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HYPERSURFACES WITH CONSTANT k-TH MEAN CURVATURE IN A LORENTZIAN SPACE FORM

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Abstract. In this paper, we study $n(n \geq 3)$-dimensional complete connected and oriented space-like hypersurfaces M^n in an (n+1)-dimensional Lorentzian space form $M_1^{n+1}(c)$ with non-zero constant k-th ($k < n$) mean curvature and two distinct principal curvatures λ and μ. We give some characterizations of Riemannian product $H^m(c_1) \times M^{n-m}(c_2)$ and show that the Riemannian product $H^m(c_1) \times M^{n-m}(c_2)$ is the only complete connected and oriented space-like hypersurface in $M_1^{n+1}(c)$ with constant k-th mean curvature and two distinct principal curvatures, if the multiplicities of both principal curvatures are greater than 1, or if the multiplicity of λ is $n-1$, $\lim_{s\to\pm\infty} \lambda^k \neq H_k$ and the sectional curvature of M^n is non-negative (or non-positive) when $c > 0$, non-positive when $c \leq 0$, where $M^{n-m}(c_2)$ denotes R^{n-m}, $S^{n-m}(c_2)$ or $H^{n-m}(c_2)$, according to $c = 0$, $c > 0$ or $c < 0$, where s is the arc length of the integral curve of the principal vector field corresponding to the principal curvature μ.

1. Introduction

Let $M_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian space form with constant sectional curvature c. According to $c > 0$, $c = 0$ or $c < 0$, it is called a de Sitter space, a Minkowski space or an anti-de Sitter space, respectively, and it is denoted by $S_1^{n+1}(c)$, R_1^{n+1} or $H_1^{n+1}(c)$. A hypersurface in a Lorentzian manifold is said to be space-like if the induced metric on the hypersurface is positive definite.

In connection with the negative settlement of the Bernstein problem due to Calabi [4], Cheng-Yau [5] and Chouque-Bruhat et al. [6] proved for $c \geq 0$ and T. Ishihara [9] proved for $c < 0$ the following theorem:

Theorem 1.1. *Let M^n be an n-dimensional $(n \geq 2)$ complete maximal space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$. Then*

(i) *if $c \geq 0$, M^n is totally geodesic;*

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(ii) *if* $c < 0$, *then* $S \leq n$ *and* $S = n$ *if and only if* $M^n = H^m(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$, *where* S *denotes the norm square of the second fundamental form of* M^n.

As a generalization of Theorem 1.1, complete space-like hypersurfaces with constant mean curvature or constant scalar curvature in a Lorentz manifold have been investigated by many mathematicians. For example, let M^n be an n-complete space-like hypersurface with constant mean curvature in a de Sitter space $S_1^{n+1}(c)$, Goddard [7] conjectured that every such hypersurface must be totally umbilical. Akutagawa [2] and Ramanthan [13] have proved independently that Goddard's conjecture is true if $H^2 \leq c$ when $n = 2$, and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Further discussions in this regard have been carried out by many other authors, we can see ([8]–[10] and [14]). Z. Hu et al. [8] studied the complete connected and oriented space-like hypersurfaces in an $(n+1)$-dimensional de Sitter space $S_1^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures and gave some characterizations of Riemannian product $H^m(c_1) \times S^{n-m}(c_2)$ in terms of the squared norm of the second fundamental form of M^n. By considering the sectional curvature of M^n, Zheng [16] proved the following result:

Theorem 1.2. *Let* M^n *be an* n*-dimensional compact space-like hypersurface in an* $(n+1)$*-dimensional de Sitter space* $S_1^{n+1}(c)$ *with constant scalar curvature* $n(n-1)r$. *If* $r < c$ *and the sectional curvature of* M^n *is non-negative, then* M^n *is isometric to a sphere.*

