$, where ϕ, ψ are the unique factorizations of the cocones ϕ_i, ψ_i, respectively, over the coproduct of domains of single identities.

(3) As a consequence, every class naturally induced by a set of identities is single-induced.

1.3. **Examples of naturally induced classes.**
In Section 1.4 we show that every equational class is naturally induced and vice versa. And since the concept of equational classes generalizes varieties in universal algebra, every variety of algebras in the classical sense is a naturally induced class. An explicit correspondence is shown in the following example.

Example 1.1. Let $\mathcal{C} = \mathcal{S}et$, Σ be a signature consisting of operation symbols σ of (possibly infinite) arities $ar(\sigma)$. Let $F = \coprod_{\sigma\in\Sigma} \hom(ar(\sigma), -)$ and $u_\sigma\colon \hom(ar(\sigma), -) \to F$ be the canonical inclusion for every $\sigma \in \Sigma$. Then the category of Σ-algebras is isomorphic to $\mathbf{Alg}\, F$. For each Σ-term τ, let X_τ be the set of variables occurring in τ and let $d(\tau)$ be the depth of τ (supremum of ordinals corresponding to the chains of the proper subterms of τ ordered by subterm-relation).

For a given term τ, we define a $d(\tau)$-ary G_τ-term ϕ_τ, where $G_\tau = \hom(X_\tau, -)$. The transformation ϕ_τ is defined inductively: if τ is a variable x, then $\phi_\tau\colon \hom(\{x\}, -) \to F_0$ is an obvious isomorphism. If $\tau = \sigma\big(\rho_i; i \in ar(\sigma)\big)$ and we have $\phi_{\rho_i}\colon \hom\big(ar(\rho_i), -\big) \to F_{d(\rho_i)}$ for each $i \in ar(\sigma)$, we can extend all transformations ϕ_{ρ_i} to $\phi'_i\colon \hom(ar(\rho_i), -) \to F_n$ where $n = \sup\{d(\rho_i) \mid i \in ar(\sigma)\}$. We define ϕ_τ in the following way. Since $X_{\rho_i} \subseteq X_\tau$ for every i, we have $p_i\colon \hom(X_\tau, -) \to \hom(X_{\rho_i}, -)$, hence the factorization over the limit cone yields a unique $r\colon \hom(X_\tau, -) \to \prod_{i\in ar(\sigma)} \hom(X_{\rho_i}, -)$. We define ϕ_τ as the following composition:

$$\begin{tikzcd}
\hom(X_\tau,-) \arrow[r, "r"] \arrow[d, "\phi_\tau"] & \prod_{i\in ar(\sigma)} \hom(X_{\rho_i},-) \arrow[rr, "\prod \phi'_i"] & & \prod_{i\in ar(\sigma)} F_n \arrow[d, equal, "iso"] \\
F_{n+1} & FF_n \arrow[l, "q_n"'] & & \hom(ar(\sigma),-)\circ F_n \arrow[ll, "u_\sigma F_n"']
\end{tikzcd}$$

Observe that $n + 1 = d(\tau)$.

To each Σ-term, we have assigned a natural term. Now, to identity (τ_1, τ_2) consisting of two Σ-terms with variables in X, we assign a pair of corresponding natural $\hom(X, -)$-terms. It is easy to see, that we get an identity which induces exactly the variety given by (τ_1, τ_2). Monoids, for example, are objects of $\mathbf{Alg}\,\big((\hom(2, -) + \hom(0, -), \{i, j, k\}\big)$, where i is a binary identity with domain $\hom(3, -)$ and stands for associativity while j, k are unary with domain Id and correspond to left and right neutrality of 1.

The concept can be used to define naturally induced classes of algebras even on some illegitimate categories.

Example 1.2. Let $\mathcal{C} = \mathbf{End}\mathcal{A}$ be an illegitimate category of endofunctors on some cocomplete category $\mathcal{A}$. For every $k \in \omega$, the composability of objects of $\mathcal{C}$ yields the existence of a "composition power functor" $S_k\colon \mathcal{C} \to \mathcal{C}$ such that

$$S_k(P) = \underbrace{P \circ P \circ \cdots \circ P}_{k \text{ times}}\ .$$

We can define analogues to universal algebras – all we need to do is to replace products of sets by composition of functors and each $\hom(k, -)$ by S_k in the description above. By analogy to monoids, we get the category **Monad** $\mathcal{A}$ of monads on $\mathcal{A}$. Namely, **Monad** $\mathcal{A} = \mathbf{Alg}\,\big((S_2 + S_0), \{i, j, k\}\big)$ where the domains of identities i, j, k are S_3, S_1, S_1, respectively. Each operation $\pi\colon (S_2 + S_0)(P) \to P$ decomposes into $\mu\colon S_2(P) = PP \to P$ and $\eta\colon S_0 = \mathrm{Id} \to P$ and the identities are satisfied exactly as required by the usual condition for μ and η.

