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A NOTE ON LINEAR PERTURBATIONS OF OSCILLATORY SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. Under suitable hypotheses on $\gamma(t)$, $\lambda(t)$, $q(t)$ we prove some stability results which relate the asymptotic behavior of the solutions of $u'' + \gamma(t)u' + \big(q(t) + \lambda(t)\big)u = 0$ to the asymptotic behavior of the solutions of $u'' + q(t)u = 0$.

1. Introduction

Let $q\colon [t_0, \infty) \to (0, \infty)$ and γ, $\lambda\colon [t_0, \infty) \to \mathbb{C}$ be continuous functions. We will consider the differential equation

$$u'' + \gamma(t)\, u' + \big(q(t) + \lambda(t)\big)\, u = 0\,, \quad t_0 \le t < \infty \tag{1.1}$$

as a perturbation of

$$u'' + q(t)\, u = 0\,, \quad t_0 \le t < \infty\,. \tag{1.2}$$

A number of papers have dealt with the linear perturbations of (1.2) assuming q, or the solutions of (1.2), suitably well-behaved as $t \to \infty$. For instance, R. Bellman [1] proved that if all solutions of (1.2) belong to $L^p[t_0, \infty) \cap L^{p'}[t_0, \infty)$, where $1 \le p \le p' \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ ($p' = \infty$, if $p = 1$) then all solutions of

$$u'' + (q(t) + \lambda(t))\, u = 0\,, \tag{1.3}$$

where λ is bounded, belong to $L^p[t_0, \infty) \cap L^{p'}[t_0, \infty)$; Z. Opial [9] showed that if q is nondecreasing, then all solutions of (1.3) are bounded as $t \to \infty$, if $\int^\infty |\lambda|\, q^{-\frac{1}{2}}\, dx < \infty$; W. F. Trench [10] demonstrated that if $\int^\infty |\lambda|\, |z_i|^2 dt < \infty$ ($i = 1, 2$), where z_1, z_2 are two linearly independent solutions of (1.2), then every solution of (1.3) can be written in the form $\alpha z_1 + \beta z_2$ with α, β suitable absolutely continuous functions. For other results of this type we may refer to [2, 3, 5].

Now, one observes immediately that many of these criteria place rather ineffective conditions, since one needs to know the behavior of solutions of the unperturbed equation (1.2) as $t \to \infty$. On the other hand, assuming q nondecreasing, in Opial's criteria [9] this a-priori knowledge is not required.

In this note, applying some results proved in [7], we will derive new effective conditions on q, γ, λ which, if q is positive and sufficiently smooth, ensure that

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all solutions of (1.1) are bounded or $p-$integrable (i.e. $\int_{t_0}^{\infty} |u|^p \, dt < \infty$ for some $p > 0$) on $[t_0, \infty)$. Precisely, under the assumption that

$$(1.4) \qquad q(t) \geq \delta > 0 \quad \text{and} \quad \frac{d^m}{dt^m}\left(q^{-\frac{1}{2}}\right) \quad \text{is of bounded variation in } [t_0, \infty)\,,$$

for some integer $m \geq 1$, we shall prove the following:

Theorem 1.1. *Assume* (1.4) *holds and that* $\int_{t_0}^{\infty} \left(|\gamma| + |\lambda|\, q^{-\frac{1}{2}}\right) d\tau < \infty$. *Then all solutions of* (1.1) *are p-integrable* $(p > 0)$ *if and only if* $\int_{t_0}^{\infty} q^{-\frac{p}{4}} \, dt < \infty$.

According to the Weyl classification, for $p = 2$ the conclusion of Th. 1.1 means that if $|\gamma| + |\lambda|\, q^{-\frac{1}{2}}$ is integrable then equation (1.1) retains the *limit circle* property.

Concerning the boundedness and the asymptotic behavior of solutions of (1.1), we introduce the energy:

$$(1.5) \qquad \mathcal{E}(u,t) \stackrel{\text{def}}{=} q(t)^{\frac{1}{2}}\, |u(t)|^2 + q(t)^{-\frac{1}{2}}\, |u'(t)|^2\,, \quad t \geq t_0\,.$$

Then, we have:

Theorem 1.2. *Assume* (1.4) *and* $\int_{t_0}^{\infty} \left(|\gamma| + |\lambda|\, q^{-\frac{1}{2}}\right) dx < \infty$. *Then for every solution u of* (1.1) *there exists the finite limit* $\lim_{t\to\infty} \mathcal{E}(u,t) \stackrel{\text{def}}{=} \mathcal{E}_u$, *with* $\mathcal{E}_u > 0$ *if* $u \not\equiv 0$.

Moreover, if z_1, z_2 *are linearly independent solutions of* (1.2), *there exist unique* $\alpha, \beta \in AC[t_0, \infty)$ (*i.e.* $\alpha', \beta' \in L^1[t_0, \infty)$) *such that*

$$(1.6) \qquad u = \alpha\, z_1 + \beta\, z_2\,, \quad u' = \alpha\, z_1' + \beta\, z_2'\,.$$

Finally, if $q(t) \to \infty$ as $t \to \infty$, we also have:

Theorem 1.3. *Assume* (1.4) *holds with* $q \to \infty$ *as* $t \to \infty$. *In addition suppose that there exists a constant* $\mathcal{C} > 2$ *such that*

$$(1.7) \qquad \limsup_{t\to\infty} \left(\int_{t_0}^{t} \left(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\right) d\tau \; - \; \frac{1}{\mathcal{C}}\, \ln q(t) \right) < \infty\,.$$

Then all solutions of (1.1) *satisfy* $\lim_{t\to\infty} u(t) = 0$. *Furthermore,* (1.6) *holds with* $\alpha, \beta \in AC_{\text{loc}}[t_0, \infty)$, *i.e.* $\alpha', \beta' \in L^1_{\text{loc}}[t_0, \infty)$.