We denote by h the second fundamental form of M^n and denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the principal curvatures at an arbitrary point of M^n. From [11], we know that the k-th mean curvature H_k of M^n is defined by

$$P_n(t) = (1 + t\lambda_1)(1 + t\lambda_2)\ldots(1 + t\lambda_n) = 1 + C_n^1 H_1 t + \cdots + C_n^n H_n t^n\,,$$

that is, the k-th mean curvature H_k is the normalized k-th symmetric function of principal curvatures of the hypersurface M^n defined by

$$C_n^k H_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k}\,, \tag{1.1}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

We should note that if $k = 1$, H_1 is the mean curvature of M^n and if $k = 2$, from (1.1) and (2.11), we have $H_2 = c - r$, where r is the normalized scalar curvature of M^n.

In this paper, we investigate complete hypersurfaces in a Lorentzian space form $M_1^{n+1}(c)$ with constant k-th mean curvature H_k and with two distinct principal curvatures. In order to state our theorem clearly, we introduce, see U.-H. Ki et al. [10], the well-known standard models of complete space-like hypersurfaces with constant k-th mean curvature in an $(n+1)$-dimensional Lorentzian space form R_1^{n+1}, $S_1^{n+1}(c)$ or $H_1^{n+1}(c)$:

$$H^m(c_1) \times R^{n-m} = \Big\{(x, y) \in R_1^{n+1} = R_1^{m+1} \times R^{n-m}:\ \ |x|^2 = -\frac{1}{c_1} > 0\Big\}\,,$$

where $c_1 < 0$ and $m = 1, \ldots, n-1$. We note that $H^m(c_1) \times R^{n-m}$ in R_1^{n+1} has two distinct principal curvatures $\sqrt{-c_1}$ with multiplicity m and 0 with multiplicity $n-m$;

$$\begin{aligned} &H^m(c_1) \times S^{n-m}(c_2) \\ &\quad = \left\{(x,y) \in S_1^{n+1}(c) \subset R_1^{n+2} = R_1^{m+1} \times R^{n-m+1} :\ |x|^2 = -\frac{1}{c_1}, |y|^2 = \frac{1}{c_2}\right\}, \end{aligned}$$

where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$ and $m = 1, \ldots, n-1$. We note that $H^m(c_1) \times S^{n-m}(c_2)$ in $S_1^{n+1}(c)$ has two distinct principal curvatures $\sqrt{c - c_1}$ with multiplicity m and $\sqrt{c - c_2}$ with multiplicity $n-m$;

$$\begin{aligned} &H^m(c_1) \times H^{n-m}(c_2) \\ &= \left\{(x,y) \in H_1^{n+1}(c) \subset R_2^{n+2} = R_1^{m+1} \times R_1^{n-m+1} :\ |x|^2 = -\frac{1}{c_1}, |y|^2 = -\frac{1}{c_2}\right\}, \end{aligned}$$

where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$ and $m = 1, \ldots, n-1$. We note that $H^m(c_1) \times H^{n-m}(c_2)$ in $H_1^{n+1}(c)$ has two distinct principal curvatures $\pm\sqrt{c - c_1}$ with multiplicity m and $\mp\sqrt{c - c_2}$ with multiplicity $n-m$.

From U.-H. Ki et al. [10], $H^1(c_1) \times S^{n-1}(c_2)$, $H^1(c_1) \times R^{n-1}$ or $H^1(c_1) \times H^{n-1}(c_2)$ is, in particular, called a hyperbolic cylinder in $S_1^{n+1}(c)$, R_1^{n+1} or $H_1^{n+1}(c)$; $H^{n-1}(c_1) \times S^1(c_2)$ or $H^{n-1}(c_1) \times R^1$ is also called a spherical cylinder or Euclidean cylinder in $S_1^{n+1}(c)$ or R_1^{n+1}.