Theorem 3.6 in [3] describes an equational presentation of the category of algebras for a monad. The following example shows its presentation by natural identities.

Example 1.3. Given a monad $\mathbf{M} = (M, \eta, \mu)$ on $\mathcal{C}$, its Eilenberg-Moore category $\mathbf{M}$**-alg** is a class of M-algebras induced by two natural identities:

$$
\begin{tikzcd}
M^2 \arrow[r, "\mu"] \arrow[dr, "Mq_0"'] & M \arrow[r, "q_0"] & M_1 \\
& MM_1 \arrow[r, "q_1"] & M_2 \,,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Id} \arrow[r, "\mathrm{id}"] \arrow[d, "\eta"] & M_0 \\
M \arrow[r, "q_0"] & M_1
\end{tikzcd}
$$

Therefore

$$\mathbf{M}\text{-}\mathbf{alg} = \mathbf{Alg}\,(M, \{(q_0 \circ \eta, \mathrm{id}_{\mathrm{Id}}), (q_1 \circ Mq_0, q_0 \circ \mu)\}\,.$$

Example 1.4. Consider the power-set monad on $\mathcal{S}et$ defined by the power-set functor $\mathcal{P}$ and transformations $\eta : \mathrm{Id}_{\mathcal{S}et} \to \mathcal{P}$, $\mu\colon \mathcal{P}^2 \to \mathcal{P}$ given by the assignments $\eta_X(x) = \{x\}$, $\mu_X(\{X_i \mid i \in I\}) = \bigcup_{i\in I} X_i$. As a concrete instance of the previous case for power-set monad $(\mathcal{P}, \eta, \mu)$, we get the category of join-complete semilattices **JCSlat**. Hence, this class is presentable by a pair of naturally induced identities– –compare with the presentation by a proper class of equation arrows (see [3, Example 3.3] – we need equation arrows $e_X\colon F_3X \to E_X$ for every set X).

1.4. Conversion theorem.

Our aim is to prove that naturally induced classes and equational classes coincide. First we show that every single-based equational class is naturally induced. Then, conversely, we prove that every class induced by a single natural identity is equational. The crucial point of the proof is the local smallness of category $\mathcal{C}$.

Remark 1.5. In the proof, we use the copower functor:
Given an object $Q \in \mathcal{C}$, there is a functor $- \bullet Q\colon \mathcal{S}et \to \mathcal{C}$ which is left adjoint to $\hom(Q, -)\colon \mathcal{C} \to \mathcal{S}et$. It assigns to a set M the coproduct of M copies of Q (the "M-th" copower of Q) and, for a mapping $h\colon M \to N$, we define $h \bullet Q$ as the unique factorization of cocone $u_{h(m)}\colon Q \to \coprod\limits_{j\in N} Q, m \in M$, over a colimit cocone $u_m\colon Q \to \coprod\limits_{j\in M} Q$.

Then we get the adjunction $(\eta, \varepsilon)\colon (- \bullet Q) \dashv \hom(Q, -)\colon \mathcal{C} \to \mathcal{S}et$ where the unit morphism $\eta_X\colon X \to \hom(Q, X \bullet Q)$ for a set X and $x \in X$ is defined by $\eta_X(x) = u_x\colon Q \to X \bullet Q$, i.e., it is the x-labeled canonical injection into the coproduct. Moreover, for an object A of $\mathcal{C}$, the counit $\varepsilon\colon \hom(Q, A) \bullet Q \to A$ is defined as the unique factorization of a cocone $\{\phi \mid \phi\colon Q \to A\}$ over the colimit.

Lemma 1.6. *Every single-based equational class is a naturally single-induced class.*

Proof. Let $\mathcal{S}$ be a single-based equational class of F-algebras defined by an equation arrow e where e is a regular epimorphism $F_nX \to E$ such that (E, e) is a coequalizer of $\phi_0, \psi_0\colon\ Q \rightrightarrows F_n X$. We define a mapping $\theta_{\phi,A}\colon\ \hom(X, A) \to \hom(Q, F_n A)$. For every $f\colon X \to A$ let $\theta_{\phi,A}(f) = F_n f \circ \phi_0\colon Q \to F_n A$. Now let

$$G = (- \bullet Q) \circ \hom(X, -)\,,$$

$$\phi_A = \widetilde{\theta_{\phi,A}}\colon\ \hom(X, A) \bullet Q \to F_n A\,.$$

Clearly, ϕ_A is a component of a natural transformation $\phi\colon G \to F_n$. Observe that, for every $f\colon X \to A$,

$$\phi_A \circ u_f = \theta_{\phi,A}(f) = F_n f \circ \phi_0\,.$$

Analogously, we define natural transformations $\theta_{\psi,-}\colon G \to F_n$ and $\psi\colon G \to F_n$ satisfying $\psi_A \circ u_f = F_n f \circ \psi_0$. Now we have the functor G and G-identity (ϕ, ψ). It remains to show that it induces exactly the equational class $\mathcal{S}$.

