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YANG-MILLS BAR CONNECTIONS OVER COMPACT KÄHLER MANIFOLDS

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Abstract. In this note we introduce a Yang-Mills bar equation on complex vector bundles E provided with a Hermitian metric over compact Hermitian manifolds. According to the Koszul-Malgrange criterion any holomorphic structure on E can be seen as a solution to this equation. We show the existence of a non-trivial solution to this equation over compact Kähler manifolds as well as a short time existence of a related negative Yang-Mills bar gradient flow. We also show a rigidity of holomorphic connections among a class of Yang-Mills bar connections over compact Käahler manifolds of positive Ricci curvature.

1. Introduction

Let M^{2n} be a compact Hermitian manifold of real dimension $2n$ and E be a complex vector bundle over M^{2n}. The following Koszul-Malgrange criterion [6], see also [2, 2.1.53, 2.1.54], establishes the equivalence between the existence of a holomorphic structure on E and a partial flatness of E.

Koszul-Malgrange criterion. *A complex vector bundle E over a complex manifold M^{2n} carries a holomorphic structure, if and only if there is a connection A on E such that the $(0,2)$-component $F_A^{0,2}$ of the curvature F_A of A vanishes.*

Thus we call a connection A satisfying the Koszul-Malgrange criterion a holomorphic connection. It is well-known (see e.g. [2]) that we can replace the connection A in the Kozsul-Malgrange criterion by a unitary connection A for any given choice of a compatible (Hermitian) metric h on E.

We introduce in Section 2 (see (2.5.1) and (2.5.2)) a Yang-Mills bar equation as the Euler-Lagrange equation for the Yang-Mills bar functional which is the square of the L_2-norm of the $(0,2)$-component $F_A^{0,2}$ of a unitary connection A on (E,h). Solutions of a Yang-Mills bar equation are called Yang-Mills bar connections. The Yang-Mills bar equation has an advantage over the equation for a holomorphic connection, because the later one is overdetermined if the complex dimension of the bundle is greater or equal to 2 and $n \geq 4$, and the first one is elliptic modulo a

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degeneracy which is formally generated by an action of the complex gauge group of the complex vector bundle E (the degeneracy is formal generated since the action of this group on the "small" space does not preserve the Yang-Mills bar functional, see 2.7.b and Remark 5.13). Thus we hope that by using this equation we will be able to find useful sufficient conditions under which a complex vector bundle carries a holomorphic structure. Appropriate sufficient conditions for the existence of a holomorphic structure on complex vector bundles over projective algebraic manifolds could be a key step in solving the Hodge conjecture, if the conjecture is correct. A particular result in this direction is our Theorem 4.25 which states that an almost holomorphic connection over a compact Kähler manifold of positive Ricci curvature is holomorphic, in particular any Yang-Mills bar connection on a 4-dimensional compact Kähler manifold of positive Ricci curvature is holomorphic.

In Section 2 after introducing the Yang-Mills equation we also discuss the symmetry of this equation in 2.7. In Section 3 we give a proof of the Hodge-Kähler identities for general unitary connections over Kähler manifolds and show the existence of non-trivial Yang-Mills bar connections. In Section 4 we derive a Bochner-Weitzenböck type identity on compact Kähler manifolds and prove Theorem 4.25. In Section 5 we introduce the notion of affine integrability condition, a negative Yang-Mills bar gradient flow and find an affine integrability condition for this flow (Theorem 5.9). Unlike previously known cases for weakly parabolic equations (Ricci flow, Yang-Mills flow), our affine integrability is not derived from an action of a group, which preserves the Lagrangian on the space, where our flow is considered (see 2.7.b and Remark 5.13i). The automorphism group of the Yang-Mill bar equation gives us only "half" of the integrability condition. In particular, the DeTurck approach to weakly parabolic equations seems inapplicable to our flow. In the last Section 6 we prove the short time existence, uniqueness and smoothness of a solution of an evolution equation with affine integrability condition, slightly extending a Hamilton's result.

Let $(V, \langle , \rangle)$ be a Euclidean space. Denote by $V_{\mathbb{C}}$ its complexification. Then $\langle , \rangle$ extends uniquely to a complex bilinear form $\langle , \rangle_{\mathbb{C}} \colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$. Denote by $(v, w) := \langle v, \bar{w} \rangle_{\mathbb{C}}$ the associated Hermitian form on $V_{\mathbb{C}}$ and by $\langle v, w \rangle = \mathrm{Re}(v, w)$ the Euclidean metric on the space $(V_{\mathbb{C}}) \otimes \mathbb{R}$. We note that the restriction of this metric to V coincides with the original metric $\langle , \rangle$. Conversely any Hermitian metric (J-invariant Euclidean metric) on a complex space (V, J) considered as a complexification of a real vector space V_0 is obtained in this way.

In this note we define by the same $(,)$ (and resp. $\langle , \rangle$) the Hermitian form (resp. the Euclidean metric) extended in the above way from any vector bundle $(E, \langle , \rangle)$ provided with a fiber-wise Euclidean metric $\langle , \rangle$ to its complexification $E_{\mathbb{C}}$ (resp. considered as a real space). If A is a connection on $(E, \langle , \rangle)$ then A can be extended to a unitary connection also denoted by A on the complexification $E_{\mathbb{C}}$ with that extended metric by setting $d_A(\sqrt{-1}\phi) := \sqrt{-1} d_A(\phi)$.

2. Yang-Mills bar equation

Now let A be a connection on a complex vector bundle (E, J) over a Hermitian manifold M^{2n}. Denote by $\Omega^{p,q}(E)$ the space of E-valued (p, q)-forms on M^{2n}: $\Omega^{p,q}(E) = \Omega^{p,q}(M) \otimes_{\mathbb{C}} E$. We have the decomposition

$$d_A = \partial_A \oplus \bar{\partial}_A \colon \Omega(E) \to \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)\,.$$

In general we have the inclusion

$$d_A(\Omega^{p,q}(E)) \subset \Omega^{p+1,q}(E) \oplus \Omega^{p,q+1}(E)\,,$$

since for $\psi \in \Omega^0(E)$ and $\phi \in \Omega^{p,q}(M^{2n})$ we have

$$d_A(\psi \otimes \phi) = d_A(\psi) \otimes \phi + \psi \otimes d\phi \in \Omega^{p+1,q}(E) \oplus \Omega^{p,q+1}(E)\,.$$

(The operator d_A is well defined on $\Omega^{p,q}(E)$, since $d_A(J\psi) = Jd_A(\psi)$.) For $\phi \in \Omega^{p,q}(E)$ we denote by $\partial_A(\phi)$ the projection of $d_A(\phi)$ on the first factor and by $\bar{\partial}_A(\phi)$ the projection on the second factor w.r.t. the above decomposition.

We note that the curvature $F_A \in \Omega^2(\operatorname{End}_J E)$ of A can be considered as an element in $\Omega^2_{\mathbb{C}}(\operatorname{End}_J(E))$.

Let (E, h) be a Hermitian vector bundle, i.e. a complex vector bundle (E, J) provided with a Hermitian metric h but E need not to be holomorphic. There is a natural (Killing) metric on the space u_E of skew-Hermitian endomorphisms of E, defined by $\langle\theta_1, \theta_2\rangle = -\operatorname{Re}\operatorname{Tr}(\theta_1 \cdot \theta_2)$. We can also write $\operatorname{End}_J E = u_E \oplus \sqrt{-1}u_E$. Thus the metric h extends to a positive definite bilinear form on $\operatorname{End}_J E$ (defined by $\langle\theta_1, \theta_2\rangle = \operatorname{Re}\operatorname{Tr}(\theta_1 \cdot \theta_2^*)$). Here θ^* is the conjugate transpose of θ, the adjoint of θ w.r.t. the unitary metric h. We note that this metric is invariant under the original complex structure on $\operatorname{End}_J(E)$ induced by J which we denoted above by multiplication with $\sqrt{-1}$. Hence by the remark at the beginning of the section, this metric extends to a metric on the space $\Omega^k_{\mathbb{C}}(\operatorname{End}_J E)$ by combining the Killing metric with the Hermitian metric on M^{2n}. The decomposition $\Omega^k_{\mathbb{C}}(\operatorname{End}_J E) = \sum_{p+q=k} \Omega^{p,q}(\operatorname{End}_J E)$ is an orthogonal decomposition w.r.t. this metric.

If A is a unitary connection on (E, h), then $F_A \in \Omega^2(u_E) \subset \Omega^2(\operatorname{End}_J E)$. We also note that in the decomposition for the curvature of unitary connection A:

$$F_A = (F_A)^{2,0} + (F_A)^{1,1} + (F_A)^{0,2}$$

we have $(F_A)^{0,2} = -((F_A)^{2,0})^*$. The Kozsul-Malgrange criterion suggests us to consider the following Yang-Mills bar functional on the space of all unitary connections A on (E, h) over M^{2n}

$$\mathcal{YM}^b(A) = (1/2)\int_{M^{2n}} \|(F_A)^{0,2}\|\,.$$

It is easy to see that the functional $\mathcal{YM}^b$ is invariant under the gauge transformation of the Hermitian vector bundle (E, h). Let us derive the first variation formula for the Yang-Mills bar equation. First we extend the usual Hodge operator $*\colon \Omega^p(M^{2n}) \to \Omega^{2n-p}(M^{2n})$ to $\bar{*}\colon \Omega^p(\operatorname{End}_J E) \to \Omega^{2n-p}(\operatorname{End}_J E)$ defined as follows. We extend $\bar{*}\colon \Omega^p(\operatorname{End}_J E) \to \Omega^{2n-p}(\operatorname{End}_J E)$ so that for each

$\alpha \in \Omega^p_{\mathbb{C}}(\mathrm{End}_J\, E)$ and $\beta \in \Omega^p_{\mathbb{C}}(\mathrm{End}_J\, E)$ we have

$$\langle \alpha(x), \beta(x)\rangle = \left\langle \mathrm{vol}_x\, M^{2n}, \alpha(x) \wedge^{(,)} (\bar{*}\beta(x))\right\rangle . \tag{2.1}$$

Here $\wedge^{(,)}$ denotes the composition of the wedge product with the contraction of the coefficients in $\mathrm{End}_J\, E$ via the natural Hermitian form $(,)$ on $\mathrm{End}_J\, E$.