From above, we know that the hyperbolic cylinders, spherical cylinder or Euclidean cylinder has two distinct principal curvatures one of which is simple. Without loss of generality, we can denote the two distinct principal curvatures by λ and μ, and say that λ with multiplicity $n-1$ and μ with multiplicity 1. Therefore, from (1.1), we obtain

$$C_n^k H_k = C_{n-1}^k \lambda^k + C_{n-1}^{k-1} \lambda^{k-1} \mu\,,$$

this implies that

$$\lambda^{k-1}[(n-k)\lambda + k\mu] = nH_k\,. \tag{1.2}$$

For the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$, we know that $\lambda = 0$ and $\mu \neq 0$. If $k \geq 2$, from (1.2), we have $H_k \equiv 0$.

We shall prove the following result:

Main Theorem. *Let M^n be an n-dimensional ($n \geq 3$) complete connected and oriented space-like hypersurface in an (n+1)-dimensional Lorentzian space form $M_1^{n+1}(c)$ with non-zero constant k-th ($k < n$) mean curvature H_k and with two distinct principal curvatures λ and μ. Then*

(1) *if the multiplicities of both principal curvatures are greater than* 1, *then M^n is isometric to the Riemannian product $H^m(c_1) \times M^{n-m}(c_2)$, where $1 < m < n-1$, $M^{n-m}(c_2)$ denotes R^{n-m}, $S^{n-m}(c_2)$ or $H^{n-m}(c_2)$, according as $c = 0$, $c > 0$ or $c < 0$.*

(2) *if the multiplicity of* λ *is* $n-1$ *and* $\lim_{s\to\pm\infty} \lambda^k \neq H_k$, *where* s *is the arc length of the integral curve of the principal vector field corresponding to the principal curvature* μ, *then*

(i) *for* $c>0$, M^n *is isometric to the hyperbolic cylinder* $H^1(c_1)\times S^{n-1}(c_2)$ *or spherical cylinder* $H^{n-1}(c_1)\times S^1(c_2)$, $\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}$, $c_1<0$, $c_2>0$, *if the sectional curvature of* M^n *is non-negative or non-positive on* M^n;

(ii) *for* $c=0$, M^n *is isometric to the Euclidean cylinder* $H^{n-1}(c_1)\times R^1$ *or the hyperbolic cylinder* $H^1(c_1)\times R^{n-1}$ *and* $k=1$, *where* $c_1<0$, *if the sectional curvature of* M^n *is non-positive on* M^n;

(iii) *for* $c<0$, M^n *is isometric to the hyperbolic cylinder* $H^1(c_1)\times H^{n-1}(c_2)$, $\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}$, $c_1<0$, $c_2<0$, *if the sectional curvature of* M^n *is non-positive on* M^n.

Remark 1.1. If $c=1$, $k=1$ and $k=2$, the result of (1) in Main Theorem was proved by A. Brasil Jr. et al. [3] and Z. Hu et al. [8], respectively.

Remark 1.2. Let M^n be an n-dimensional ($n\geq 3$) space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ ($c\leq 0$). We should note that there is no space-like hypersurface in $H_1^{n+1}(c)$ or R_1^{n+1} with non-negative sectional curvature. In fact, if $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the principle curvatures of M^n, then the sectional curvature of the plane section spanned by $\{e_i,e_j\}$ is $R_{ijij}=c-\lambda_i\lambda_j$, $i\neq j$. For $c\leq 0$, if the sectional curvature is non-negative, we have $R_{ijij}=c-\lambda_i\lambda_j\geq 0$, this is, $\lambda_i\lambda_j\leq c\leq 0$. We infer that $\lambda_1,\lambda_2,\ldots,\lambda_n$ must have not the same sign each other, this implies that $n=2$. Since we assume that $n\geq 3$, we have a contradiction.