Let (A, α) satisfy the equation arrow e. Then, for every $f\colon X \to A$, there is an $h\colon E \to A$ such that $\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e$. Then we have

$$\begin{aligned}\epsilon_{n,(A,\alpha)} \circ \phi_A \circ u_f &= \epsilon_{n,(A,\alpha)} \circ F_n f \circ \phi_0\\ &= h \circ e \circ \phi_0 = h \circ e \circ \psi_0\end{aligned}$$

and, by symmetry, we get $\epsilon_{n,(A,\alpha)} \circ \phi_A \circ u_f = \epsilon_{n,(A,\alpha)} \circ \psi_A \circ u_f$. Since f was chosen arbitrarily and the injections u_f form a colimit cocone, we have $\epsilon_{n,(A,\alpha)} \circ \phi_A = \epsilon_{n,(A,\alpha)} \circ \psi_A$, i.e., (A, α) satisfies the G-identity (ϕ, ψ).

Now let (B, β) be an F-algebra in the class induced by the G-identity (ϕ, ψ). Let $g : X \to B$ be a morphism in $\mathcal{C}$. Then we have

$$\begin{aligned}\epsilon_{n,(B,\beta)} \circ F_n g \circ \phi_0 &= \epsilon_{n,(B,\beta)} \circ \phi_B \circ u_g\\ &= \epsilon_{n,(B,\beta)} \circ \psi_B \circ u_g\end{aligned}$$

and, again by symmetry, we get $\epsilon_{n,(B,\beta)} \circ F_n g \circ \phi_0 = \epsilon_{n,(B,\beta)} \circ F_n g \circ \psi_0$, hence $\epsilon_{n,(B,\beta)} \circ F_n g$ coequalizes the pair (ϕ_0, ψ_0) and there is a unique $h : E \to B$ such that $\epsilon_{n,(B,\beta)} \circ F_n g = h \circ e$. Thus (B, β) satisfies the equation arrow e. □

Lemma 1.7. *Every naturally single-induced class is equational.*

Proof. Let G be a $\mathcal{C}$-endofunctor. Let $\mathcal{N}$ be a class induced by a G-identity (ϕ, ψ). Due to Remark 1.4 we may assume that ϕ and ψ have the same arity, say n. Therefore, both are the natural transformations $G \to F_n$. Let (E, e) be the coequalizer of ϕ and ψ. Then, for every object X of $\mathcal{C}$, we have a morphism $e_X : F_n X \to EX$. Let $\mathcal{E} = \{e_X \mid X \in \mathrm{Ob}\,\mathcal{C}\}$. We will prove $\mathcal{N} = \mathbf{Alg}\,(F, \mathcal{E})$.

Let (A, α) satisfy (ϕ, ψ). Then, for every $X \in \mathrm{Ob}\,\mathcal{C}$ and $f\colon X \to A$, we have

$$\begin{aligned}\epsilon_{n,(A,\alpha)} \circ F_n f \circ \phi_X &= \epsilon_{n,(A,\alpha)} \circ \phi_A \circ G f\\ &= \epsilon_{n,(A,\alpha)} \circ \psi_A \circ G f\\ &= \epsilon_{n,(A,\alpha)} \circ F_n f \circ \psi_X\,.\end{aligned}$$

Therefore, we have a coequalizing morphism $\epsilon_{n,(A,\alpha)} \circ F_n f$ for (ϕ_X, ψ_X). Since the colimits of functors are calculated componentwise, e_X is a coequalizer of (ϕ_X, ψ_X), which means that there is a unique $h\colon EX \to A$ such that $\epsilon_{n,(A,\alpha)} \circ F_n f = h \circ e_X$.