Next we note that A induces naturally a connection, also denoted by A, on the Hermitian vector bundle $\mathrm{End}_J\, E$ provided with the metric described above. It is known that the curvature F_A of this induced connection acts on the space $\Omega^0(\mathrm{End}_J\, E)$ as follows

$$F_A(\phi) = F_A \wedge \phi := [F_A, \phi]\, , \tag{2.2}$$

see e.g. [1, (2.7)]. (The wedge product of differential forms with coefficients in a Lie algebra bundle is the composition of the wedge product and the Lie bracket).

Now we define the operator $\bar\partial^*_A \colon \Omega^{p,q}(\mathrm{End}_J\, E) \to \Omega^{p,q-1}(\mathrm{End}_J\, E)$ as follows (see also [5, Chapter III, (2.19)], or [3, Chapter 1, §2], for the case that E is absent)

$$(\bar\partial^*_A)\beta^{p,q} := (-1)\bar{*}\bar\partial_A\bar{*}\beta^{p,q}\, . \tag{2.3}$$

Using the following identity for the formal adjoint d^*_A of d_A on an even dimensional manifold M^{2n} (see e.g. [1, (2.27)], for the real case, the complex case can be proved by the same way by using the Stocks formula locally):

$$(d^*_A)\beta = (-1)\bar{*}d_A\bar{*}\beta$$

and taking into account (2.3) which implies that $\bar\partial^*_A$ is the component with correct bi-degree of d^*_A, we conclude that $\bar\partial^*_A$ is the formal adjoint of $\bar\partial_A$. Now using the formula $(F_{A+ta})^{0,2} = (F_A)^{0,2} + t\bar\partial_A a^{0,1} + t^2 a^{0,1} \wedge a^{0,1}$ and taking into account (2.2) we get immediately

Lemma 2.4. *Let M^{2n} be a compact Hermitian manifold with (possibly empty) boundary. The first variation of the Yang-Mills bar functional is given by the formula*

$$\frac{d}{dt}_{|t=0}\mathcal{YM}^b(A+ta) = \int_{M^{2n}} \langle (\bar\partial_A)^* F^{0,2}_A, a\rangle + \int_{\partial M^{2n}} \langle \mathrm{vol}_x, a \wedge^{(,)} \bar{*}F^{0,2}_A\rangle\, .$$

We call a smooth unitary connection A a Yang-Mills bar connection, if it satisfies the following two conditions

$$(\bar\partial_A)^* F^{0,2}_A = 0\, , \tag{2.5.1}$$

$$(\bar{*}F^{0,2}_A)_{|\partial M^{2n}} = 0\, . \tag{2.5.2}$$

Let $\triangle^{\bar\partial}_A := \bar\partial_A(\bar\partial_A)^* + (\bar\partial_A)^*\bar\partial_A$. Using the Bianchi identity $\bar\partial_A F^{0,2}_A = 0$, which follows from the usual Bianchi identity, and using the equality $\langle \mathrm{vol}_x, a \wedge^{(,)} \bar{*}b\rangle = \langle \mathrm{vol}_x, b \wedge^{(,)} \bar{*}a\rangle$, we conclude that we can replace (2.5.1) in the system of two equations (2.5.1) and (2.5.2) by the following condition

$$\triangle^{\bar\partial}_A (F_A)^{0,2} = 0\, , \tag{2.6.1}$$

to get an equivalent system of equations.

2.7. Symmetries of the Yang-Mills bar equation. a) We can vary the Yang-Mills bar functional among all compatible Hermitian metrics h' on (E, J) in order to get an invariant of the complex vector bundle E. Let A_t be a family of unitary connections w.r.t. a compatible metric h_t. We note that we can write $h_t = g_t(h)$, where g_t is a (complex) gauge transformation of (E, J). Clearly $(g_t)^{-1}A_t$ is a unitary connection w.r.t. h (i.e. $d_{(g_t)^{-1}A_t}h = 0$). Now we have $F^{0,2}_{A_t} = \mathrm{Ad}_{g_t} F^{0,2}_{(g_t)^{-1}A_t}$. Moreover

$$\|F^{0,2}_{A_t}\|_{h_t} = \|Ad^{-1}_{g_t} F^{0,2}_{A_t}\|_h = \|F^{0,2}_{(g_t)^{-1}(A_t)}\|_h \,. \tag{2.7.1}$$

(We can get (2.7.1) easily by noticing that the inner products on $\mathrm{End}_J\, E$ induced by h and $g(h)$ satisfy the following relation

$$\langle A, B\rangle_{g(h)} = \sum_i \langle A(g(e_i)), B(g(e_i))\rangle_{g(h)} = \sum_i \langle Ad_{g^{-1}} A(e_i), Ad_{g^{-1}} B(e_i)\rangle_h$$

where e_i is an orthonormal basis in E w.r.t. h.)

Hence the infimum of the Yang-Mills bar functional is a constant which does not depend on the unitary metric h.

b) The linearization of the Yang-Mills bar equation is not elliptic because the equation is invariant under the gauge group $\mathcal{G}(E, h)$ of (E, h), see (2.7.1). The complexification of this group is the gauge group $\mathcal{G}(E)$. This complexified group acts also on the space $\mathcal{A}(E, h)$ of all unitary connections w.r.t. a fixed compatible metric h [2, (6.1.4)]. For $g \in \mathcal{G}(E)$ we denote by $\hat{g}$ the new (non-canonical) action of g on $\mathcal{A}(E, h)$ defined as follows

$$\bar\partial_{\hat{g}(A)} = g\bar\partial_A g^{-1} = \bar\partial_A - (\bar\partial_A g)g^{-1} \,,$$

$$\partial_{\hat{g}(A)} = \partial_A + [(\bar\partial_A g)g^{-1}]^* \,.$$

Though this action of $\hat{\mathcal{G}}(E)$ does not preserve the Yang-Mills bar functional, infinitesimally it fails to do it at a connection A only by a quadratic term in $F^{0,2}_A$ (see (5.3)).

3. Yang-Mills bar connections over compact Kähler manifolds

Suppose that A is a unitary connection on a Hermitian vector bundle E over a Kähler manifold M^{2n} with a Kähler form ω. As before denote by $\bar\partial^*_A$ the formal adjoint of $\bar\partial_A \colon \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ defined by (2.3), and by ∂^*_A the formal adjoint of $\partial_A \colon \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$ defined in the same way.

Denote by $\Lambda \colon \Omega^{p,q}(E) \to \Omega^{p-1,q-1}(E)$ the adjoint of the wedge multiplication by ω, an algebraic operator. The following Hodge-Kähler identities

$$\bar\partial^*_A = \sqrt{-1}[\partial_A, \Lambda] \,, \tag{3.1}$$

$$\partial^*_A = -\sqrt{-1}[\bar\partial_A, \Lambda] \,, \tag{3.2}$$

are well-known for the case of a holomorphic bundle E and A being a unitary holomorphic connection [5, Chapter III, (2.39)], or [3, Chapter 0, §7, Chapter 1, §2], where they are called the Hodge identities. These identities have been called

Kähler identities in [2, §6.1]. We give a proof of the general case here, assuming the validity of (3.1) and (3.2) for unitary holomorphic connections A.

Note that it suffices to prove these identities locally, so we can assume that the bundle is $U(n)$-trivial and $\partial_A = \partial + A^{1,0}$, where $A^{1,0} = \sum_{i=1}^n A_i dz_i$, $A_i \in \mathrm{End}_J(E)$. Similarly $\bar\partial_A = \bar\partial + A^{0,1}$ with $A^{1,0} = \sum_{i=1}^n -(A_i)^* d\bar z_i$. Here we define $\bar\partial$ and ∂ to be the $(1,0)$ and $(0,1)$ components of the unique unitary connection which is compatible with the trivial holomorphic structure.

Since the Hodge-Kähler identities are valid for $A = 0$, it is easy to see that (3.1) and (3.2) are equivalent to following algebraic identities

$$[A^{0,1}]^* = \sqrt{-1}[A^{1,0}, \Lambda]\,, \tag{3.3}$$

$$[A^{1,0}]^* = -\sqrt{-1}[A^{0,1}, \Lambda]\,. \tag{3.4}$$

In view of the Hermitian linearity of LHS of (3.3) and (3.4):

$$(\lambda A + \gamma B)^* = \bar\lambda A^* + \bar\gamma B^*$$

for $\lambda, \gamma \in \mathbb{C}$, and taking into account the unitary of A which implies $A^{1,0} = -(A^{0,1})^*$, it suffices to prove these identities for a $\mathbb{C}$-basic $\{A^{1,0} = e_{ij} dz_k, |\ 1 \le i, j \le \dim_{\mathbb{C}} E,\ 1 \le k \le \dim_{\mathbb{C}} M^{2n} = n\}$ of $(0,1)$-forms in $\Omega^{0,1}(\mathrm{End}_J E)$. Here e_{ij} is an elementary matrix in $\mathrm{End}_J(E)$. We also assume that the Kähler metric at a given point x is $\sum_i dz_i d\bar z_i$. Denote by i_k and $\bar i_k$ the adjoint of the multiplication operators $dz_k\wedge$ and $d\bar z_k\wedge$ correspondingly. Then we have

$$[A^{1,0}]^* = (e_{ji} i_k),\ [A^{0,1}]^* = -(e_{ij}\bar i_k)$$

$$\Lambda = -\frac{\sqrt{-1}}{2}\sum_{k=1}^n \bar i_k i_k\,.$$

Substituting these identities in LHS of (3.3) and (3.4) we conclude that (3.3) and (3.4) are equivalent to the following identities for all i, j, k

$$-(e_{ij}\bar i_k) = \sqrt{-1}\Big[e_{ij} dz_k, -\frac{\sqrt{-1}}{2}\sum_{k=1}^n \bar i_k i_k\Big]\,, \tag{3.5}$$

$$(e_{ji} i_k) = -\sqrt{-1}\Big[-e_{ji} d\bar z_k, -\frac{\sqrt{-1}}{2}\sum_{k=1}^n \bar i_k i_k\Big]\,, \tag{3.6}$$

In their turn (3.5) and (3.6) are immediate consequences of the following identities

$$-\bar i_k = \frac{1}{2}\Big[dz_k\wedge, \sum_{k=1}^n \bar i_k i_k\Big]\,. \tag{3.7}$$

$$-i_k = \frac{1}{2}\Big[d\bar z_k\wedge, \sum_{k=1}^n \bar i_k i_k\Big]\,, \tag{3.8}$$

To prove (3.7) (and (3.8) resp.) we compare the action of LHS of (3.7) (and of (3.8) resp.) and the action of RHS of (3.7) (and of (3.8) resp.) on $\phi = dz_J \wedge d\bar z_K$.