We do not know if the condition $\mathcal{C} > 2$ in (1.7) is the best possible. However, we can show that it is not sufficient to require that (1.7) holds for an arbitrary constant $\mathcal{C} > 0$. See Example 5.4 below.

Remark 1.4. It is possible to prove all the previous results under slightly different assumptions on q. More precisely, the following holds:

Assume $q(t) > 0$ and $(q^{-\frac{1}{2}})^{(m)} \in AC_{\text{loc}}[t_0, \infty)$ for some integer $m \geq 1$. Then Th. 1.1, 1.2, 1.3 remain to hold if, instead of (1.4), we suppose:

$$(1.8) \qquad \lim_{t\to\infty} \left(q^{-\frac{1}{2}}(t)\right)^{1-\frac{1}{h}} \left|(q^{-\frac{1}{2}})^{(h)}(t)\right|^{\frac{1}{h}} = 0\,, \quad 1 \leq h \leq m\,,$$

and

$$(1.9) \qquad q^{-\eta_0/2} \left(\frac{d}{dt} q^{-\frac{1}{2}}\right)^{\eta_1} \cdots \left(\frac{d^{m+1}}{dt^{m+1}} q^{-\frac{1}{2}}\right)^{\eta_{m+1}} \in L^1(t_0, \infty)\,,$$

for all integers $\eta_0, \dots, \eta_{m+1} \geq 0$ such that

$$\sum_{0\leq h\leq m+1} \eta_h = m\,, \qquad \sum_{1\leq h\leq m+1} h\,\eta_h = m+1\,. \tag{1.10}$$

See [7, Prop. 6.1, Cor. 6.3]. One can also show that q satisfies (1.8)–(1.10) if (1.4) holds and $(q^{-\frac{1}{2}})^{(m)} \in AC_{\mathrm{loc}}[t_0,\infty)$. In some cases the conditions (1.8)–(1.10) are less restrictive than (1.4). See [8], [7, Section 7] and Remark 5.3 below.

2. Some preliminaries

To demonstrate Th. 1.1, 1.2 and 1.3 we will apply some results of [7] (see also [6, 8]) on the asymptotic behavior of solutions of the unperturbed equation (1.2). Below we briefly state the main results which will be needed in the proofs.

Theorem 2.1 ([7, Th. 1.1]). *Assume that* (1.4) *holds. Then all solutions of* (1.2) *are $p-$integrable, $p>0$, if and only if $\int_{t_0}^{\infty} q^{-\frac{p}{4}}\,dt < \infty$.*

Theorem 2.2 ([7, Th. 1.2]). *Assume that* (1.4) *holds and let u be a solution of* (1.2)*. Then there exists the finite limit*

$$\lim_{t\to\infty} \mathcal{E}(u,t) \overset{\mathrm{def}}{=} \mathcal{E}_u\,, \quad \textit{with} \quad \mathcal{E}_u > 0 \quad \textit{if} \quad u \not\equiv 0\,. \tag{2.1}$$

Remark 2.3. All these statements remain true if, instead of (1.4), we assume one of the following conditions:

– q satisfies the conditions (1.8)–(1.10), see [7];

– $0 < \delta \leq q(t) \leq \hat{\delta} < \infty$ and $q^{(m)}$ is of bounded variation for some $m \geq 1$; if $m = 1$ it is enough to suppose $q(t) \geq \delta > 0$. See [8].

On the other hand if, instead of (1.4) with $m \geq 1$, we only suppose $q \geq \delta > 0$ and $q^{-\frac{1}{2}}$ of bounded variation, the conclusions of Th. 2.1 and Th. 2.2 are, in general, false. This happens even if we further require that $q(t) \to \infty$ as $t \to \infty$. See [4].

Notation. Given $a, b \in \mathbb{R}$, we shall use the symbol $a \vee b$ for $\max\{a,b\}$.

From now on we fix

$$z_1,\, z_2 \colon [t_0,\infty) \to \mathbb{C}\,, \tag{2.2}$$

two linearly independent solutions of (1.2). Namely we suppose that, for $i = 1, 2$,

$$z_i'' + q(t) z_i = 0 \quad \text{in} \quad [t_0,\infty)\,, \tag{2.3}$$

with nonzero wronskian, i.e. $W(z_i, z_2) = z_1 z_2' - z_1' z_2 \neq 0$.

Applying Th. 2.2 we deduce the following:

Lemma 2.4. *Assume that* (1.4) *holds. Then there exists the finite limit*

$$\lim_{t\to\infty} \Big(q^{\frac{1}{2}}\, z_1 \bar{z}_2 + q^{-\frac{1}{2}}\, z_1' \bar{z}_2' \Big) \overset{\mathrm{def}}{=} \mathcal{E}_{12}\,. \tag{2.4}$$

In addition, setting $\mathcal{E}_i \overset{\mathrm{def}}{=} \lim_{t\to\infty} \mathcal{E}(z_i,t)$ $(i = 1,2)$, the quadratic form

$$Q(a,b) \overset{\mathrm{def}}{=} \mathcal{E}_1\,|a|^2 + \mathcal{E}_2\,|b|^2 + 2\,\mathrm{Re}\big(\mathcal{E}_{12}\,a\bar{b}\big)\,, \quad (a,b) \in \mathbb{C}^2\,, \tag{2.5}$$

is positive definite.