2. Preliminaries

Let M^n be an n-dimensional space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$. We choose a local field of semi-Riemannian orthonormal frames $\{e_1,\ldots,e_{n+1}\}$ in $M_1^{n+1}(c)$ such that at each point of M^n, $\{e_1,\ldots,e_n\}$ span the tangent space of M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1\leq A,B,C,\ldots\leq n+1\,;\quad 1\leq i,j,k,\ldots\leq n\,.$$

Let $\{\omega_1,\ldots,\omega_{n+1}\}$ be the dual frame field so that the semi-Riemannian metric of $M_1^{n+1}(c)$ is given by $d\bar{s}^2=\sum_i\omega_i^2-\omega_{n+1}^2=\sum_A\epsilon_A\omega_A^2$, where $\epsilon_i=1$ and $\epsilon_{n+1}=-1$.

The structure equations of $M_1^{n+1}(c)$ are given by

$$d\omega_A=\sum_B\epsilon_B\omega_{AB}\wedge\omega_B\,,\quad \omega_{AB}+\omega_{BA}=0\,,\tag{2.1}$$

$$d\omega_{AB}=\sum_C\epsilon_C\omega_{AC}\wedge\omega_{CB}+\Omega_{AB}\,,\tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D\,, \tag{2.3}$$

$$K_{ABCD} = \epsilon_A\epsilon_B c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,. \tag{2.4}$$

Restricting these forms to M^n, we have

$$\omega_{n+1} = 0\,. \tag{2.5}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j\,, \quad h_{ij} = h_{ji}\,. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,, \tag{2.8}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk})\,, \tag{2.9}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{2.10}$$

is the second fundamental form of M^n.
From the above equation, we have

$$n(n-1)(r-c) = S - n^2H^2\,, \tag{2.11}$$

where $n(n-1)r$ is the scalar curvature of M^n, H is the mean curvature, and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form of M^n.

We choose $e_1, \dots, e_n$ such that $h_{ij} = \lambda_i\delta_{ij}$. From (2.6) we have

$$\omega_{n+1i} = \lambda_i\omega_i\,, \quad i = 1, 2, \dots, n\,. \tag{2.12}$$

Hence, we have from the structure equations of M^n

$$\begin{aligned} d\omega_{n+1i} &= d\lambda_i \wedge \omega_i + \lambda_i d\omega_i \\ &= d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j\,. \end{aligned} \tag{2.13}$$

On the other hand, we have on the curvature forms of $M_1^{n+1}(c)$,

$$\begin{aligned}\Omega_{n+1i} &= -\frac{1}{2}\sum_{C,D} K_{n+1iCD}\omega_C \wedge \omega_D \\ &= \frac{1}{2}\sum_{C,D} c(\delta_{n+1C}\delta_{iD} - \delta_{n+1D}\delta_{iC})\omega_C \wedge \omega_D \\ &= c\omega_{n+1} \wedge \omega_i = 0\,. \end{aligned} \tag{2.14}$$

Therefore, from the structure equations of $M_1^{n+1}(c)$, we have

$$\begin{aligned} d\omega_{n+1i} &= \sum_j \omega_{n+1j} \wedge \omega_{ji} - \omega_{n+1n+1} \wedge \omega_{n+1i} + \Omega_{n+1i} \\ &= \sum_j \lambda_j \omega_{ij} \wedge \omega_j\,. \end{aligned} \tag{2.15}$$

From (2.13) and (2.15), we obtain

$$d\lambda_i \wedge \omega_i + \sum_j (\lambda_i - \lambda_j)\omega_{ij} \wedge \omega_j = 0\,. \tag{2.16}$$

Putting

$$\psi_{ij} = (\lambda_i - \lambda_j)\omega_{ij}\,, \tag{2.17}$$

we have $\psi_{ij} = \psi_{ji}$. (2.16) can be rewritten as

$$\sum_j (\psi_{ij} + \delta_{ij} d\lambda_j) \wedge \omega_j = 0\,. \tag{2.18}$$

By E. Cartan's Lemma, we get

$$\psi_{ij} + \delta_{ij} d\lambda_j = \sum_k Q_{ijk}\omega_k\,, \tag{2.19}$$

where Q_{ijk} are uniquely determined functions such that for all index i, j, k

$$Q_{ijk} = Q_{ikj}\,. \tag{2.20}$$

3. Proof of Main Theorem

We firstly state a Proposition which is well-known due to Otsuki [12] for Riemannian space forms (and for Lorentzian space forms see [8] or [3]).