We use the following formulas proved in p.112–113 of [3]

$$i_k(dz_J \wedge d\bar{z}_K) = 0, \text{ if } k \notin J\,, \tag{3.9}$$

$$i_k(dz_k \wedge dz_J \wedge d\bar{z}_K) = 2dz_J \wedge d\bar{z}_K\,, \tag{3.10}$$

$$\bar{i}_k(dz_J \wedge d\bar{z}_K) = 0, \text{ if } k \notin K\,, \tag{3.11}$$

$$\bar{i}_k(d\bar{z}_k \wedge dz_J \wedge d\bar{z}_K) = 2dz_J \wedge d\bar{z}_K\,. \tag{3.12}$$

With help of (3.9)–(3.12) we get (3.7) immediately. It is easy to see that (3.8) can be obtained from (3.7) by changing the complex orientation.

Set $\triangle_A^{\partial} := \partial_A\partial_A^* + \partial_A^*\partial_A$, $\triangle_A^{\bar\partial} := \bar\partial_A\bar\partial_A^* + \bar\partial_A^*\bar\partial_A$.

Corollaries 3.13. *For $\phi, \psi \in \Omega^{0,p}(E)$ we have the following simple expressions*

$$\bar\partial_A^*\phi = -\sqrt{-1}\Lambda\partial_A(\phi), \tag{3.13.1}$$

$$\begin{aligned}\int_{M^{2n}} \langle\sqrt{-1}\Lambda F_A^{1,1}\phi, \psi\rangle = \int_{M^{2n}} &-\langle\bar\partial_A^*\phi, \bar\partial_A^*\psi\rangle\\ &+ \langle\partial_A\phi, \partial_A\psi\rangle - \langle\bar\partial_A\phi, \bar\partial_A\psi\rangle\,.\end{aligned} \tag{3.13.2}$$

More generally, for all $\phi \in \Omega^{p,q}(E)$ we have

$$(\triangle_A^{\partial} - \triangle_A^{\bar\partial})\phi = -\sqrt{-1}[F_A^{1,1}\wedge, \Lambda]\phi\,; \tag{3.13.3}$$

$$\triangle_A^{\bar\partial}\phi = \frac{1}{2}(\triangle_A^d + \sqrt{-1}[-F_A^{0,2} + F_A^{2,0} + F_A^{1,1}, \Lambda])\phi\,. \tag{3.13.4}$$

Proof. 1) The first statement follows immediately from the Hodge-Kähler identity (3.1).

2) Substituting $F_A^{1,1} = \bar\partial_A\partial_A + \partial_A\bar\partial_A$ we get

$$\int_{M^{2n}} \langle\sqrt{-1}\Lambda F_A^{1,1}\phi, \psi\rangle = \int_{M^{2n}} \langle\sqrt{-1}\Lambda(\bar\partial_A\partial_A + \partial_A\bar\partial_A)\phi, \psi\rangle\,.$$

Now applying the Hodge-Kähler identities to this equation we get

$$\begin{aligned}\int_{M^{2n}} \langle\sqrt{-1}\Lambda F_A^{1,1}\phi, \psi\rangle &= \int_{M^{2n}} \langle\sqrt{-1}(\bar\partial_A\Lambda\partial_A - \sqrt{-1}\partial_A^*\partial_A)\phi, \psi\rangle - \int_{M^{2n}} \langle\bar\partial_A^*\bar\partial_A\phi, \psi\rangle\\ &= \int_{M^{2n}} \langle\sqrt{-1}\Lambda\partial_A\phi, \bar\partial_A^*\psi\rangle + \int_{M^{2n}} \langle\partial_A\phi, \partial_A\psi\rangle - \int_{M^{2n}} \langle\bar\partial_A\phi, \bar\partial_A\psi\rangle\end{aligned} \tag{3.14}$$

Using (3.13.1) we get Corollary 3.13.2 immediately from (3.14).

3) Using the Hodge-Kähler identities (3.1) and (3.2), we get

$$\begin{aligned}-\sqrt{-1}\triangle_A^{\partial} &= \partial_A(\Lambda\bar\partial_A - \bar\partial_A\Lambda) + (\Lambda\bar\partial_A - \bar\partial_A\Lambda)\partial_A\\ &= \partial_A\Lambda\bar\partial_A - \partial_A\bar\partial_A\Lambda + \Lambda\bar\partial_A\partial_A - \bar\partial_A\Lambda\partial_A\,.\end{aligned} \tag{3.15}$$

In the same way we have

$$\begin{aligned}\sqrt{-1}\triangle^{\bar\partial}_A &= \bar\partial_A(\Lambda\partial_A - \partial_A\Lambda) + (\Lambda\partial_A - \partial_A\Lambda)\bar\partial_A \\ &= \bar\partial_A\Lambda\partial_A - \bar\partial_A\partial_A\Lambda + \Lambda\partial_A\bar\partial_A - \partial_A\Lambda\bar\partial_A\,. \end{aligned}\tag{3.16}$$

Using the identities

$$-(\partial_A\bar\partial_A + \bar\partial_A\partial_A) = -F^{1,1}_A\wedge$$

we get from (3.15) and (3.16)

$$-\sqrt{-1}(\triangle^{\partial}_A - \triangle^{\bar\partial}_A) = -[F^{1,1}_A\wedge,\Lambda]\,.$$

which yields (3.13.3) immediately.

4) We have

$$\begin{aligned}\triangle^d_A &= (\partial_A + \bar\partial_A)(\partial^*_A + \bar\partial^*_A) + (\partial^*_A + \bar\partial^*_A)(\partial_A + \bar\partial_A) \\ &= \triangle^{\partial}_A + \triangle^{\bar\partial}_A + (\partial_A\bar\partial^*_A + \bar\partial_A\partial^*_A + \partial^*_A\bar\partial_A + \bar\partial^*_A\partial_A)\end{aligned}\tag{3.17}$$

Using the Hodge-Kähler identity (3.1), and replacing $\partial_A\partial_A$ by $F^{2,0}_A\wedge$, we get

$$\begin{aligned}(\partial_A\bar\partial^*_A + \bar\partial^*_A\partial_A) &= -\sqrt{-1}\partial_A(\Lambda\partial_A - \partial_A\Lambda) - \sqrt{-1}(\Lambda\partial_A - \partial_A\Lambda)\partial_A \\ &= \sqrt{-1}[F^{2,0}_A\wedge,\Lambda]\,.\end{aligned}\tag{3.18}$$

Similarly

$$\begin{aligned}(\bar\partial_A\partial^*_A + \partial^*_A\bar\partial_A) &= -\sqrt{-1}\bar\partial_A(\bar\partial_A\Lambda - \Lambda\bar\partial_A) - \sqrt{-1}(\bar\partial_A\Lambda - \Lambda\bar\partial_A)\bar\partial_A \\ &= -\sqrt{-1}[F^{0,2}_A\wedge,\Lambda]\,.\end{aligned}\tag{3.19}$$

Using Corollary (3.13.3), we get from (3.17), (3.18), (3.19)

$$\triangle^d_A = 2\triangle^{\bar\partial}_A - \sqrt{-1}[F^{1,1}_A\wedge,\Lambda] + \sqrt{-1}[F^{0,2}_A\wedge,\Lambda] - \sqrt{-1}[F^{0,2}_A\wedge,\Lambda]$$

which yields (3.13.4) immediately. □

Remark 3.20. Clearly (3.13.2) follows directly from (3.13.3). Furthermore, taking into account (2.2), we conclude that all the formulas in Corollaries (3.15) are valid, if we replace bundle E by bundle $\mathrm{End}_J\,E$.

Using Corollary (3.13.1), we observe that a connection A over a compact Kähler manifold is Yang-Mills bar, iff $\Lambda\partial_A F^{0,2}_A = 0$. We call a connection A almost holomorphic, if $\partial_A F^{0,2}_A = 0$. Using the Bianchi identity $\bar\partial_A F^{0,2}_A = 0$, we get that $\partial_A F^{0,2} = 0$, iff $d_A F^{0,2}_A = 0$. Since $F^{2,0}_A = -(F^{0,2}_A)^*$, we observe that $d_A F^{0,2}_A = 0$, iff $d_A F^{2,0} = 0$. Using the Bianchi identity $d_A F_A = 0$, we observe that A is almost holomorphic, iff $dF^{1,1}_A = 0$. If $F^{1,1}_A = 0$ we call A almost flat holomorphic connection.

If dimension of M equals 4, it is easy to check that

$$\Lambda\partial_A F^{0,2}_A = 0 \Longleftrightarrow \partial_A F^{0,2}_A = 0\,.$$

Thus any Yang-Mills bar connection over M^4 is an almost holomorphic connection.

3.21. Existence of almost holomorphic connections. Let T^4 be a 2-dimensional complex torus with coordinates $z_1 = x_1 + \sqrt{-1}y_1$, $z_2 = x_2 + \sqrt{-1}y_2$. Let L be a complex line bundle whose Chern class is represented by the cohomology class c_1 of $dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2$. Let A be a unitary connection of L. Then $F_A = \sqrt{-1}(dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2) + \sqrt{-1}d\alpha$, where $\alpha \in \Omega^1(T^4)$. The new connection $A' = A - \alpha$ has the curvature $\sqrt{-1}(dz_1 \wedge dz_2 + d\bar{z}_1 \wedge d\bar{z}_2)$, whose component $F^{1,1}_{A'}$ vanishes. Thus A' is an almost flat holomorphic connection. We observe that by the Hodge theorem L carries no holomorphic structure.

The same argument provides us a differential-geometric proof of the Hodge conjecture for Hodge classes of dimension 2.

To get an almost holomorphic connection in vector bundles of higher dimension we can take the sum of line bundles or a tensor product of a complex line bundle with a holomorphic vector bundles.