Given an F-algebra (B, β) satisfying all equation arrows from $\mathcal{E}$, it satisfies the arrow $e_B\colon F_n B \to EB$ and there is $h\colon EB \to B$ (chosen for $\mathrm{id}_B\colon B \to B$) such that $\epsilon_{n,(B,\beta)} = h \circ e_B$. Thus, the property is satisfied since e_B coequalizes the pair (ϕ_B, ψ_B). □

Theorem 1.8. *Let F be an endofunctor on a cocomplete category $\mathcal{C}$. Then the equational classes of F-algebras coincide with the naturally induced classes of F-algebras.*

Proof. Every equational class $\mathcal{S}$ is a (possibly large) intersection of single-based ones and these are, by the Lemma 1.6, naturally induced, more precisely single-induced. Hence, $\mathcal{S}$ is naturally induced by the class of corresponding natural identities. Conversely, the naturally induced class $\mathcal{N}$ is a (possibly large) intersection of the ones induced by a single natural identity, which, due to Lemma 1.7, are equational classes induced by a class of equation arrows. The union of these classes defines the class of all equation arrows defining the class $\mathcal{N}$ as an equational class. □

Definition 1.5. A class of algebras induced by equations or natural identities is called a *variety.*

2. Free algebras in a variety

Our aim is to answer the question of existence of free algebras in varieties. First we recall well-known results involving free algebras, which will be used to solve this problem.

2.1. **Free algebras and monads.**

We will work with $\mathcal{C}$-endofunctors preserving the colimits of λ-chains where λ is an infinite limit ordinal – let the class containing these functors be denoted by $\mathbf{End}_\lambda\mathcal{C}$. Since colimits commute with colimits, we get the following property (see also [7, 2.4]).

Proposition 2.1. *The class $\mathbf{End}_\lambda\mathcal{C}$ is closed under colimits and compositions.*

Definition 2.1. Let G be an endofunctor on $\mathcal{C}$. The natural G-identity is called *accessible* if G preserves the colimits of λ-chains for some infinite limit ordinal λ.

Definition 2.2. A functor F admitting free F-algebras is called a *varietor.*

Let F preserve the colimits of λ-chains. Then, as shown in [1], F is a varietor. Since F preserves the colimits of λ-chains, FF_λ is a colimit of chain $\{FF_n \mid n < \lambda\}$. Hence, one can see that $w_{\lambda,\lambda+1}$ is an isomorphism. In such a case we say that *the free F-algebra construction stops after λ steps.* If we set $\upsilon = \operatorname*{colim}_{n<\lambda} q_n$, on every object A, we get the free F-algebra

$$\mathcal{V}_F(A) = (F_\lambda A, \upsilon_A)\,.$$

If necessary, we write the name of functor F in the superscript: $\upsilon = \upsilon^F$.

This construction gives rise to the functor $\mathcal{V}_F : \mathcal{C} \to \mathbf{Alg}F$, $\mathcal{V}_F = (F_\lambda, \upsilon)$ together with the transformation $\epsilon_\lambda : \mathcal{V}_F\mathcal{Z}_F \to \mathrm{Id}_{\mathbf{Alg}\,F}$, $\epsilon_{\lambda,(A,\alpha)} : (F_\lambda A, \upsilon_A) \to (A, \alpha)$. Hence, we have obtained the free functor $\mathcal{V}_F$ and adjunction $\mathcal{V}_F \dashv \mathcal{Z}_F$, where the unit and counit are ι_λ and ϵ_λ, respectively.

It is a well known fact, that this adjunction yields the free monad over a functor F – see [2, Theorem 20.56]. Hence the free monad over F is

$$\mathcal{M}(F) = (F_\lambda, \eta^F, \mu^F)$$

where $\eta^F = \iota_\lambda^F$ and $\mu^F = \mathcal{Z}_F \epsilon_\lambda \mathcal{V}_F$ and the universal morphism for F is $y_\lambda^F : F \to F_\lambda$. A more detailed approach to the theory of monads can be found in [2], [6] and [7].

Now we will use another functor G in $\mathbf{End}_\lambda \mathcal{C}$ and work with its algebras. We point out the important consequences of the discussion above:

Proposition 2.2. *Let there be a transformation $\rho\colon G \to F_\lambda$. Then there is a transformation $\sigma\colon G_\lambda \to F_\lambda$, subject to the conditions:*

(1) *$\sigma = \overline{\rho}$ is given by the freeness of $\mathcal{M}(F)$ as the unique monad transformation $\mathcal{M}(G) \to \mathcal{M}(F)$ corresponding to $\rho\colon G \to F_\lambda$; thus*
$$\sigma \circ y_\lambda^G = \rho\,.$$