Proposition 3.1. *Let M^n be a space-like hypersurface in an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Proof of Main Theorem. (1) Let λ and μ be the two distinct principal curvatures of multiplicities m and $n-m$ respectively, where $1 < m < n-1$. From (1.1), we have

$$C_n^k H_k = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \lambda_{i_1} \ldots \lambda_{i_k} ,$$

where the principal curvatures $\lambda_i = \lambda$ or μ $(i = 1, \ldots, n)$. This is always a equality of H_k, λ and μ, we can denote it by

$$C_n^k H_k = \mathcal{F}(\lambda, \mu) . \tag{3.1}$$

Denote by D_λ and D_μ the integral submanifolds of the corresponding distribution of the space of principal vectors corresponding to the principal curvature λ and μ, respectively. From Proposition 3.1, we know that λ is constant on D_λ. Since the k-th mean curvature H_k is constant, (3.1) implies that μ is constant on D_λ. By making use of Proposition 3.1 again, we have μ is constant on D_μ. Therefore, we know that μ is constant on M^n. By the same assertion we know that λ is constant on M^n. Therefore M^n is isoparametric. By the congruence Theorem of Abe, Koike and Yamaguchi [1], we know that M^n is isometric to the Riemannian product $H^m(c_1) \times M^{n-m}(c_2)$, where $M^{n-m}(c_2)$ denotes R^{n-m}, $S^{n-m}(c_2)$ or $H^{n-m}(c_2)$, according as $c = 0$, $c > 0$ or $c < 0$.

(2) From now on, we consider $n(n \ge 3)$-dimensional complete connected and oriented space-like hypersurface with non-zero constant k-th mean curvature H_k and with two distinct principal curvatures, one of which is simple. Without loss of generality, we may assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda , \quad \lambda_n = \mu ,$$

where λ_i for $i = 1, 2, \ldots, n$ are the principal curvatures of M^n. Therefore, we obtain

$$C_n^k H_k = C_{n-1}^k \lambda^k + C_{n-1}^{k-1} \lambda^{k-1} \mu ,$$

this implies that

$$\lambda^{k-1}[(n-k)\lambda + k\mu] = nH_k . \tag{3.2}$$

For $k \ge 2$, if $\lambda = 0$ at some point, from (3.2), we have $H_k = 0$ at this point, this is a contraction. Therefore, we have for all k

$$\mu = \frac{n}{k} H_k \lambda^{1-k} - \frac{n-k}{k} \lambda . \tag{3.3}$$

Since

$$\lambda - \mu = n \frac{\lambda^k - H_k}{k\lambda^{k-1}} \ne 0 ,$$

we know that $\lambda^k - H_k \ne 0$.

Let $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$. We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Since $\{\omega_1, \ldots, \omega_n\}$ is the dual frame field of M^n, putting

$$d\lambda = \sum_{k=1}^n \lambda_{,k} \, \omega_k , \quad d\mu = \sum_{k=1}^n \mu_{,k} \, \omega_k , \tag{3.4}$$

from Proposition 3.1, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x)\,. \tag{3.5}$$

From (3.3), we have

$$d\mu = \Big[\frac{n(1-k)}{k} H_k \lambda^{-k} - \frac{n-k}{k}\Big]\, d\lambda\,. \tag{3.6}$$

Thus, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x)\,. \tag{3.7}$$

In this case, we may consider locally λ is a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.19) and (3.5), we have for $1 \leq j \leq n-1$,

$$\begin{aligned} d\lambda = d\lambda_j &= \sum_{k=1}^{n} Q_{jjk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n = \lambda_{,n}\, \omega_n\,. \end{aligned} \tag{3.8}$$