In the next section we show that if M^{2n} is a Kähler manifold of positive Ricci curvature, then any almost holomorphic connection is a holomorphic connection (Theorem 4.25), in particular any almost flat holomorphic connection is a flat connection.

In general, the Hodge theory implies that on any Hermitian complex line bundle over a Kähler manifold there is a Yang-Mills bar connection which realizes the infimum of the Yang-Mills bar functional.

4. Yang-Mills bar equation over compact Kähler manifolds of positive Ricci curvature

Suppose that A is a unitary connection on a Hermitian vector bundle E over a Kähler manifold M^{2n}. Let D be the Levi-Civita connection on T^*M^{2n}:

$$D\colon \Omega^1(M^{2n}) \to \Omega^1(M^{2n}) \otimes T^*M^{2n}\,.$$

The connection D extends $\mathbb{C}$-linearly to a connection also denoted by $D\colon \Omega^1_{\mathbb{C}}(M^{2n}) \to \Omega^1_{\mathbb{C}}(M^{2n}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n} =_{\mathbb{R}} \Omega^1_{\mathbb{C}}(M^{2n}) \otimes_{\mathbb{R}} T^*M^{2n}$. Since M^{2n} is Kähler, we have $D_v(\phi \pm \sqrt{-1}J\phi) = D_v(\phi) \pm \sqrt{-1}JD_v(\phi)$ for all $v \in T^*_{\mathbb{C}}M^{2n}$ and for all $\phi \in \Omega^{0,1}(M^{2n})$. It follows that $D(\Omega^{0,1}(M^{2n})) \subset \Omega^{0,1}(M^{2n}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n}$, and iterating we have $D(\Omega^{0,p}(M^{2n})) \subset \Omega^{0,p}(M^{2n}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n}$ for all p. Now we denote by $\bar{D}$ the composition $\pi^{0,1} \circ D\colon \Omega^{0,p}(M^{2n}) \to \Omega^{0,p}(M^{2n}) \otimes_{\mathbb{C}} T^{0,1}M^{2n}$, where $\pi^{0,1}$ is the projection to the corresponding component with $(0,1)$-forms. Clearly for all $\phi \in \Omega^{0,p}(M)$ the following formula holds

$$D_{v^{0,1}}(\phi) = \bar{D}_{v^{0,1}}(\phi)\,, \tag{4.1}$$

where $v^{0,1}$ denotes the $(0,1)$-component of v: $v^{0,1} = (1/2)(v + \sqrt{-1}Jv)$. Similarly, we use the notation $v^{1,0} = (1/2)(v - \sqrt{-1}Jv)$.

Combining $\bar{D}$ with $\bar{\partial}_A\colon \Omega(E) \to \Omega^{0,1}(E)$, we define the following partial connection

$$\bar{\nabla}_A\colon \Omega^{0,p}(E) \to \Gamma(E \otimes_{\mathbb{C}} \Lambda^{0,p}T^*_{\mathbb{C}}M^{2n} \otimes_{\mathbb{C}} T^{0,1}M^{2n})\,.$$

In view of (4.1) we have

$$\bar{\nabla}_A = \pi^{0,1} \circ \nabla A_{|\Omega^{0,p}(E)}\,,$$

where ∇_A is the tensor product of d_A and D, which preserves the natural induced metric on the bundle $E \otimes_{\mathbb{C}} \Lambda^p T^*_{\mathbb{C}} M^{2n}$:

$$\nabla_A \colon \Omega^p_{\mathbb{C}}(E) \to \Gamma(E \otimes_{\mathbb{C}} \Lambda^p T^*_{\mathbb{C}} M^{2n} \otimes_{\mathbb{C}} T^*_{\mathbb{C}} M^{2n}) \,.$$

In view of (4.1) we also have $\nabla_A\big(\Omega^{0,p}(E)\big) \subset \Omega^{0,p}(E) \otimes_{\mathbb{C}} T^*_{\mathbb{C}} M^{2n}$.

For any element $\phi \in \Omega^p(E)$ the expression $\phi_{v_1,\dots,v_p}$ denotes the value of ϕ at $(v_1, \dots, v_p) \in \Lambda^p(T_* M^{2n})$.

Now we define a basic zero order operator $\mathcal{R}^A \colon \Omega^1_{\mathbb{C}}(\mathrm{End}_J\, E) \to \Omega^1_{\mathbb{C}}(\mathrm{End}_J\, E)$ by setting

$$\mathcal{R}^A(\phi)_X = \sum_{j=1}^{2n} [(F_A)_{e_j, X}, \phi_{e_j}] \in \mathrm{End}_J\, E \,, \tag{4.2}$$

where $(e_1, \dots, e_{n+k} = Je_k, \dots, e_{2n})$ is a unitary basis of the tangent space $T_x M^{2n}$ at the point x in question. We also regard F_A as an element in $\Omega^2_{\mathbb{C}}\big(\mathrm{End}_J(E)\big)$.

Recall that the Ricci transformation $\mathrm{Ric} \colon T_x M^{2n} \to T_x M^{2n}$ is defined by

$$\mathrm{Ric}(X) = \sum_{j=1}^{2n} R_{X, e_j} e_j \,,$$

where R denotes the curvature tensor of the Levi-Civita connection on the tangent space TM^{2n}. We denote by the same Ric the $\mathbb{C}$-linear extension of Ric from $T_x M^{2n}$ to $(T_x M^{2n})_{\mathbb{C}}$.

We modify this transformation by setting

$$\mathrm{Ric}^-(X) := \sum_{j=1}^{2n} R_{X, e_j} e_j^{0,1} \in (TM^{2n})_{\mathbb{C}} \,.$$

Since $J \circ R = R \circ J$, we have $\mathrm{Ric}^-(X) = \pi^{0,1} \circ \mathrm{Ric}(X)$. Here $\pi^{0,1}$ denotes the projection on the $(0,1)$-component.

Given $\phi \in \Omega^1(\mathrm{End}_J\, E)$ we define a new 1-form $\phi \circ \mathrm{Ric} \in \Omega^1(\mathrm{End}_J\, E)$ by requiring that for all $X \in TM$ we have

$$(\phi \circ \mathrm{Ric})_X := \phi_{\mathrm{Ric}(X)} \,.$$

We also define $\phi \circ \mathrm{Ric}^- \in \Omega^1(\mathrm{End}_J\, E)$ by requiring that for all $X \in TM$ we have

$$(\phi \circ \mathrm{Ric}^-)_X := \phi_{\mathrm{Ric}^-(X)} \,.$$

If $\phi \in \Omega^{0,1}(\mathrm{End}_J\, E)$, it is easy to see that $\phi \circ \mathrm{Ric}^- = \phi \circ \mathrm{Ric}$.

Lemma 4.3. *Suppose that (E, h) is a Hermitian vector bundle provided with a unitary connection A. We have the following simple formulas for any $\phi \in \Omega^{0,p}(E)$*

and for arbitrary $(0,1)$*-vectors* X_i

$$(\bar\partial_A\phi)_{X_0,\ldots,X_p} = \sum_{k=0}^{p}(-1)^k((\nabla_A)_{X_k}\phi)_{X_0,\ldots,\hat X_k,\ldots X_p}\,, \tag{4.3.1}$$

$$(\bar\partial_A^*\phi)_{X_1,\ldots,X_{p-1}} = -\sum_{j=1}^{2n}((\nabla_A)_{e_j^{1,0}}\phi)_{e_j^{0,1},X_1,\ldots,X_{p-1}}\,, \tag{4.3.2}$$

where $(e_1,\ldots,e_{n+k}=Je_k,\ldots,e_{2n})$ *is an unitary frame at a given point.*

Proof. First we extend a well-known formula for real forms (see e.g. [1, (2.12), (2.13)]) to complex forms $\phi\in\Omega^k_{\mathbb{C}}(E)$ and $X_i\in T^*_{\mathbb{C}}M^{2n}$:

$$(d_A\phi)_{X_0,\ldots,X_p} = \sum_{k=0}^{p}(-1)^k\big((\nabla_A)_{X^k}\phi\big)_{X_0,\ldots,\hat X_k,\ldots X_p}\,, \tag{4.4}$$

Formula (4.4) holds, since it holds for all real forms $\phi\in\Omega^k(E)\subset\Omega^k_{\mathbb{C}}(E)$ and for all $X_i\in T_*M^{2n}$, and because both LHS and RHS of (4.4) are $\mathbb{C}$-linear w.r.t. to variables ϕ and X_k.

By definition the LHS of (4.3.1) equals the LHS of (4.4) and clearly the RHS of (4.3.1) equals the RHS of (4.4). Hence we get (4.3.1).

Now let us prove (4.3.2). For $\phi\in\Omega^{0,p}(E)$ and for a set of a $(1,0)$-vector X_0 and $(0,1)$-vectors X_i, $1\le i\le p$, using (4.4) , we have

$$(\partial_A\phi)_{X_0,X_1\ldots,X_p} = \sum_{k=0}^{p}(-1)^k\big((\nabla_A)_{X_k}\phi\big)_{X_0,\ldots,\hat X_k,\ldots X_p}\,, \tag{4.5}$$

since LHS of (4.5) coincides with the value $(d_A\phi)_{X_0,X_1\ldots,X_p}$. Since $\phi\in\Omega^{0,p}(E)$, we get

$$\sum_{k=0}^{p}(-1)^k((\nabla_A)_{X_k}\phi)_{X_0,\ldots,\hat X_k,\ldots,X_p} = \big((\nabla_A)_{X_0}\phi\big)(X_1,\ldots,X_p)\,. \tag{4.6}$$

Thus we get

$$\sum_{k=0}^{p}(-1)^k(((\partial_A\phi) = \sum_{i=1}^{n} dz_i\wedge(\nabla_A)_{e_i^{1,0}}\phi\,. \tag{4.7}$$

Now using the Kähler identity $\bar\partial_A^* = -\sqrt{-1}\Lambda\partial_A$, we get from (4.7)

$$\begin{aligned}(\bar\partial_A^*\phi)_{X_1,\ldots,X_{p-1}} &= \frac{-1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n}\big[\bar i_k i_k dz_j\wedge\big((\nabla_A)_{e_j^{1,0}}\phi\big)\big]_{X_1,\ldots,X_{p-1}}\\ &= -\sum_{j=1}^{n}\big[\bar i_j\big((\nabla_A)_{e_j^{1,0}}\phi\big)\big]_{X_1,\ldots,X_p}\,.\end{aligned} \tag{4.8}$$

Clearly, the last term of (4.8) equals the RHS of (4.3.2). This completes the proof of Lemma 4.3. □

The following proposition is a complex analogue of Theorem 3.2 in [1].