(2) *$\sigma_A = \widetilde{\eta_A^F}$ is given by the adjunction $\mathcal{V}_G \dashv \mathcal{Z}_G$ as the unique G-algebra morphism $\mathcal{V}_G(A) \to P(A)$ corresponding to $\eta_A^F\colon A \to F_\lambda A = \mathcal{Z}_G P(A)$ where $P\colon \mathcal{C} \to \mathbf{Alg}\, G$ is the functor assigning to an object A an algebra $(F_\lambda A, \beta_A)$ and $\beta_A = (\mu^F \circ \rho F_\lambda)_A$. Hence*
$$\sigma \circ \iota_\lambda^G = \eta^F\,.$$

(3) *For $k \leq \lambda$, let $\epsilon_{k,P}^G\colon G_k F_\lambda \to F_\lambda$ be the obvious transformation with the components $\epsilon_{\lambda,P(A)}^G$. Then the following equation holds:*
$$\sigma = \epsilon_{\lambda,P}^G \circ G_\lambda \eta^F\,.$$

We will show that σ defined above from the transformation $\rho\colon G \to F_\lambda$ can be gained via the colimit construction, which will be useful later on.

Definition 2.3. For all $k \in \mathbb{N}$, we define the transformations $\rho_k\colon G_k \to F_\lambda$ inductively: $\rho_1 = [\rho, \eta^F]$, $\rho_{k+1} = [\mu^F \circ \rho F_\lambda \circ G\rho_k, \eta^F]$

Lemma 2.3. *For every $j < k \in \mathbb{N}$, $\rho_k \circ w_{j,k}^G = \rho_j$.*

Proof. For every natural $j < k$ we have

$$\begin{aligned}
\rho_k \circ w_{j,k}^G &= [\mu^F \circ \rho F_\lambda \circ G\rho_k, \eta^F] \circ w_{j,k}^G \\
&= [\mu^F \circ \rho F_\lambda \circ G\rho_{k-1} \circ G w_{j-1,k-1}^G, \eta^F] \\
&= [\mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w_{j-1,k-1}^G), \eta^F]
\end{aligned}$$

If $j = 1$, then

$$\begin{aligned}
\mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w_{j-1,k-1}^G) &= \mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w_{0,k-1}^G) \\
&= \mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ \iota_{k-1}^G) \\
&= \mu^F \circ \rho F_\lambda \circ G\eta^F \\
&= \mu^F \circ F_\lambda \eta^F \circ \rho = \rho
\end{aligned}$$

hence the property holds for $j = 1$ and every $k > 1$.

Now let $1 < j < k$ and assume the validity of $\rho_{k-1} \circ w^G_{j-1,k-1} = \rho_{j-1}$. Then we have: $\mu^F \circ \rho F_\lambda \circ G(\rho_{k-1} \circ w^G_{j-1,k-1}) = \mu^F \circ \rho F_\lambda \circ G\rho_{j-1}$ hence $\rho_k \circ w^G_{j,k} = \rho_j$ and the proof is complete. □

Since the transformations ρ_k form a compatible cocone for $w^G_{j,k}$, we can extend it to the infinite limit case:

Definition 2.4. For a limit ordinal l let ρ_l will be defined as $\operatorname*{colim}_{k<l} \rho_k$.

Since the w-compatibility clearly holds, our construction extends to the ordinal chain. The isolated step is given by an analogy to the finite-instance definition. The definition yields the property for every $k \leq \lambda$:

$$\rho_k \circ \iota^G_k = \iota^F_\lambda \,. \tag{6}$$

To prove that this chain of transformations converges to σ, we will show that its colimit is a G-algebra morphism.

Lemma 2.4. *The transformation $\rho_\lambda : G_\lambda \to F_\lambda$ underlies the natural transformation $\mathcal{V}_G(A) \to P(A)$ of the functor $\mathcal{C} \to \mathbf{Alg}\, G$ where P is the functor used in Proposition 2.2.*

Proof. What we need to prove is that, for an object A in $\mathcal{C}$, the morphism $\rho_{\lambda,A} \colon G_\lambda A \to F_\lambda A$ is a G-algebra morphism $(G_\lambda A, \upsilon^G_A) \to (F_\lambda A, \beta_A)$. It suffices to prove the equality of natural transformations: $\beta \circ G\rho_\lambda = \rho_\lambda \circ \upsilon^G$. Let $k < \lambda$, then we have

$$\begin{aligned}
\rho_\lambda \circ \upsilon^G \circ Gw^G_{k,\lambda} &= \rho_\lambda \circ w^G_{k+1,\lambda} \circ q^G_k \\
&= \rho_{k+1} \circ q^G_k \\
&= \mu^F \circ \rho F_\lambda \circ G\rho_k \\
&= \beta \circ G\rho_k \\
&= \beta \circ G\rho_\lambda \circ Gw^G_{k,\lambda}
\end{aligned}$$

and, since G preserves the colimits of λ-chains, $\{Gw^G_{k,l} \mid k \leq l < \lambda\}$ is the colimit cocone. From the uniqueness of the factorization over the colimit, we get the required equality. □