Therefore, we have

$$Q_{jjk} = 0\,, \quad 1 \leq k \leq n-1\,, \quad \text{and} \quad Q_{jjn} = \lambda_{,n}\;. \tag{3.9}$$

By (2.19) and (3.7), we have

$$\begin{aligned} d\mu = d\lambda_n &= \sum_{k=1}^{n} Q_{nnk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n = \sum_{i=1}^{n} \mu_{,i}\, \omega_i = \mu_{,n}\, \omega_n\,. \end{aligned} \tag{3.10}$$

Hence, we obtain

$$Q_{nnk} = 0\,, \quad 1 \leq k \leq n-1\,, \quad \text{and} \quad Q_{nnn} = \mu_{,n}\;. \tag{3.11}$$

From (3.6), we get

$$Q_{nnn} = \mu_{,n} = \Big[\frac{n(1-k)}{k} H_k \lambda^{-k} - \frac{n-k}{k}\Big]\lambda_{,n}\;. \tag{3.12}$$

From the definition of ψ_{ij}, if $i \neq j$, we have $\psi_{ij} = 0$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$. Therefore, from (2.19), if $i \neq j$ and $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$ we have

$$Q_{ijk} = 0\,, \quad \text{for any} \quad k\,. \tag{3.13}$$

Since for all index i, j, k (2.20) holds, we have from (3.11) that $Q_{jnn} = 0$. By (2.19), (3.9), (3.11), (3.12) and (3.13), we get

$$\psi_{jn} = \sum_{k=1}^{n} Q_{jnk}\omega_k = Q_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda_{,n}\, \omega_j\,. \tag{3.14}$$

From (2.19), (3.3) and (3.14)we have

$$\omega_{jn} = \frac{\psi_{jn}}{\lambda-\mu} = \frac{\lambda_{,n}}{\lambda-\mu}\omega_j = \frac{k\lambda^{k-1}\lambda_{,n}}{n(\lambda^k - H_k)}\omega_j\,. \tag{3.15}$$

Therefore, from the structure equations of M^n we have

$$d\omega_n = \sum_{k=1}^{n-1}\omega_k\wedge\omega_{kn} + \omega_{nn}\wedge\omega_n = 0\,.$$

Therefore, we may put $\omega_n = ds$. By (3.6) and (3.10), we get

$$d\lambda = \lambda_{,n}\ ds\,, \qquad \lambda_{,n} = \frac{d\lambda}{ds}\,,$$

and

$$d\mu = \mu_{,n}\ ds\,, \qquad \mu_{,n} = \frac{d\mu}{ds}\,.$$

Then we have

$$\begin{aligned}\omega_{jn} &= \frac{k\lambda^{k-1}\lambda_{,n}}{n(\lambda^k - H_k)}\omega_j = \frac{k\lambda^{k-1}\frac{d\lambda}{ds}}{n(\lambda^k - H_k)}\omega_j \\ &= \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\omega_j\,. \end{aligned}\tag{3.16}$$

From (3.16) and the structure equations of $M_1^{n+1}(c)$, we have

$$\begin{aligned}d\omega_{jn} &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn}\wedge\omega_{nn} - \omega_{jn+1}\wedge\omega_{n+1n} + \Omega_{jn} \\ &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} - \omega_{jn+1}\wedge\omega_{n+1n} - c\omega_j\wedge\omega_n \\ &= \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k - (c-\lambda\mu)\omega_j\wedge\ ds\,.\end{aligned}$$

From (3.16), we have

$$\begin{aligned}d\omega_{jn} &= \frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2}\,ds\wedge\omega_j + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\,d\omega_j \\ &= \frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2}\,ds\wedge\omega_j + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k \\ &= \Big\{-\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} + \Big[\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\Big]^2\Big\}\omega_j\wedge ds \\ &\quad + \frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k\,.\end{aligned}$$

From the above two equalities, we have

$$\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} - \left\{\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\right\}^2 - (c - \lambda\mu) = 0\,. \tag{3.17}$$

Since we define $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$, we obtain from the above equation

$$\frac{d^2\varpi}{ds^2} + \varpi(c - \lambda\mu) = 0\,. \tag{3.18}$$

Now we prove the second part of Main Theorem.