Proposition 4.9. *For any $\phi \in \Omega^{0,1}(\mathrm{End}_J E)$ the following identity holds*

$$\triangle^{\bar\partial}_A \phi = \bar\nabla^*_A \bar\nabla_A(\phi) + \phi \circ \mathrm{Ric} + \mathcal{R}^A(\phi)\,. \tag{4.9.1}$$

Proof. Let $X \in T^{0,1}_x(M^{2n})$. We extend X locally on M^{2n} so that $DX(x) = 0$. We also extend the unitary frame $\{e_1, \dots, e_{n+k} := Je_k, \dots, e_{2n}\}$ locally so that $\mathrm{De}_i(x) = 0$. Using (4.3.1) and (4.3.2), and taking into account $(Je_j)^{0,1} = -\sqrt{-1}e_j^{0,1}$, $(Je_j)^{1,0} = \sqrt{-1}e_j^{1,0}$, we get at the point x

$$\begin{aligned}(\bar\partial_A \bar\partial^*_A \phi)_X &= (\nabla_A)_X\{\bar\partial^*_A\phi\} = -(\nabla_A)_X\Big\{\sum_{j=1}^{2n}\big[(\nabla_A)_{e_j}\phi\big]_{e_j^{0,1}}\Big\}\\
&= -\sum_{j=1}^{2n}\big[(\nabla_A)_X(\nabla_A)_{e_j}\phi\big]_{e_j^{0,1}}\,.\end{aligned}\tag{4.10}$$

$$\begin{aligned}(\bar\partial^*_A \bar\partial_A \phi)_X &= -\sum_{j=1}^{2n}\{(\nabla_A)_{e_j}(\bar\partial_A\phi)\}_{e_j^{0,1},X}\\
&= -\sum_{j=1}^{2n}(\nabla_A)_{e_j}\big\{[(\nabla_A)_{e_j^{0,1}}\phi]_X - [(\nabla_A)_X\phi]_{e_j^{0,1}}\big\}\\
&= -\sum_{j=1}^{2n}\big\{[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi]_X - [(\nabla_A)_{e_j}(\nabla_A)_X\phi]_{e_j^{0,1}}\big\}\,.\end{aligned}\tag{4.11}$$

Summing (4.10) and (4.11), we get

$$(\triangle^{\bar\partial}_A\phi)_X = -\sum_{j=1}^{2n}\Big\{[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi]_X + \sum_{j=1}^{2n}(R^A_{e_j,X}\phi)_{e_j^{0,1}}\Big\}\,. \tag{4.12}$$

Here we denote by R^A the curvature of the tensor product connection on the bundle $T^*_{\mathbb{C}}M^{2n} \otimes_{\mathbb{C}} \mathrm{End}_J E = (T^*M^{2n} \otimes_{\mathbb{R}} \mathrm{End}_J E)_{\mathbb{C}}$. This curvature coincides with the one on $T^*M^{2n} \otimes_{\mathbb{R}} \mathrm{End}_J E$, if we consider $\Omega^2(T^*M \otimes_{\mathbb{R}} \mathrm{End}_J E)$ as a subspace in $\Omega^2_{\mathbb{C}}(T^*_{\mathbb{C}}M^{2n} \otimes_{\mathbb{C}} \mathrm{End}_J E)$. Now we observe that for $\psi \in \Omega^{0,1}(\mathrm{End}_J E)$ we get

$$\begin{aligned}&\Big\langle -\sum_{j=1}^{2n}[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi], \psi\Big\rangle\\
&\qquad = -\sum_{j=1}^{2n}\big[(\nabla_A)_{e_j}\langle(\nabla_A)_{e_j^{0,1}}\phi, \psi\rangle - \langle(\nabla_A)_{e_j^{0,1}}\phi, (\nabla_A)_{e_j}\psi\rangle\big]\,.\end{aligned}\tag{4.13}$$

We define a 1-form σ, depending on ϕ and ψ, on M by

$$\sigma(X) := \langle(\nabla_A)_{X^{0,1}}\phi, \psi\rangle\,.$$

Then

$$-\sum_{j=1}^{2n}(\nabla_A)_{e_j}\langle(\nabla_A)_{e_j^{0,1}}\phi, \psi\rangle(x) = (-d^*\sigma)(x)\,, \tag{4.14}$$

and

$$\left\langle (\nabla_A)_{e_j^{0,1}}\phi, (\nabla_A)_{e_j}\psi \right\rangle = \langle \bar\nabla_A\phi, \bar\nabla_A\psi\rangle + \sum_{i=1}^{2n} \left\langle (\nabla_A)_{e_j^{0,1}}\phi, (\nabla_A)_{e_j^{1,0}}\psi \right\rangle .$$

Since $(Je_j)^{0,1} = -\sqrt{-1}e_j^{0,1}$ and $(Je_j)^{1,0} = \sqrt{-1}e_j^{1,0}$, we get

$$\begin{aligned} \left\langle (\nabla_A)_{e_i^{0,1}}\phi, (\nabla_A)_{e_j^{1,0}}\psi \right\rangle + \left\langle (\nabla_A)_{(Je_i)^{0,1}}\phi, (\nabla_A)_{(Je_j)^{1,0}}\psi \right\rangle = 0 \\ \Longrightarrow \sum_{i=1}^{2n} \left\langle (\nabla_A)_{e_j^{0,1}}\phi, (\nabla_A)_{e_j^{1,0}}\psi \right\rangle = 0 \end{aligned}$$

$$\Longrightarrow \left\langle (\nabla_A)_{e_j^{0,1}}\phi, (\nabla_A)_{e_j}\psi \right\rangle = \langle \bar\nabla_A\phi, \bar\nabla_A\psi\rangle . \tag{4.15}$$

From (4.13), (4.14), (4.15) we get

$$\int_{M^{2n}} -\left\langle (\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi, \psi \right\rangle = \int_{M^{2n}} -d^*\sigma + \int_{M^{2n}} \langle \bar\nabla_A\phi, \bar\nabla_A\psi\rangle . \tag{4.16}$$

Next we have

$$(R^A_{e_j,X}\phi)_{e_j^{0,1}} = (F_A)_{e_j,X}\phi_{e_j^{0,1}} - \phi(R_{e_j,X}e_j^{0,1}) . \tag{4.17}$$

Clearly Proposition 4.9 follows from (4.12), (4.13) and (4.17). □

Denote by $\mathcal{R}^A$ the following linear operator: $\Omega^{0,2}(\mathrm{End}_J E) \to \Omega^{0,2}(\mathrm{End}_J E)$ such that for all $(0,1)$-vectors X, Y we have

$$\left(\mathcal{R}^A(\phi)\right)_{X,Y} = \sum_{j=1}^{2n} \left\{ [(F_A)_{e_j,X}, \phi_{e_j,Y}] - [(F_A)_{e_j,Y}, \phi_{e_j,X}] \right\} . \tag{4.18}$$

We also associate to each $\phi \in \Omega^{0,2}(\mathrm{End}_J E)$ a new $(0,2)$-form $\phi \circ (\mathrm{Ric} \wedge I) \in \Omega^{0,2}(\mathrm{End}_J E)$ by setting

$$\left(\phi \circ (\mathrm{Ric} \wedge I)\right)_{X,Y} := \phi\big(\mathrm{Ric}(X), Y\big) - \phi\big(\mathrm{Ric}(Y), X\big) .$$

Proposition 4.19. *For any $\phi \in \Omega^{0,2}(\mathrm{End}_J E)$ the following identity holds*

$$\triangle^{\bar\partial}_A\phi = \bar\nabla_A^*\bar\nabla_A\phi + \phi \circ (\mathrm{Ric} \wedge I) + \mathcal{R}^A(\phi) . \tag{4.19.1}$$

Proof. (Cf. [1, Theorem 3.10]) We use the notations $X, Y, e_1, \dots, e_n$ as in the proof of Proposition 4.9. Then at the point x and for $(0,1)$-vectors X and Y we have

$$\begin{aligned} (\bar\partial_A\bar\partial_A^*\phi)_{X,Y} &= \left((\bar\partial_A)_X\bar\partial_A^*\phi\right)_Y - \left((\bar\partial_A)_Y\bar\partial_A^*\phi\right)_X \\ &= -(\nabla_A)_X\Big\{\sum_{j=1}^{2n}\left((\nabla_A)_{e_j}\phi\right)_{e_j^{0,1},Y}\Big\} + (\nabla_A)_Y\Big\{\sum_{j=1}^{2n}\left((\nabla_A)_{e_j}\phi\right)_{e_j^{0,1},X}\Big\} \end{aligned}$$

$$= -\sum_{j=1}^{2n}\left\{ [(\nabla_A)_X(\nabla_A)_{e_j}\phi]_{e_j^{0,1},Y} - [(\nabla_A)_Y(\nabla_A)_{e_j}\phi]_{e_j^{0,1},X} \right\} . \tag{4.20}$$

We also have

$$
\begin{aligned}
(\bar\partial_A^*\bar\partial_A\phi)_{X,Y} &= -\sum_{j=1}^{2n}((\nabla_A)_{e_j}\bar\partial_A\phi)_{e_j^{0,1},X,Y}\\
&= -\sum_{j=1}^{2n}(\nabla_A)_{e_j}\big\{((\nabla_A)_{e_j^{0,1}}\phi)_{X,Y} + ((\nabla_A)_Y\phi)_{e_j^{0,1},X} + ((\nabla_A)_X\phi)_{Y,e_j^{0,1}}\big\}\\
&= -\sum_{j=1}^{2n}\big\{[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi]_{X,Y} + [(\nabla_A)_{e_j}(\nabla_A)_Y\phi]_{e_j^{0,1},X}\\
&\qquad - [(\nabla_A)_{e_j}(\nabla_A)_X\phi]_{e_j^{0,1},Y}\big\}\,. \qquad (4.21)
\end{aligned}
$$