Lemma 2.5. *The transformations $\rho_\lambda, \sigma \colon G_\lambda \to F_\lambda$ coincide.*

Proof. Let a $\mathcal{C}$-object A be given. Then, due to the previous lemma, $\rho_{\lambda,A}$ is a G-algebra morphism $\rho_{\lambda,A} \colon (G_\lambda A, \upsilon^G_A) \to (F_\lambda A, \beta_A)$ and, by (6), we have $\rho_\lambda \circ \iota^G_\lambda = \iota^F_\lambda = \eta^F$, hence, by the uniqueness of the factorization of $\eta^F_A \colon A \to \mathcal{Z}_G(F_\lambda A, \beta_A)$ over $\eta^G_A = \iota^G_\lambda$, we get $\rho_{\lambda,A} = \widetilde{\eta^F_A}$ which, due to Proposition 2.2, is equal to σ_A. □

Lemma 2.6. *Let $\phi, \psi : G \to F_\lambda$ be natural transformations. Then, for every $k \leq \lambda$, we have the algebraic equivalence:*

$$(\phi, \psi) \approx (\phi_k, \psi_k)$$

where ϕ_k, ψ_k are derived from ϕ, ψ, respectively, as in Definition 2.3, 2.4.

Proof. Let (A,α) be an F-algebra. Then, for every $k \le \lambda$,

$$\epsilon_{\lambda,(A,\alpha)} \circ \phi_{k,A} \circ \iota_k^G = \epsilon_{\lambda,(A,\alpha)} \circ \psi_{k,A} \circ \iota_k^G \,, \tag{$*$}$$

since $\epsilon_{\lambda,(A,\alpha)} \circ \phi_{k,A} \circ \iota_k^G = \epsilon_{\lambda,(A,\alpha)} \circ \eta^F = \epsilon_{\lambda,(A,\alpha)} \circ w_{0,\lambda,A} = \epsilon_{0,(A,\alpha)} = \mathrm{id}$.

Let $(A,\alpha) \models (\phi,\psi)$, i.e.,

$$\epsilon_{\lambda,(A,\alpha)} \circ \phi_A = \epsilon_{\lambda,(A,\alpha)} \circ \psi_A \,. \tag{h}$$

We will show by induction that then $(A,\alpha) \models (\phi_k,\psi_k)$ for every $k \le \lambda$. In each step, we shorten the computations using the $(\phi-\psi)$-symmetry of expressions.

$\underline{k=1}$: Since $\epsilon_{\lambda,(A,\alpha)} \circ \phi_{1,A} \circ q^G_{0,A} = \epsilon_{\lambda,(A,\alpha)} \circ \phi_A$, from (h) and symmetry, we get $\epsilon_{\lambda,(A,\alpha)} \circ \phi_{1,A} \circ q^G_{0,A} = \epsilon_{\lambda,(A,\alpha)} \circ \psi_{1,A} \circ q^G_{0,A}$, which, together with $(*)$, yields $\epsilon_{\lambda,(A,\alpha)} \circ \phi_{1,A} = \epsilon_{\lambda,(A,\alpha)} \circ \psi_{1,A}$.

$\underline{1 < k < \lambda,\ k \text{ isolated}}$: Assume the hypothesis

$$\epsilon_{\lambda,(A,\alpha)} \circ \phi_{k,A} = \epsilon_{\lambda,(A,\alpha)} \circ \psi_{k,A} \tag{$\mathrm{h_k}$}$$

Recall, that $\epsilon_{\lambda,(A,\alpha)} \colon F_\lambda A \to A$ is a morphism $(F_\lambda A, \mu^F) \to (A, \epsilon_{\lambda,(A,\alpha)})$, i.e., $\epsilon_{\lambda,(A,\alpha)} \circ \mu^F = \epsilon_{\lambda,(A,\alpha)} \circ F_\lambda \epsilon_{\lambda,(A,\alpha)}$. Then we have:

$$\begin{aligned}
\epsilon_{\lambda,(A,\alpha)} \circ \phi_{k+1,A} \circ q^G_{k,A} &= \epsilon_{\lambda,(A,\alpha)} \circ \mu^F_A \circ \phi_{F_\lambda A} \circ G\phi_{k,A} \\
&= \epsilon_{\lambda,(A,\alpha)} \circ F_\lambda \epsilon_{\lambda,(A,\alpha)} \circ F_\lambda \phi_{k,A} \circ \phi_{G_k A} \\
&\overset{(h_k)}{=} \epsilon_{\lambda,(A,\alpha)} \circ F_\lambda \epsilon_{\lambda,(A,\alpha)} \circ F_\lambda \psi_{k,A} \circ \phi_{G_k A} \\
&= \epsilon_{\lambda,(A,\alpha)} \circ F_\lambda \epsilon_{\lambda,(A,\alpha)} \circ \phi_{F_\lambda A} \circ G\psi_{k,A} \\
&= \epsilon_{\lambda,(A,\alpha)} \circ \phi_A \circ G\epsilon_{\lambda,(A,\alpha)} \circ G\psi_{k,A} \\
&\overset{(h)}{=} \epsilon_{\lambda,(A,\alpha)} \circ \psi_A \circ G\epsilon_{\lambda,(A,\alpha)} \circ G\psi_{k,A} \\
&\overset{\text{symmetry}}{=} \epsilon_{\lambda,(A,\alpha)} \circ \psi_{k+1,A} \circ q^G_{k,A}
\end{aligned}$$

and, together with $(*)$, we get $\epsilon_{\lambda,(A,\alpha)} \circ \phi_{k+1,A} = \epsilon_{\lambda,(A,\alpha)} \circ \psi_{k+1,A}$.

$\underline{l \le \lambda,\ l \text{ limit}}$: Assume $(\mathrm{h_k})$ for every $k < l$. Then, from the uniqueness of factorization of the cocone $\epsilon_{\lambda,(A,\alpha)} \circ \phi_{k,A} \colon G_k \to A$ over the colimit cocone $w^G_{k,l,A}$, we get $\epsilon_{\lambda,(A,\alpha)} \circ \phi_{l,A} = \epsilon_{\lambda,(A,\alpha)} \circ \psi_{l,A}$.

We have proved for every k:

$$(A,\alpha) \models (\phi,\psi) \ \Rightarrow\ (A,\alpha) \models (\phi_k,\psi_k)\,.$$

However, $(A,\alpha) \models (\phi_k,\psi_k)$ easily implies $(A,\alpha) \models (\phi,\psi)$ since $\phi = \phi_1 \circ q^G_0 = \phi_k \circ w^G_{1,k} \circ q^G_0 = \phi_k \circ y^G_k$. □

2.2. **Algebras for a diagram of monads.**

This section refers to the paper [7, Chapter VIII] by G. M. Kelly, which deals with colimits of monads. It is well known (see e.g. [2, Corollary 20.57]) that for every varietor F the categories of its algebras and algebras for a monad $\mathcal{M}(F)$ are concretely isomorphic via the comparison functor.

Let $D \colon \mathcal{D} \to \mathbf{Monad}\ \mathcal{C}$ be a diagram and $D(x) = (M_x, \eta^x, \mu^x)$ for every object $x \in \mathcal{D}$. Consider the category D-**alg** of *algebras for a diagram D of monads* whose objects are collections of $\mathcal{C}$-morphisms $\{\alpha_x \colon M_x A \to A \mid x \in \mathcal{D}\}$, where α_x is in $D(x)$-**alg** for every object $x \in \mathcal{D}$ and, for each $f \colon x \to y$ in $\mathcal{D}$, the D-compatibility condition $\alpha_y \circ D(f)_A = \alpha_x$ is satisfied. The morphisms in D-**alg** are the morphisms of algebras for each x, i.e., $\phi \colon (A, \alpha) \to (B, \beta)$ is a morphism if $\phi \circ \alpha_x = \beta_x \circ M_x(\phi)$ for every x. If there is a monad $\mathbf{K}$ such that $\mathbf{K}$-**alg** $\cong_{\mathcal{C}}$ D-**alg**, then this monad is called *algebraic colimit of D*.

Kelly asked about the existence of this algebraic colimit, which came out to be equivalent to the existence of the free objects in D-**alg**. He proved this existence in his Theorem 27.1 in [7] under the general assumptions of existence of suitable factorization systems and some smallness requirements for the monads. Using the trivial factorization system $(\mathrm{Iso}, \mathrm{Mor})$ and preservation of colimits of λ-chains, we get this theorem in the following form:

Theorem 2.7. *Let the underlying functor of each $D(x)$ preserve the colimits of λ-chains. Then D-**alg** has free objects.*

This theorem will be used to prove the existence of free object in a variety induced by accessible identities. Let F be a functor in $\mathbf{End}_\kappa \mathcal{C}$ for some infinite limit ordinal κ and consider the variety of F-algebras induced by a set of accessible natural identities. Since the free F-algebra construction stops after κ steps, we may consider the arity of each natural term to be less or equal to κ. Then, due to Remark 1.4, the set of natural identities can be replaced by a single identity (ϕ, ψ). Its domain, denoted by G, is the coproduct of domains of single identities, hence, due to 2.1, it preserves colimits of ν-chains for some large enough limit ordinal ν. Let $\lambda = \max\{\kappa, \nu\}$, then $F, G \in \mathbf{End}_\lambda \mathcal{C}$. Hence the arity of (ϕ, ψ) can be set to λ.