(i) For $c > 0$, if the sectional curvature of M^n is non-negative, that is, for $i \neq j$, $R_{ijij} = c - \lambda_i\lambda_j \geq 0$, we have $c - \lambda\mu \geq 0$. From (3.18), we have $\frac{d^2\varpi}{ds^2} \leq 0$. Thus, $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, by the similar assertion in Wei [15], we have $\varpi(s)$ must be monotonic when s tends to infinity. Since $\lambda^k \neq H_k$ and λ is continuous, we know that there is no $s_0 \in (-\infty, +\infty)$, such that $\lim\limits_{s\to s_0} \lambda^k = H_k$. From the definition of $\varpi(s)$ and $\lim\limits_{s\to\pm\infty} \lambda^k \neq H_k$, we infer that the positive function $\varpi(s)$ is bounded. Since $\varpi(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim\limits_{s\to-\infty} \varpi(s)$ and $\lim\limits_{s\to+\infty} \varpi(s)$ exist and then we get

$$\lim_{s\to-\infty} \frac{d\varpi(s)}{ds} = \lim_{s\to+\infty} \frac{d\varpi(s)}{ds} = 0\,. \tag{3.19}$$

From the monotonicity of $\frac{d\varpi(s)}{ds}$, we have $\frac{d\varpi(s)}{ds} \equiv 0$ and $\varpi(s) =$ constant. From $\varpi = |\lambda^k - H_k|^{-\frac{1}{n}}$ and (3.2), we have λ and μ are constant, that is, M^n is isoparametric. Therefore, by the congruence Theorem of Abe, Koike and Yamaguchi [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times S^{n-1}(c_2)$ or spherical cylinder $H^{n-1}(c_1) \times S^1(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 > 0$.

If the sectional curvature of M^n is non-positive, that is, for $i \neq j$, $R_{ijij} = c - \lambda_i\lambda_j \leq 0$, we have $c - \lambda\mu \leq 0$. From (3.18), we have $\frac{d^2\varpi}{ds^2} \geq 0$. Similar to the assertion of the proof above, we know that Main Theorem is true.

(ii) For $c = 0$, if the sectional curvature of M^n is non-positive, that is, for $i \neq j$, $R_{ijij} = -\lambda_i\lambda_j \leq 0$, we have $-\lambda\mu \leq 0$. From (3.18), we have $\frac{d^2\varpi}{ds^2} \geq 0$. Thus, $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Combining $\frac{d^2\varpi}{ds^2} \geq 0$ with the boundedness of $\varpi(s)$, similar to the assertion of the proof in (i), we know that λ and μ are constant, that is, M^n is isoparametric. Therefore, by the congruence Theorem of Abe, Koike and Yamaguchi [1] and the discussion of Section 1, we know that M^n is isometric to the Euclidean cylinder $H^{n-1}(c_1) \times R^1$ or the hyperbolic cylinder $H^1(c_1) \times R^{n-1}$, in this case $k = 1$, where $c_1 < 0$.

(iii) For $c < 0$, if the sectional curvature of M^n is non-positive, that is, for $i \neq j$, $R_{ijij} = c - \lambda_i\lambda_j \leq 0$, we have $c - \lambda\mu \leq 0$. From (3.18), we have $\frac{d^2\varpi}{ds^2} \geq 0$. Thus, $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Combining $\frac{d^2\varpi}{ds^2} \geq 0$ with the boundedness of $\varpi(s)$, similar to the assertion of the proof in (i), we know that λ and μ are constant, that is, M^n is isoparametric. Therefore, by the congruence Theorem of Abe, Koike and Yamaguchi [1], we know that M^n is isometric to the hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$, $c_2 < 0$. This completes the proof of Main Theorem. $\square$

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