Summing (4.20) and (4.21) we get

$$
\begin{aligned}
(\triangle_A^{\bar\partial}\phi)_{X,Y} &= -\sum_{j=1}^{2n}\big[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi\big]_{X,Y}\\
&\quad + \sum_{j=1}^{2n}\big\{[R^A_{e_j,X}\phi]_{e_j^{0,1},Y} - [R^A_{e_j,Y}\phi]_{e_j^{0,1},X}]\big\}\,. \qquad (4.22)
\end{aligned}
$$

As in the proof of Proposition 4.9 (see (4.13)) we have for $\psi \in \Omega^{0,2}(\mathrm{End}_J E)$

$$
\int_{M^{2n}}\Big\langle -\sum_{j=1}^{2n}(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi, \psi\Big\rangle = \int_{M^{2n}}\langle\bar\nabla_A\phi, \bar\nabla_A\psi\rangle\,. \qquad (4.23)
$$

Combining the following identity

$$
(R^A_{X,Y}\phi)_{Z,W} = \big[(F_A)_{X,Y}, \phi_{Z,W}\big] - \phi(R_{X,Y}Z, W) - \phi(Z, R_{X,Y}W)
$$

with (4.23) to rewrite (4.22) as follows

$$
\begin{aligned}
(\triangle_A^{\bar\partial}\phi)_{X,Y} &= (\bar\nabla_A^*\bar\nabla_A\phi)_{X,Y} + \mathcal{R}^A(\phi)_{X,Y} + \phi(\mathrm{Ric}^-(X), Y)\\
&\quad - \sum_{j=1}^{2n}\phi_{e_j^{0,1},R_{e_j,X}Y} - \phi\big(\mathrm{Ric}^-(Y), X) + \sum_{j=1}^{2n}\phi_{e_j^{0,1},R_{e_j,Y}X}\,. \qquad (4.24)
\end{aligned}
$$

Using the Bianchi identity

$$
-R_{e_j,X}Y - R_{Y,e_j}X = R_{X,Y}e_j\,,
$$

and taking into account that the following quantity vanishes for all $\phi \in \Omega^{0,2}(\mathrm{End}_J E)$ and for all $X, Y \in T^{0,1}M^{2n}$

$$
(\phi\circ R)_{X,Y} := \sum_{j=1}^{2n}\phi(e_j, R_{X,Y}e_j)\,,
$$

because $(Je_j)^{0,1} = -\sqrt{-1}e_j^{0,1}$ and $(Je_j)^{1,0} = \sqrt{-1}e_j^{0,1}$, we get Proposition 4.19 from (4.24) immediately. $\square$

Theorem 4.25. *Let M be a compact Kähler manifold with positive Ricci curvature. If A is an almost holomorphic connection, then A is holomorphic.*

Proof. First let us prove the following formula for $\phi \in \Omega^{0,2}(\mathrm{End}_J E)$.

$$\mathcal{R}^A(\phi) = -\sqrt{-1}\{\Lambda F_A^{1,1} \wedge \phi - (\Lambda F_A^{1,1})\phi\} := \bar{R}(A)\phi\,. \tag{4.26}$$

Let us rewrite the expression in (4.18) as follows

$$\mathcal{R}^A(\phi) = \sum_{1\le k<l\le n}\sum_{j=1}^{2n}\{[(F_A)_{e_j,e_k^{0,1}}, \phi_{e_j,e_l^{0,1}}] - [(F_A)_{e_j,e_l^{0,1}}, \phi_{e_j,e_k^{0,1}}]\}d\bar{z}_k d\bar{z}_l\,. \tag{4.27}$$

For any $\phi \in \Omega^{k,p}(\mathrm{End}_J E)$ denote by

$$\phi_{i_1\dots i_k,\bar{j}_1\dots\bar{j}_p} := \phi(e_{i_1}^{1,0},\dots,e_{i_k}^{1,0},e_{j_1}^{0,1},\dots,e_{j_p}^{0,1})\,.$$

Since $\phi \in \Omega^{0,2}(\mathrm{End}_J E)$, we get from (4.27)

$$\begin{aligned}\mathcal{R}^A(\phi) = \sum_{1\le k<l\le n}\sum_{j=1}^{2n}\big\{&[(F_A)_{j\bar{k}}, \phi_{\bar{j}\bar{l}}] - [(F_A)_{j\bar{l}}, \phi_{\bar{j}\bar{k}}]\\ &+ [(F_A)_{\bar{j}\bar{k}}, \phi_{\bar{j}\bar{l}}] - [(F_A)_{\bar{j}\bar{l}}, \phi_{\bar{j}\bar{k}}]\}d\bar{z}_k d\bar{z}_l\,.\end{aligned} \tag{4.28}$$

Since $\big(J(e_j)\big)^{0,1} = -\sqrt{-1}e_j^{0,1}$ and $\big(J(e_j)\big)^{1,0} = \sqrt{-1}e_j^{1,0}$, we get from (4.28)

$$\mathcal{R}^A(\phi) = 2\sum_{1\le k<l\le n}\sum_{j=1}^{n}\{[(F_A)_{j\bar{k}}, \phi_{\bar{j}\bar{l}}] - [(F_A)_{j\bar{l}}, \phi_{\bar{j}\bar{k}}]\}\, d\bar{z}_k d\bar{z}_l\,. \tag{4.29}$$

Now expanding the following expression in local coordinates

$$\begin{aligned}\sqrt{-1}\Lambda F^{1,1}\wedge\phi &= \frac{1}{2}\sum_{p=1}^{n}\bar{i}_p i_p\Big\{\sum_{i,j}\sum_{k<l}[(F_A)_{i\bar{j}}, \phi_{\bar{k}\bar{l}}]\, dz_i\, d\bar{z}_j\, d\bar{z}_k\, d\bar{z}_l\Big\}\\ &= \sqrt{-1}(\Lambda F_A^{1,1})\phi - 2\sum_{1\le i\le n}\ \sum_{1\le j,l\le n}[(F_A)_{i\bar{j}}, \phi_{\bar{i}\bar{l}}]\, d\bar{z}_j\, d\bar{z}_l\,,\end{aligned}$$

and comparing it with the RHS of (4.29), we get (4.26) immediately.

Now let A be a Yang-Mills bar connection. Applying (4.19.1) to $F_A^{0,2}$ and using (4.26) we get

$$\begin{aligned}0 = \int_{M^{2n}}&\langle\bar{\nabla}_A F_A^{0,2}, \bar{\nabla}_A F_A^{0,2}\rangle + \big\langle F_A^{0,2}\circ(\mathrm{Ric}\wedge I), F_A^{0,2}\big\rangle\\ &+\int_M\big\langle\bar{R}(A)F_A^{0,2}, F_A^{0,2}\big\rangle\,.\end{aligned} \tag{4.30}$$

Since A is a Yang-Mills bar connection, differentiating (2.7.1), we get

$$\big\langle(\Lambda F_A^{1,1})F_A^{0,2}, F_A^{0,2}\big\rangle = 0\,. \tag{4.31}$$

Now let A be an almost holomorphic connection. Using (4.26), (4.30), (4.31), (3.13.2) (see also Corollary 3.20, we get $F_A^{0,2} = 0$ immediately. $\square$

Remark 4.32. Theorem 4.25 implies that any Yang-Mills bar connection on a compact 4-dimensional Kähler manifold of positive Ricci curvature is holomorphic.

5. Short time existence of a Yang-Mills bar gradient flow over a compact Kähler manifold

5.1. Affine integrability condition. The following identity holds for any $\theta \in \Omega(\mathrm{End}_J E)$ and any unitary connection A

$$\int_{M^{2n}} \big\langle [\theta, F_A^{0,2}], F_A^{0,2}\big\rangle = -\int_{M^{2n}} \big\langle [F_A^{0,2}, \theta], F_A^{0,2}\big\rangle . \tag{5.2}$$

We prove that at any point $x \in M^{2n}$

$$\big\langle [\theta, F_A^{0,2}], F_A^{0,2}\big\rangle = -2\langle \theta, \Lambda\Lambda F_A^{0,2} \wedge F_A^{2,0}\rangle . \tag{5.3}$$

We write $\theta = \theta^+ + \sqrt{-1}\theta^-$ where $\theta^+, \theta^- \in u_E$. In the same way at a fixed point $x \in M^{2n}$ we can take coordinates such that the Kähler metric g has the form $g(x) = \sum dz_i \otimes d\bar z_i$. We write

$$F_A^{0,2} = \sum_{1\le i<j\le n} (F_{ij}^+ + \sqrt{-1} F_{ij}^-)\, d\bar z_i\, d\bar z_j\, ,$$

where $F_{ij}^{\pm} \in u_E$. Then $F_A^{2,0} = \sum_{ij}(F_{ij}^+ - \sqrt{-1}F_{ij}^-) dz_i dz_j$. Recall that $\|d\bar z_i d\bar z_j\|^2 = 4$. A direct computation at a point x shows

$$\begin{aligned}
\big\langle [\theta, F_A^{0,2}], F_A^{0,2}\big\rangle &= \sum_{1\le i<j\le n} \big\langle [\theta^-, F_{ij}^+]\, d\bar z_i\, d\bar z_j, F_{ij}^-\, d\bar z_i\, d\bar z_j\big\rangle \\
&\quad + \sum_{1\le i<j\le n} \big\langle -[\theta^-, F_{ij}^-]\, d\bar z_i\, d\bar z_j, F_{ij}^+\, d\bar z_i\, d\bar z_j\big\rangle \\
&= 8\Big\langle \theta^-, \sum_{1\le i<j\le n} [F_{ij}^+, F_{ij}^-]\Big\rangle .
\end{aligned} \tag{5.4}$$

Now we compute

$$\begin{aligned}
\langle \theta, \Lambda\Lambda F_A^{0,2} \wedge F_A^{2,0}\rangle &= -2 \sum_{1\le i<j\le n} \big\langle \theta^-, \Lambda\Lambda[F_{ij}^+, F_{ij}^-]\, dz_i\, dz_j\, d\bar z_i\, d\bar z_j\big\rangle \\
&= -4\sqrt{-1} \sum_{1\le i<j\le n} \big\langle \theta^-, \Lambda[F_{ij}^+, F_{ij}^-](dz_j\, d\bar z_j + dz_i\, d\bar z_i)\big\rangle \\
&= -16\Big\langle \theta^-, \sum_{1<i<j\le n} [F_{ij}^+, F_{ij}^-]\Big\rangle .
\end{aligned} \tag{5.5}$$

Clearly (5.3) follows from (5.4) and (5.5).