Proof. By Th. 2.2 there exist finite, the limits as $t \to \infty$, of

$$\mathcal{E}(z_1,t),\ \mathcal{E}(z_2,t),\ \mathcal{E}(z_1+z_2,t),\ \mathcal{E}(z_1+iz_2,t)\,. \tag{2.6}$$

Observing that

$$\mathcal{E}(z_1+z_2,t)=\mathcal{E}(z_1,t)+\mathcal{E}(z_2,t)+2\mathrm{Re}\Big(q^{\frac12}\,z_1\bar z_2+q^{-\frac12}\,z_1'\bar z_2'\Big)\,, \tag{2.7}$$

we deduce that there exits the finite limit

$$\lim_{t\to\infty}\ \mathrm{Re}\Big(q^{\frac12}\,z_1\bar z_2+q^{-\frac12}\,z_1'\bar z_2'\Big)\,. \tag{2.8}$$

Moreover, since

$$\mathcal{E}(z_1+iz_2,t)=\mathcal{E}(z_1,t)+\mathcal{E}(z_2,t)+2\,\mathrm{Im}\Big(q^{\frac12}\,z_1\bar z_2+q^{-\frac12}\,z_1'\bar z_2'\Big)\,, \tag{2.9}$$

we also deduce that there exits the finite limit

$$\lim_{t\to\infty}\ \mathrm{Im}\Big(q^{\frac12}\,z_1\bar z_2+q^{-\frac12}\,z_1'\bar z_2'\Big)\,. \tag{2.10}$$

Thus, it is clear that there exists the finite limit (2.4).

To continue, if $u=az_1+bz_2$ $(a,b\in\mathbb{C})$ is any solution of (1.2), we easily have

$$\begin{aligned}\lim_{t\to\infty}\mathcal{E}(u,t)&=\lim_{t\to\infty}\mathcal{E}(az_1+bz_2,t)\\&=|a|^2\lim_{t\to\infty}\mathcal{E}(z_1,t)\ +\ |b|^2\lim_{t\to\infty}\mathcal{E}(z_2,t)\\&\quad+2\,\mathrm{Re}\lim_{t\to\infty}a\bar b\Big(q^{\frac12}\,z_1\bar z_2+q^{-\frac12}\,z_1'\bar z_2'\Big)\\&=\mathcal{E}_1\,|a|^2+\mathcal{E}_2\,|b|^2+2\,\mathrm{Re}\big(\mathcal{E}_{12}\,a\bar b\big)\,,\end{aligned} \tag{2.11}$$

Since $\lim_{t\to\infty}\mathcal{E}(u,t)>0$ if $u\not\equiv0$, by definition (2.5) it follows that

$$Q(a,b)>0\,,\quad \forall\,(a,b)\in\mathbb{C}^2\setminus\{(0,0)\}\,. \tag{2.12}$$

Thus the quadratic form (2.5) is positive definite. □

Further, we also have:

Lemma 2.5. *Assume that* (1.4) *holds. Given* $\Lambda>1$ *there exists* $t_\Lambda\ge t_0$ *such that for all solutions* u *of* (1.2) *one has*

$$\Lambda^{-1}\,\mathcal{E}(u,t_1)\le\mathcal{E}(u,t_2)\le\Lambda\,\mathcal{E}(u,t_1)\,, \tag{2.13}$$

for all $t_1,t_2\ge t_\Lambda$.

Proof. It is clearly sufficient to prove the second inequality in (2.13).

Since the quadratic form (2.5) is positive definite, there exists $\rho>0$ such that

$$Q(a,b)>2\,\rho\,(|a|^2+|b|^2)\,,\quad \forall\,(a,b)\in\mathbb{C}^2\,. \tag{2.14}$$

By a continuity argument, this in turn implies that

$$\mathcal{E}(az_1+bz_2,t)\ge\rho\,(|a|^2+|b|^2)\,,\quad \forall\,(a,b)\in\mathbb{C}^2\,, \tag{2.15}$$

provided t is large enough, say $t \geq \bar{t} \geq t_0$. Moreover, $\forall \varepsilon > 0$ there exists $t_\varepsilon \geq t_0$ such that

$$|\mathcal{E}(az_1 + bz_2, t_2) - \mathcal{E}(az_1 + bz_2, t_1)| \leq \varepsilon\,(|a|^2 + |b|^2)\,, \tag{2.16}$$

for all $(a,b) \in \mathbb{C}^2$, for all $t_1, t_2 \geq t_\varepsilon$. Hence

$$|\mathcal{E}(az_1 + bz_2, t_2) - \mathcal{E}(az_1 + bz_2, t_1)| \leq \varepsilon\,\rho^{-1}\,\mathcal{E}(az_1 + bz_2, t_1) \tag{2.17}$$

if $t_1, t_2 \geq (t_\varepsilon \vee \bar{t}\,)$. From this, we obtain that

$$\mathcal{E}(az_1 + bz_2, t_2) \leq \left(1 + \varepsilon\rho^{-1}\right)\,\mathcal{E}(az_1 + bz_2, t_1)\,, \tag{2.18}$$

for all $(a,b) \in \mathbb{C}^2$, if $t_1, t_2 \geq (t_\varepsilon \vee \bar{t}\,)$.