Let $\mathcal{D}$ be a category consisting of two objects $0, 1$, their identities and two more morphisms $f, g \colon 0 \to 1$. Let $D \colon \mathcal{D} \to \mathbf{Monad}\ \mathcal{C}$ be a diagram such that $D(0) = \mathcal{M}(G)$, $D(1) = \mathcal{M}(F)$, $D(f) = \overline{\phi}$, $D(g) = \overline{\psi}$, where $\overline{\phi}, \overline{\psi}$ are the monad transformations given by Proposition 2.2. We will prove the concrete equivalence of $\mathbf{Alg}\,(F, (\phi, \psi))$ and D-**alg**.

Lemma 2.8. *For the λ-ary G-identity (ϕ, ψ) and diagram D defined above, we have:*

$$\mathbf{Alg}\,(F, (\phi, \psi)) \cong_{\mathcal{C}} D\text{-}\mathbf{alg}\,.$$

Proof. Consider the comparison functor $I \colon \mathbf{Alg}\, F \to \mathcal{M}(F)\text{-}\mathbf{alg}$ assigning to an F-algebra (A, α) the F_λ-algebra $(A, \epsilon_{\lambda,(A,\alpha)})$. Then due to Lemma 2.6

$$(A, \alpha) \models (\phi, \psi) \quad \Leftrightarrow \quad (A, \alpha) \models (\phi_\lambda, \psi_\lambda) \quad \Leftrightarrow \quad \epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda = \epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda\,.$$

For every F_λ-algebra (A, β), $\epsilon^{F_\lambda}_{1,(A,\beta)} \circ q_0^{F_\lambda} = \beta$ holds , hence

$$\epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda = \epsilon_{\lambda,(A,\alpha)} \circ \phi_\lambda \quad \Leftrightarrow \quad I(A, \alpha) \models (\phi^*, \psi^*)$$

where $\phi^* = q_0^{F_\lambda} \circ \phi_\lambda$. Since I is the isomorphism with an inverse given by $(A, \beta) \mapsto (A, \beta \circ y_{\lambda,A})$, we get

$$\mathbf{Alg}\,(F, (\phi, \psi)) \cong_{\mathcal{C}} \mathcal{M}(F)\text{-}\mathbf{alg} \cap \mathbf{Alg}\,\big(F_\lambda, (\phi^*, \psi^*)\big)\,.$$

Due to Proposition 2.2, we have $\phi_\lambda = \overline{\phi}$, which is a monad transformation (and analogously for ψ), hence we get $\mathcal{M}(F)$-**alg** $\cap$ **Alg** $(F_\lambda, (\phi^*, \psi^*))$ to be concretely isomorphic to D-**alg**. $\square$

Now we can use Kelly's theorem to conclude our investigation:

Theorem 2.9. *Let F preserve the colimits of λ-chains for some limit ordinal λ. Then the free algebra exists in every variety induced by a set of accessible identities.*

To express the consequence for the varieties presented by equation arrows, recall the notion of presentability of an object (see [4]):

Definition 2.5. Let λ be a regular cardinal. An object A of a category is called *λ-presentable* provided that its hom-functor $\hom(A, -)$ preserves λ-directed colimits. An object is called *presentable* if it is λ-presentable for some λ.

Theorem 2.10. *Let F preserve the colimits of λ-chains for some limit ordinal λ. Then the free algebra exists in every variety induced by a set of equation arrows with presentable variable-objects.*

Proof. As shown in the proof of Lemma 1.6, an equation arrow $e\colon F_nX \to E$ converts to a natural identity with the domain $G = (- \bullet Q) \circ \hom(X, -)$ for some $Q \in \mathrm{Ob}\,\mathcal{C}$. If the variable-object X is presentable, $\hom(X, -)$ preserves κ-directed colimits for some κ and, since $(- \bullet Q)$ is left adjoint, G preserves κ-directed colimits, too. Therefore the colimits of κ-chains are preserved and due to Theorem 2.9 the corresponding variety has free objects. The rest is obvious. $\square$

Acknowledgement. The author acknowledges partial support from Ministry of Education of the Czech Republic, research plan no. MSM0021630518.

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