Now substituting $[F_A^{0,2}, \theta] = \bar\partial_A \bar\partial_A \theta$ in the RHS of (5.2), and taking into account (5.3), we get

$$\int_{M^{2n}} \langle \theta, 2\Lambda\Lambda F_A^{0,2} \wedge F_A^{2,0}\rangle = \int_{M^{2n}} \langle \theta, \bar\partial_A^* \bar\partial_A^* F_A^{0,2}\rangle . \tag{5.6}$$

Thus we get the following identity

$$\bar\partial_A^*\bar\partial_A^* F_A^{0,2} - 2\Lambda\Lambda F_A^{0,2}\wedge F_A^{2,0} = 0\,. \tag{5.7}$$

Define the following operator $P_A\colon \Omega^{0,1}(\mathrm{End}_J E)\times\Omega^{0,1}(\mathrm{End}_J E)\to\Omega(\mathrm{End}_J E)$

$$P_A(a)\phi := \bar\partial_{A+a}^*\phi - 2\Lambda\Lambda F_{A+a}^{0,2}\wedge F_{A+a}^{2,0}\,. \tag{5.8}$$

Clearly $P_A(a)\phi$ is a differential operator of order 1 in a and order 1 in ϕ. Moreover $P_A(a)\phi$ is an affine differential operator w.r.t. ϕ, i.e. $P_A(a)\phi = L_A(a)\phi + C_A(a)$, where $L_A(a)\phi$ is a linear differential operator w.r.t. ϕ. By (5.6) we have $P_A(a)\bar\partial_{A+a}^* F_{A+a}^{0,2} = 0$. Thus we call $P_A(a)$ an affine integrability condition for the differential operator $\bar\partial_{A+a}^* F_{A+a}^{0,2}\colon \Omega^{0,1}(\mathrm{End}_J E)\to\Omega^{0,1}(\mathrm{End}_J E)$.

Proposition 5.9. *Let $\xi\in T_x^*M^{2n}\setminus\{0\}$. All the eigenvalues of the eigenspace of the symbol $\sigma_\xi D(-1)\partial_{A+a}^* F_{A+a}^{0,2}\colon \Omega^{0,1}(\mathrm{End}_J E)\to\Omega^{0,1}(\mathrm{End}_J E)$ in* Null $\sigma_\xi P_A(a)$ *are positive. Hence the evolution equation*

$$\frac{da}{dt} = -\bar\partial_{A+a}^* F_{A+a}^{0,2}\,, \tag{5.9.1}$$

has a unique smooth solution for a short time which may depend on a.

Proof. Since $F_{A+a+th}^{0,2} = F_A^{0,2} + t\bar\partial_{A+a}\wedge h + t^2 h\wedge h$ for $h\in\Omega^{0,1}(\mathrm{End}_J E)$, we have the following expression for the linearization of $\bar\partial_{A+a}^* F_{A+a}^{0,2}$ at point $a\in\Omega^{0,1}(\mathrm{End}_J E)$

$$D_a(\bar\partial_{A+a}^* F_{A+a}^{0,2})(h) = \bar\partial_{A+a}^*\bar\partial_{A+a}h + \{\text{terms of lower order}\}\,. \tag{5.10}$$

We may assume that $\xi = dx_1$. Then a direct computation using the Hodge-Kähler identity $\bar\partial_{A+a}^* = -\sqrt{-1}\Lambda\partial_{A+a}$ and (5.10) shows

$$-\sigma_\xi D_a(\bar\partial_{A+a}^* F_{A+a}^{0,2})(\alpha_1\, d\bar z_1,\dots,\alpha_n\, d\bar z_n) = (0,\alpha_2\, d\bar z_2,\dots,\alpha_n\, d\bar z_n)\,. \tag{5.11}$$

Clearly the linearization $D_\phi P_A(a)\phi$ with respect to the variable ϕ is

$$\big[D_\phi P_A(a)\phi\big]h = \frac{d}{dt}_{|t=0}\bar\partial_{A+a}^*(\phi+th) - 2\Lambda\Lambda F_{A+a}^{0,2}\wedge F_{A+a}^{2,0} = \bar\partial_{A+a}^*(h)\,.$$

We note that this linearization does not depend on ϕ. A short computation shows

$$\sigma_\xi D_\phi P_A(a)(\alpha_1 d\bar z_1,\dots,\alpha_n d\bar z_n) = \sqrt{-1}\alpha_1\,. \tag{5.12}$$

Now (5.11) and (5.12) imply the first statement of Proposition 5.9. The second statement follows from Hamilton's theory for evolution equation with integrability condition [4, Theorem 5.1], actually from its slightly extended version in Theorem 6.6 below. □

Remarks 5.13. 1. By taking derivative of (2.7.1) in the time t we also get (5.2) and hence (5.7). In the same way we can get (5.2) (and hence (5.7)) as an infinitesimal consequence of the non-canonical action of the complex gauge group on the space of unitary connections w.r.t. a fixed Hermitian metric on the bundle.

2. It is likely that $\bar\partial_A^* F_A^{0,2}$ also satisfies an affine integrability condition analogous to (5.8), if the ground manifold M^{2n} is Hermitian but not necessary Kähler.

6. Evolution equations with affine integrability condition

In his work [4] Hamilton introduced the notion of an evolution equation with integrability condition. Let us rapidly recall the Hamilton concept from Section 5 of that paper. We try to keep most of Hamilton's notations in that paper, which may have quite different meanings from that ones we used in previous sections.

We consider an evolution equation

$$\frac{df}{dt} = E(t)\,,$$

where $E(f)$ is a non-linear differential operator of degree 2 in f. We suppose that the values of f belong to an open set U in a vector bundle F over a compact manifold X, and $E(f)$ takes its values in F also. (For the case we are dealing in this note, we take $U = F$. We shall write later, following Hamilton, $f \in U$, meaning that the values of f belong to U.) Then E is a smooth map

$$E\colon C^\infty(X,U) \subset C^\infty(X,F) \to C^\infty(X,F)$$

of an open set in a Fréchet space to itself.

We shall consider problems where some of the eigenvalues of the symbol $\sigma DE(f)\xi$ are zero. This happens when $E(f)$ satisfies an integrability condition.

Definition 6.1 ([4]). Let $g = L(f)h\colon C^\infty(X,U) \times C^\infty(F) \to C^\infty(G)$ be a differential operator of degree 1 on sections $f \in U \subset F$, $h \in F$, and G another vector bundle over X. We call $L(f)h$ the integrability condition for $E(f)$, if the operator $Q(f) = L(f)E(f)$ only has degree at most one in f.

Suppose that $L(f)h$ is an integrability condition for $E(f)$. Taking a variation in $\tilde{f}$ we see that

$$L(f)DE(f)\tilde{f} + DL(f)\{E(f),\tilde{f}\} = DQ(f)\tilde{f}\,. \tag{6.2}$$

Since $DQ(f)\tilde{f}$ as well as $L(f)DE(f)\tilde{f}$ only have degree 1 in f the operator $L(f)DE(f)\tilde{f}$ also have degree 1. hence $\sigma L(f)(\xi)\sigma DE(f)(\xi) = 0$. Therefore we get

$$\operatorname{Im} \sigma DE(f)(\xi) \subset \operatorname{Null} \sigma L(f)(\xi)\,. \tag{6.3}$$

Theorem 6.4 ([4, Theorem 5.1]). *Let $df/dt = E(f)$ be an evolution equation with integrability condition $L(f)$. Suppose that all the eigenvalues of the eigenspaces of $\sigma DE(f)(\xi)$ in* Null *$\sigma L(f)(\xi)$ is positive. Then the initial value problem $f = f_0$ at $t = 0$ has a unique smooth solution for a short time $0 \le t \le \varepsilon$ where ε may depend on f_0.*

Remark 6.5. Hamilton's notation in (6.2) indicates that $L(f)h$ is a linear w.r.t. h. (In fact, in section 4 of that paper Hamilton stressed that $L(f)h$ is linear w.r.t. h.) A closer look at Hamilton's proof (see also our proof of Theorem 6.6 below) shows that, the linearity of $L(f)h$ w.r.t. h is important. We shall call such integrability condition $L(f)h$ linear in the argument (and f shall be considered as parameter). Now we shall call an integrability condition $L(f)h$ an affine integrability condition, if $L(f)h = L_0(f)h + A(f)$, where $L_0(f)h$ is linear w.r.t. h. The linearization $(D_\phi L(f)h)\tilde{h} = L_0(f)\tilde{h}$ does not depend on h.

Theorem 6.6. *Let $df/dt = E(f)$ be an evolution equation with affine integrability condition $L(f)$: $L(f)h = L_0(f)h + A(f)$. Suppose that all the eigenvalues of the eigenspaces of $\sigma DE(f)(\xi)$ in* Null *$\sigma L_0(f)(\xi)$ is positive. Then the initial value problem $f = f_0$ at $t = 0$ has a unique smooth solution for a short time $0 \le t \le \varepsilon$ where ε may depend on f_0.*

Proof of Theorem 6.6. We follow Hamilton's argument, replacing $L(f)h$ in his proof by $L_0(f)h$ in some places, and re-arranging parameters which do not depend on h. To keep our notations as close as possible with those of Hamilton, we denote by DL the derivative of $L(f)h$ w.r.t. the parameter f. We divide the proof in 3 steps. $\square$

STEP 1. *Reduction of Theorem 6.6 to a version of the Nash-Moser inverse function theorem.*

In this step we reduce Theorem 6.6 to the following

Lemma 6.7. *Suppose that $\bar f$ is a solution of the perturbed evolution equation by a term $\bar h(t,x)$*

$$\frac{d\bar f(t,x)}{dt} = E(\bar f(t,x)) + \bar h(t,x),$$

$$\bar f(0,x) = \bar f_0(x)$$

over the interval $0 \le t \le 1$. Then for any f_0 near $\bar f_0$ and h near $\bar h$ there exists a unique solution of the perturbed equation

$$\frac{df(t,x)}{dt} = E\big(f(t,x)\big) + h(t,x)\,,$$

$$f(0,x) = f_0(x)$$

over the interval $0 \le t \le 1$.