Hence, if

$$u = az_1 + bz_2 \tag{2.19}$$

is any solution of (1.2), we have

$$\mathcal{E}(u, t_2) \leq \left(1 + \varepsilon\rho^{-1}\right)\,\mathcal{E}(u, t_1)\,, \quad \forall\, t_1, t_2 \geq (t_\varepsilon \vee \bar{t}\,)\,. \tag{2.20}$$

Finally, given $\Lambda > 1$, setting

$$\varepsilon = \rho\,(\Lambda - 1) \quad \text{and} \quad t_\Lambda = (t_\varepsilon \vee \bar{t}\,)\,, \tag{2.21}$$

we obtain the second inequality of (2.13). □

3. Proofs of Theorems 1.1, 1.2

Let z_1, z_2 be the independent solutions of (1.2) fixed in (2.2)–(2.3). Denoting with

$$W \overset{\text{def}}{=} W(z_1, z_2) = z_1\,z_2' - z_1'\,z_2 \tag{3.1}$$

the wronskian, we clearly have

$$W(t) = W(t_0) \neq 0\,, \quad \forall\, t \in [t_0, \infty)\,. \tag{3.2}$$

Then, recalling (1.5), we introduce the quantity

$$A \overset{\text{def}}{=} \sup_{t \geq t_0}\,\left[\mathcal{E}(z_1, t) \vee \mathcal{E}(z_2, t)\right]. \tag{3.3}$$

By Th. 2.2, we know that $0 < A < \infty$.

Besides, since $\mathcal{E}(z_i, t) \overset{\text{def}}{=} q(t)^{\frac{1}{2}}\,|z_i(t)|^2 + q(t)^{-\frac{1}{2}}\,|z_i'(t)|^2$, for all $t \geq t_0$ we have:

$$\begin{aligned} |z_1|,\, |z_2| &\leq \sqrt{A}\,q^{-\frac{1}{4}}\,, \\ |z_1'|,\, |z_2'| &\leq \sqrt{A}\;q^{\frac{1}{4}}\,, \\ |z_1\,z_1'|,\, |z_2\,z_2'| &\leq \frac{A}{2}\,, \end{aligned} \tag{3.4}$$

where the last inequality of (3.4) is a consequence of the fact that

$$|z_i(t)\,z_i'(t)| \leq 2^{-1}\mathcal{E}(z_i, t) \quad (i = 1, 2)\,, \tag{3.5}$$

via the classical inequality: $ab \leq (a^2 + b^2)/2$ for $a, b \in \mathbb{R}$.

Now, let

$$u : [t_0, \infty) \to \mathbb{C} \tag{3.6}$$

be a given solution of (1.1). Following the argument of W. F. Trench [10], we look for α, β: $[t_0, \infty) \to \mathbb{C}$ such that

$$u = \alpha z_1 + \beta z_2\,, \quad u' = \alpha z_1' + \beta z_2'\,. \tag{3.7}$$

If (3.7) holds, then α, β are uniquely determined by

$$\alpha = \frac{u\, z_2' - u'\, z_2}{W} \quad \text{and} \quad \beta = \frac{u'\, z_1 - u\, z_1'}{W}\,. \tag{3.8}$$

On the other hand, differentiating the expression $\alpha z_1 + \beta z_2$ twice and substituting into (1.1), we easily see that (3.7) holds if and only if α, β verify

$$\begin{cases} \alpha'\, z_1' + \beta'\, z_2' = -\gamma\left(\alpha z_1' + \beta z_2'\right) - \lambda\left(\alpha z_1 + \beta z_2\right), \\ \alpha'\, z_1 + \beta'\, z_2 = 0\,, \end{cases} \tag{3.9}$$

with initial data, at $t = t_0$,

$$\alpha(t_0) = \frac{u\, z_2' - u'\, z_2}{W}\Big|_{t=t_0}, \quad \beta(t_0) = \frac{u'\, z_1 - u\, z_1'}{W}\Big|_{t=t_0}. \tag{3.10}$$

Solving (3.9) with respect to α', β' we obtain the first order, linear system

$$\begin{cases} \alpha' = \frac{\alpha}{W}\left(\gamma\, z_2\, z_1' + \lambda\, z_1\, z_2\right) + \frac{\beta}{W}\left(\gamma\, z_2\, z_2' + \lambda\, z_2^2\right) \\ \beta' = -\frac{\alpha}{W}\left(\gamma\, z_1\, z_1' + \lambda\, z_1^2\right) - \frac{\beta}{W}\left(\gamma\, z_1\, z_2' + \lambda\, z_1\, z_2\right) \end{cases} \qquad t \geq t_0\,. \tag{3.11}$$

Since the Cauchy problem (3.10)–(3.11) has a unique solution in $[t_0, \infty)$, we conclude that there exist α, β such that (3.7) holds.

Now, using the integral representation

$$\begin{aligned} \alpha(t) &= \alpha(t_0) + \frac{1}{W}\int_{t_0}^{t} \left[\alpha\big(\gamma\, z_2\, z_1' + \lambda\, z_1\, z_2\big) + \beta\big(\gamma\, z_2\, z_2' + \lambda\, z_2^2\big)\right] ds\,, \\ \beta(t) &= \beta(t_0) - \frac{1}{W}\int_{t_0}^{t} \left[\alpha\big(\gamma\, z_1\, z_1' + \lambda\, z_1^2\big) + \beta\big(\gamma\, z_1\, z_2' + \lambda\, z_1\, z_2\big)\right] ds\,, \end{aligned} \tag{3.12}$$

for all $t \geq t_0$, we can estimate α, β.