Now we explain how to get Theorem 6.6 from Lemma 6.7. Let $\bar f(t,x)$ be any function satisfying

$$\frac{d\bar f(t,x)}{dt}_{|t=0} = E\big(f(0,x)\big)\,,$$

$$\bar f(0,x) = f_0(x)\,.$$

Set

$$\bar h(t,x) := \frac{d\bar f(t,x)}{dt} - E(\bar f(t,x))\,.$$

Then $\bar h(0,x) = 0$.

Since X is compact, for any $\delta > 0$ there exist a number $\varepsilon > 0$ and a function $h(t,x)$ such that $H(t,x)$ is δ-close to $\bar h(t,x)$ and moreover $h(t,x) = 0$ for a short time $0 \le t \le \varepsilon$. Applying Lemma 6.7 to the pair $(\bar h, h)$ we conclude that the equation

$$\frac{df(t,x)}{dt} = E\big(f(t,x)\big) + h(t,x)\,,$$
$$f(0,x) = f_0(x)$$

has solution up to time ε. This solution in the interval $(0, \varepsilon)$ is a solution of our original equation in that time interval. This completes the first step.

STEP 2. *Reduction of Lemma 6.7 to a case of a weakly parabolic linear system of* (6.14.1) *and* (6.14.2). We can apply the Nash-Moser inverse function theorem to the operator

$$\mathcal{E} \colon C^\infty(X \times [0,1], F) \to C^\infty(X \times [0,1], F) \times C^\infty(X, F)\,,$$
$$\mathcal{E}(f) = \big(df/dt - E(f),\ f|\{t = 0\}\big)\,.$$

Its derivative is the operator

$$D\mathcal{E}(f)\tilde{f} = \Big(\frac{d\tilde{f}}{dt} - DE(f)\tilde{f}, \tilde{f}|\{t = 0\}\Big)\,.$$

We must show that the linearized equation

$$d\tilde{f}/dt - DE(f)\tilde{f} = \tilde{h} \tag{6.8}$$

has a unique solution for the initial value problem $\tilde{f} = \tilde{f}_0$ at $t = 0$, and verify that the solution $\tilde{f}$ is a smooth tame function of $\tilde{h}$ and $\tilde{f}_0$.

We make the substitution $\tilde{g} = L(f)\tilde{f}$. Then $\tilde{g}$ satisfies the evolution equation

$$\frac{d\tilde{g}}{dt} = L_0(f)\frac{d\tilde{f}}{dt} + DL(f)\{\tilde{f}, \frac{df}{dt}\}\,. \tag{6.9}$$

Now differentiating the integrability condition $L(f)E(f) = Q(f)$ we get

$$L_0(f)DE(f)\tilde{f} = -DL(f)\big\{E(f), \tilde{f}\big\} + DQ(f)\tilde{f}\,. \tag{6.10}$$

Substituting $d\tilde{f}/dt = DE(f)\tilde{f} + \tilde{h}$ from (6.8) into (6.9) and taking into account (6.10) we rewrite (6.9) as follows

$$\frac{d\tilde{g}}{dt} - M(f)\tilde{f} = \tilde{k}\,, \tag{6.11}$$

where $\tilde{k} = L_0(f)\tilde{h}$ and

$$M(f)\tilde{f} = DL(f)\Big\{\tilde{f}, \frac{df}{dt}\Big\} - DL(f)\big\{E(f), \tilde{f}\big\} + DQ(f)\tilde{f}$$
$$\overset{(6.10)}{=} DL(f)\Big\{\tilde{f}, \frac{df}{dt}\Big\} + L_0(f)DE(f)\tilde{f}\,. \tag{6.12}$$

is a linear differential operator in $\tilde{f}$ of degree 1 whose coefficients depend smoothly on f and its derivatives.

If we choose a measure on X and inner product on the vector bundle F and G, we can form a differential operator $L_0^*(f)g = h$ of degree 1 in f and g which is the adjoint of $L_0(f)$. Let us write

$$P(f)h := DE(f)h + L_0^*(f)L(f)h\,.$$

We claim that the equation $d\tilde{f}/dt = P(f)\tilde{f}$ is parabolic (for a given f). To see this we examine the symbol

$$\sigma P(f)\xi = \sigma DE(f)\sigma + \sigma L_0^*(f)(\xi)\cdot\sigma L_0(f)(\xi)\,. \tag{6.13}$$

Suppose v is an eigenvector in F with eigenvalue λ. Then $\sigma P(f)(\xi)v = 0$. But $\sigma L_0(f)(\xi)\cdot\sigma DE(f)(\xi) = 0$, so applying $\sigma L_0(f)$ to the LHS and RHS of (6.13) we get

$$\sigma L_0(f)(\xi)\cdot\sigma L_0^*(f)\xi\cdot\sigma L_0(f)(\xi)v = \lambda\sigma L_0(f)(\xi)v\,.$$

Taking inner product of the above equality with $\sigma L_0(f)(x)v$ we get

$$|\sigma L_0^*(f)(\xi)\cdot\sigma L_0(f)(\xi)v|^2 = \lambda|\sigma L_0(f)(\xi)v|^2\,.$$

Now if $\sigma L_0^*(f)\cdot\sigma L_0(f)(\xi)v = 0$ then $\sigma L_0(f)(\xi)v = 0$, and otherwise λ is real and strictly positive. When $\sigma L_0(f)(\xi)v = 0$, then $\sigma DE(f)(\xi)v = \lambda v$ by (6.13) and λ has strictly positive real part by our hypothesis in Theorem 6.6. Thus $P(f)$ is parabolic.

We proceed to solve the system of equations

$$\frac{d\tilde{f}}{dt} - P(f)\tilde{f} + L_0^*(f)\tilde{g} = \tilde{h}\,, \tag{6.14.1}$$

$$\frac{d\tilde{g}}{dt} - M(f)\tilde{f} = \tilde{k} \tag{6.14.2}$$

for the unknown function $\tilde{f}$ and $\tilde{g}$ for given $\tilde{h}$ and $\tilde{k}$ and given f, with initial data $\tilde{f} = \tilde{f}_0$ and $\tilde{g} = \tilde{g}_0 = L(f_0)\tilde{f}_0$ at $t = 0$.

In Step 3 below we prove that the solution $(\tilde{f}, \tilde{g})$ exists and is unique, and is a smooth tame function of $(f, \tilde{h}, \tilde{k}, \tilde{f}_0, \tilde{g}_0)$. Then putting $\tilde{l} = \tilde{g} - L(f)\tilde{f}$ and substituting $\tilde{k} = L_0(f)\tilde{h}$ we get

$$\begin{aligned}
\frac{d\tilde{l}}{dt} &= \frac{d\tilde{g}}{dt} - L_0(f)\frac{d\tilde{f}}{dt}\\
&= L_0(f)DE(f)\tilde{f} + \tilde{k} - L_0(f)\frac{d\tilde{f}}{dt}\\
&\overset{(6.14.1)}{=} -L_0(f)DE(f)\tilde{f} - L_0(f)P(f)\tilde{f} + L_0(f)L_0^*(f)\tilde{g}\\
&\overset{(6.13)}{=} L_0(f)\big[-L_0^*(f)L(f)\tilde{f} + L_0^*(f)L(f)(\tilde{l} + L(f)\tilde{f})\big]\\
&= L_0(f)L_0^*(f)\tilde{l}\,, \qquad (6.15)
\end{aligned}$$

and $\tilde{l} = 0$ at $t = 0$. But then (6.15) implies the obvious integral inequality

$$\frac{d}{dt}\int_X |\tilde{l}|^2 d\mu + 2\int_X |L_0^*(f)\tilde{l}| d\mu = 0\,.$$

Hence $\tilde{l} = 0$. Then it follows that $\tilde{g} = L(f)\tilde{f}$. Using this and we get from (6.14.1)

$$\frac{d\tilde{f}}{dt} - DE(f)\tilde{f} = \tilde{h}\,.$$

This completes Step 2.

STEP 3. *The system* (6.14.1) *and* (6.14.2) *is a weakly parabolic linear system whose smooth solution uniquely exists.*

Set $P_0(f)h := DE(f)h + L_0^*(f)L_0(f)h$. Then $P_0(f)h$ is a linear differential operator in h and $P(f)h = P_0(f)h + L_0^*(f)A(f)$. Set $h = \tilde{h} - L_0^*(f)A(f)$. Since f in the system of (6.14.1) and (6.14.2) is given, we shall re-denote a given constant $\tilde{k}$ by k, variables $\tilde{f}$, $\tilde{g}$, by f, g and linear differential operators $P_0(f)$, $L_0^*(f), M(f)$ by P, L, M. Then the system of (6.14.1) and (6.14.2) is equivalent to the following system of linear evolution equations on $0 \le t \le T$ for sections f of F and g of G

$$\frac{df}{dt} = Pf + Lg + h\,, \qquad \frac{dg}{dt} = Mf + k\,. \tag{6.16}$$

Clearly the existence, uniqueness and smoothness of a solution of (6.16) is a consequence of Hamilton's theorem [4, Theorem 6]. He considered the following equation

$$\frac{df}{dt} = Pf + Lg + h, \qquad \frac{dg}{dt} = Mf + Ng + k \tag{6.17}$$

where P, L, M and N are linear differential operators involving only space derivatives whose coefficients are smooth functions of both space and time. He assumed that P has degree 2, L and M have degree 1 and N has degree 0.

Theorem 6.18 ([4, Theorem 6])**.** *Suppose the equation $df/dt = Pf$ is parabolic. Then for any given (f_0, g_0, h, k) there exists a unique smooth solution (f, g) of the system* (6.17) *with $f = f_0$ and $g = g_0$ at $t = 0$.*

The proof of this Theorem occupies the whole Section 6 in Hamilton's paper.

Finally we formulate a conjecture which might be solved by using the Yang-Mills bar equation and might be helpful for understanding the Hodge conjecture. A unitary connection A on a Hermitian bundle E over a projective algebraic manifold M is holomorphic, if the L^p_q-norm of the component $F_A^{0,2}$ less than some positive constant $\varepsilon(M)$, where p, q are some integers depending on the dimension of M.

In a subsequent paper we shall show the long time existence of a Yang-Mills bar gradient-like flow and discuss its consequences.

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