In fact, setting

$$Z(t) \stackrel{\text{def}}{=} \left|\alpha(t)\right| + \left|\beta(t)\right|, \tag{3.13}$$

from (3.4) and (3.12) it follows that

$$Z(t) \leq Z(t_0) + \frac{A}{2\,|W|}\int_{t_0}^{t} Z\big(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\big)\, ds\,, \tag{3.14}$$

for all $t \geq t_0$. Then, applying Gronwall's Lemma to (3.14), we finally deduce

$$Z(t) \,\leq\, Z(t_0)\, \exp\, \frac{A}{2\,|W|}\int_{t_0}^{t} \left(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\right) ds\,, \tag{3.15}$$

for all $t \geq t_0$.

Remark 3.1. More generally, given $t_1, t_2 \geq t_0$, one can also prove that

$$Z(t_2) \leq Z(t_1) \exp \frac{A}{2\,|W|}\Big| \int_{t_1}^{t_2} \Big(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\Big)\, ds\Big| . \tag{3.16}$$

For $t_2 \geq t_1$ it is clear that (3.16) holds; for $t_2 < t_1$ it sufficient to apply Gronwall's Lemma backward in time. In particular, if $\int_{t_0}^{\infty} \big(|\gamma| + |\lambda|\, q^{-\frac{1}{2}} \big) dx < \infty$, it follows from (3.16) that there exists the finite limit

$$\lim_{t\to\infty} Z(t) \stackrel{\text{def}}{=} Z_\infty\,, \quad \text{with} \quad Z_\infty > 0 \quad \text{if} \quad u \not\equiv 0\,.$$

We are now in position to prove Th. 1.2 and then Th. 1.1.

3.1. **The Proof of Th. 1.2.** The assumption

$$\int_{t_0}^{\infty} \big(|\gamma| + |\lambda|\, q^{-\frac{1}{2}} \big)\, dt < \infty \tag{3.17}$$

and inequality (3.15) imply that $Z(t) \leq C$ in $[t_0, \infty)$, for a suitable $C \geq 0$. Thus

$$\big|\alpha(t)\big|,\ \big|\beta(t)\big| \leq C \quad \text{in} \quad [t_0, \infty)\,. \tag{3.18}$$

From this, we easily see that

$$\mathcal{E}(u, t) \leq 2\, C^2\, A \quad \text{for all} \quad t \geq t_0\,. \tag{3.19}$$

Further, from (3.4), (3.17) and (3.18), it turns out that the integrals in the right hand-side of (3.12) are absolutely convergent. This means that $\alpha,\, \beta \in AC[t_0, \infty)$, i.e. α', $\beta' \in L^1[t_0, \infty)$. In particular, it follows that there exist the finite limits

$$\lim_{t\to\infty} \beta(t) = \beta_\infty\,, \quad \lim_{t\to\infty} \alpha(t) = \alpha_\infty\,. \tag{3.20}$$

By Th. 2.2 and Lemma 2.4, we know that there exist the finite limits:

$$\lim_{t\to\infty} \mathcal{E}(z_i, t) = \mathcal{E}_i \quad (i = 1, 2)\,, \tag{3.21}$$

$$\lim_{t\to\infty} \Big(q^{\frac{1}{2}}\, z_1 \bar{z}_2 + q^{-\frac{1}{2}}\, z_1' \bar{z}_2'\Big) = \mathcal{E}_{12}\,. \tag{3.22}$$

Then, by (3.7), one has

$$\begin{aligned} \lim_{t\to\infty} \mathcal{E}(u, t) &= \lim_{t\to\infty} \mathcal{E}(\alpha z_1 + \beta z_2, t) \\ &= \lim_{t\to\infty} |\alpha|^2 \mathcal{E}(z_1, t) + \lim_{t\to\infty} |\beta|^2\, \mathcal{E}(z_2, t) \\ &\quad + 2 \operatorname{Re} \lim_{t\to\infty} \alpha \bar{\beta}\big(q^{\frac{1}{2}}\, z_1 \bar{z}_2 + q^{-\frac{1}{2}}\, z_1' \bar{z}_2'\big) \\ &= \mathcal{E}_1\, |\alpha_\infty|^2 + \mathcal{E}_2\, |\beta_\infty|^2 + 2 \operatorname{Re}\big(\mathcal{E}_{12}\, \alpha_\infty \bar{\beta}_\infty\big) \\ &= Q(\alpha_\infty, \beta_\infty)\,, \end{aligned} \tag{3.23}$$

where $Q(\cdot, \cdot)$ is the quadratic form (2.5). This means that $\mathcal{E}(u, t)$ tends to a finite limit as $t \to \infty$. Moreover, by Remark 3.1, we know that

$$|\alpha_\infty| + |\beta_\infty| = Z_\infty > 0 \quad \text{if} \quad u \not\equiv 0\,. \tag{3.24}$$

Since $Q(\cdot, \cdot)$ is positive definite, the limit (3.23) is strictly positive if $u \not\equiv 0$.

3.2. **The Proof of Th. 1.1.** Assuming (1.4), by Th. 2.1 the condition $\int_{t_0}^{\infty} q^{-\frac{p}{4}}\,dt < \infty$ $(p > 0)$ is equivalent to the p-integrability of z_1, z_2, namely

$$\int_{t_0}^{\infty} |z_1|^p\,dt < \infty\,, \qquad \int_{t_0}^{\infty} |z_2|^p\,dt < \infty\,. \tag{3.25}$$

Besides, the assumption $\int_{t_0}^{\infty}(|\gamma| + |\lambda|\,q^{-\frac{1}{2}}\,)dt < \infty$ and (3.15) lead to (3.18). Hence, by (3.7), we obtain

$$\int_{t_0}^{\infty} |u|^p\,dt \le 2^p\,C^p \int_{t_0}^{\infty} \Big(|z_1|^p + |z_2|^p\Big)\,dt\ < \infty\,. \tag{3.26}$$

Conversely, let us suppose that all solutions of (1.1) are p-integrable. By Th. 1.2, we know that for every solution $u \not\equiv 0$ of (1.1) there exists a finite and positive the limit

$$\lim_{t\to\infty} \mathcal{E}(u,t) \stackrel{\text{def}}{=} \mathcal{E}_u > 0\,. \tag{3.27}$$

This implies that if u_1, u_2 are two linearly independent solutions of (1.1) then

$$\liminf_{t\to\infty} \sqrt{q(t)}\,\big(|u_1(t)|^2 + |u_2(t)|^2\big)\ > 0\,. \tag{3.28}$$

In fact, if (3.28) does not hold, there exists a sequence $\{t_n\}_{n\ge 1}$, $t_n \to \infty$, such that $\sqrt{q(t_n)}\,(|u_1(t_n)|^2 + |u_2(t_n)|^2) \to 0$ as $n \to \infty$. Then, by Th. 1.2,

$$\lim_{n\to\infty} \frac{|u_1'(t_n)|^2}{\sqrt{q(t_n)}}\ =\ \mathcal{E}_{u_1}\,, \qquad \lim_{n\to\infty} \frac{|u_2'(t_n)|^2}{\sqrt{q(t_n)}}\ =\ \mathcal{E}_{u_2}\,, \tag{3.29}$$

with $0 < \mathcal{E}_{u_1}\,,\, \mathcal{E}_{u_2} < \infty$. In particular $|u_1'(t_n)|\,,\,|u_2'(t_n)| > 0$ for n large enough, and

$$\lim_{n\to\infty} \frac{|u_1'(t_n)|}{|u_2'(t_n)|} = \mathcal{E}_{u_1}^{\frac{1}{2}}\,\mathcal{E}_{u_2}^{-\frac{1}{2}}\,. \tag{3.30}$$

Hence, for a suitable subsequence $\{\tau_n\}_{n\ge 1} \subset \{t_n\}_{n\ge 1}$ we may suppose that $u_2'(\tau_n) \neq 0$ for all $n \ge 1$ and that

$$\lim_{n\to\infty} \frac{u_1'(\tau_n)}{u_2'(\tau_n)}\ = \zeta \quad \text{with} \quad |\zeta| = \mathcal{E}_{u_1}^{\frac{1}{2}}\,\mathcal{E}_{u_2}^{-\frac{1}{2}}\,. \tag{3.31}$$

Next, we consider

$$v(t) \stackrel{\text{def}}{=} u_1(t) - \zeta\,u_2(t)\,. \tag{3.32}$$

Since u_1, u_2 are linearly independent, v is a non-zero solution of (1.1). It follows that

$$\begin{aligned} &\limsup_{n\to\infty} \sqrt{q(\tau_n)}\ |v(\tau_n)|^2 \\ &\qquad \le 2 \lim_{n\to\infty}\ \sqrt{q(\tau_n)}\ \big(|u_1(\tau_n)|^2 + |\zeta|^2|u_2(\tau_n)|^2\big) = 0\,. \end{aligned} \tag{3.33}$$

Moreover, by (3.29) and (3.31)–(3.32) we have also

$$\begin{aligned}\lim_{n\to\infty}\frac{|v'(\tau_n)|^2}{\sqrt{q(\tau_n)}} &= \lim_{n\to\infty}\frac{|u_1'(\tau_n)-\zeta\,u_2'(\tau_n)|^2}{\sqrt{q(\tau_n)}}\\ &= \lim_{n\to\infty}\frac{|u_2'(\tau_n)|^2}{\sqrt{q(\tau_n)}}\,\left|\frac{u_1'(\tau_n)}{u_2'(\tau_n)}-\zeta\right|^2 = 0\,.\end{aligned}\tag{3.34}$$

From (3.33) and (3.34) it follows that $\lim_{n\to\infty}\mathcal{E}(v,\tau_n)=0$. On the other hand, by Th. 1.2, we must have $\lim_{t\to\infty}\mathcal{E}(v,t)=\mathcal{E}_v>0$ because $v\not\equiv 0$.

This contradiction proves that (3.28) holds.

Now we can show that $\int_{t_0}^{\infty} q^{-\frac{p}{4}}\,dt<\infty$, if all the solutions of (1.2) are p-integrable. In fact, (3.28) implies that there exists $\varepsilon>0$ such that $\sqrt{q(t)}\,\big(|u_1(t)|^2+|u_2(t)|^2\big)\geq\varepsilon$ for t large enough, say $t\geq\bar t\geq t_0$. Hence, since $p>0$, we have the inequalities

$$\begin{aligned}\int_{\bar t}^{\infty} q(t)^{-\frac{p}{4}}\,dt &\leq \Big(\frac{1}{\varepsilon}\Big)^{\frac{p}{2}}\int_{\bar t}^{\infty}\big(|u_1(t)|^2+|u_2(t)|^2\big)^{\frac{p}{2}}\,dt\\ &\leq \Big(\frac{2}{\varepsilon}\Big)^{\frac{p}{2}}\int_{\bar t}^{\infty}\big(|u_1(t)|^p+|u_2(t)|^p\big)\,dt<\infty\,.\end{aligned}\tag{3.35}$$

4. Proof of Theorem 1.3

First of all we prove that the solutions of (1.1) are bounded if (1.7) holds. To this end, we select suitable linearly independent solutions of (1.2). More precisely, fixed $\tau\geq t_0$, we denote by v_τ, w_τ the solutions of (1.2) satisfying, for $t=\tau$, the initial conditions

$$\begin{cases}v_\tau(\tau)=q(\tau)^{-\frac12}\\ v_\tau'(\tau)=0\end{cases}\qquad\qquad\begin{cases}w_\tau(\tau)=0\\ w_\tau'(\tau)=1\end{cases}\tag{4.1}$$

Denoting with $W_\tau\stackrel{\text{def}}{=}v_\tau\,w_\tau'-v_\tau'\,w_\tau$ the wronskian of v_τ, w_τ, from (4.1) we clearly have

$$W_\tau(t)=q(\tau)^{-\frac12}\,,\qquad\forall\,t\in[t_0,\infty)\,.\tag{4.2}$$

Taking (1.5) into account, we introduce the quantity

$$A_\tau\stackrel{\text{def}}{=}\sup_{t\geq\tau}\Big[\mathcal{E}(v_\tau,t)\vee\mathcal{E}(w_\tau,t)\Big]\,.\tag{4.3}$$

By Th. 2.2, $0<A_\tau<\infty$. In addition, we have:

$$\begin{aligned}|v_\tau|,\,|w_\tau|\;&\leq\sqrt{A_\tau}\,q^{-\frac14}\,,\\ |v_\tau'|,\,|w_\tau'|\;&\leq\sqrt{A_\tau}\;q^{\frac14}\,,\\ |v_\tau\,v_\tau'|,\,|w_\tau\,w_\tau'|\;&\leq\frac{A_\tau}{2}\,,\end{aligned}\tag{4.4}$$

for all $t\geq\tau$. Now, let

$$u\colon[t_0,\infty)\to\mathbb{C}\tag{4.5}$$

be a solution of (1.1). We look for $\tilde{\alpha}$, $\tilde{\beta}\colon [t_0,\infty) \to \mathbb{C}$ such that

$$u = \tilde{\alpha} v_\tau + \tilde{\beta} w_\tau\,, \quad u' = \tilde{\alpha} v'_\tau + \tilde{\beta} w'_\tau\,. \tag{4.6}$$

As in the proofs of Th. 1.1 and 1.2, differentiating the expression $u = \tilde{\alpha} v_\tau + \tilde{\beta} w_\tau$ twice (with respect to t) and substituting into (1.1), we easily see that $\tilde{\alpha}$, $\tilde{\beta}$ must satisfy the integral equations

$$\begin{aligned} \tilde{\alpha}(t) &= \tilde{\alpha}(\tau) + \frac{1}{W_\tau} \int_\tau^t \big[\tilde{\alpha}\big(\gamma\, w_\tau\, v'_\tau + \lambda\, v_\tau\, w_\tau\big) + \tilde{\beta}\big(\gamma\, w_\tau\, w'_\tau + \lambda\, w_\tau^2\big)\big]\, ds\,, \\ \tilde{\beta}(t) &= \tilde{\beta}(\tau) - \frac{1}{W_\tau} \int_\tau^t \big[\tilde{\alpha}\big(\gamma\, v_\tau\, v'_\tau + \lambda\, v_\tau^2\big) + \tilde{\beta}\big(\gamma\, v_\tau\, w'_\tau + \lambda\, v_\tau\, w_\tau\big)\big]\, ds \end{aligned} \tag{4.7}$$

with initial data, at $t = \tau$,

$$\begin{cases} \tilde{\alpha}(\tau) = u(\tau)\, q(\tau)^{\frac{1}{2}}\,, \\ \tilde{\beta}(\tau) = u'(\tau)\,. \end{cases} \tag{4.8}$$

From (4.4) and (4.7), it follows that

$$\begin{aligned} \big|\tilde{\alpha}(t)\big| + \big|\tilde{\beta}(t)\big| \le& \big|\tilde{\alpha}(\tau)\big| + \big|\tilde{\beta}(\tau)\big| \\ &+ \frac{A_\tau}{2\, W_\tau} \int_\tau^t \big(|\tilde{\alpha}| + |\tilde{\beta}|\big)\big(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\big)\, ds\,, \end{aligned} \tag{4.9}$$

for all $t \ge \tau$. Thus, by Gronwall's Lemma, we obtain:

$$\big|\tilde{\alpha}(t)\big| + \big|\tilde{\beta}(t)\big| \le \big(|\tilde{\alpha}(\tau)| + |\tilde{\beta}(\tau)|\big)\, \exp \frac{A_\tau}{2\, W_\tau} \int_\tau^t \big(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\big)\, ds\,. \tag{4.10}$$

Then, from (4.4), (4.6), (4.8) and (4.10) we have

$$\big|u(t)\big| \le B_\tau\, q(t)^{-\frac{1}{4}}\, \exp \frac{A_\tau}{2\, W_\tau} \int_\tau^t \big(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\big)\, ds\,, \tag{4.11}$$

for all $t \ge \tau$, with $B_\tau = \sqrt{A_\tau}\, \big(|u(\tau)|\, q(\tau)^{\frac{1}{2}} + |u'(\tau)|\big)$.

We can now prove that u remains bounded as $t \to \infty$. In fact, by (4.11), u is uniformly bounded in $[t_0,\infty)$ if the quantity

$$K_\tau(t) \overset{\text{def}}{=} \int_\tau^t \big(3\,|\gamma| + 4\,|\lambda|\, q^{-\frac{1}{2}}\big) dx - \frac{W_\tau}{2\, A_\tau}\, \ln q(t) \tag{4.12}$$

remains bounded as $t \to \infty$, i.e. if (1.7) is verified for some $\mathcal{C} \ge \frac{2\, A_\tau}{W_\tau}$. Hence, it is clearly enough that (1.7) holds for some $\mathcal{C}$ such that

$$\mathcal{C} > 2 \inf_{\tau \ge t_0} \frac{A_\tau}{W_\tau}\,. \tag{4.13}$$

We claim that the greatest lower bound of the quotient A_τ / W_τ is equal to one. To see this, we observe that the initial conditions (4.1)–(4.2) give

$$\mathcal{E}(v_\tau, \tau) = \mathcal{E}(w_\tau, \tau) = q(\tau)^{-\frac{1}{2}}\,. \tag{4.14}$$

Thus, by (4.2)–(4.3), we have $A_\tau / W_\tau \ge 1$ for all $\tau \ge t_0$.

On the other hand, by Lemma 2.5, for all $\Lambda > 1$ there exists $t_\Lambda \geq t_0$ such that

$$\mathcal{E}(v_\tau, t) \leq \Lambda\, \mathcal{E}(v_\tau, \tau)\,, \quad \mathcal{E}(w_\tau, t) \leq \Lambda\, \mathcal{E}(w_\tau, \tau)\,, \tag{4.15}$$

for all $t, \tau \geq t_\Lambda$. Hence, by (4.3) and (4.14), we have

$$A_\tau \leq \Lambda\, q(\tau)^{-\frac{1}{2}}\,, \quad \text{for} \quad \tau \geq t_\Lambda\,. \tag{4.16}$$

It follows that $A_\tau / W_\tau \leq \Lambda$ for $\tau \geq t_\Lambda$ and we my conclude that

$$\lim_{\tau\to\infty} \frac{A_\tau}{W_\tau} = 1\,. \tag{4.17}$$

From (4.13) we deduce that u remains bounded if (1.7) holds with $\mathcal{C} > 2$.

Finally, let us prove that $u(t) \to 0$, as $t \to \infty$, if (1.7) is verified with $\mathcal{C} > 2$. In fact, by (4.17) we may fix $\tau_o \geq t_0$ such that

$$\mathcal{C} > \frac{2\, A_{\tau_0}}{W_{\tau_0}}\,. \tag{4.18}$$

Then, by (1.7), there exists $\rho \in \mathbb{R}$ such that

$$K_{\tau_0}(t) \leq \rho + \Big(\frac{1}{\mathcal{C}} - \frac{W_{\tau_0}}{2\, A_{\tau_o}}\Big) \ln q(t)\,, \quad \forall\, t \geq \tau_0\,, \tag{4.19}$$

where K_τ is the quantity introduced in (4.12). Hence, since $q(t) \to \infty$, from (4.18)–(4.19) we see that $K_{\tau_0}(t) \to -\infty$ as $t \to \infty$. Then, setting $\tau = \tau_o$ in (4.11), we deduce that $u(t) \to 0$ as $t \to \infty$.

5. Some applications

We give here some applications of Th. 1.1, 1.2 and 1.3. We will also compare these results with the criteria of R. Bellman [1] and Z. Opial [9] stated in the introduction.

In the following examples, C will stands for a generic positive constant, independent of t; in addition, r, r', s will always indicate real numbers.

Example 5.1. Let us consider equation (1.1) in $[e, \infty)$, with $q(t) = (2 + \sin(t^s)) \ln t$, $0 \leq s < 1$; $\gamma(t) = \frac{\phi_1(t)}{t\, \ln t}$, $\lambda(t) = \frac{\phi_2(t)}{t\, \sqrt{\ln t}}$ where ϕ_1, ϕ_2 are bounded, continuous functions.

Then $q(t) \geq \ln t \geq 1$ in $[e, \infty)$, moreover

$$\big|\big(q^{-\frac{1}{2}}\big)^{(h)}\big| \leq C\, \frac{t^{h(s-1)}}{\sqrt{\ln t}} \quad \text{in} \quad [e, \infty)\,, \tag{5.1}$$

for all integers $h \geq 1$. Hence (1.4) is verified taking a positive integer $m > \frac{s}{1-s}$.

It is easy to show that (1.7) holds if $3\|\phi_1\|_{L^\infty} + 4\|\phi_2\|_{L^\infty} < \frac{1}{2}$. In this case, applying Th. 1.3 we deduce that every solution u of (1.1) tends to 0 as $t \to \infty$. The assumptions of [9] are not verified, because q is not monotone. Since $\int^\infty q^{-\frac{p}{4}}\, dt = +\infty$ for all $p > 0$, by Th. 2.1 for every $p > 0$ there exists at least a solution of (1.2) which is not p-integrable on $[t_0, \infty)$. Thus the criterium of [1] is not applicable.

Example 5.2. Consider the equation

$$u'' + \frac{\phi}{t}\, u' + \big[t^r + a\, t^{r'} \sin(t^s)\big]\, u = 0 \quad \text{for} \quad t \geq t_0 \geq 1\,, \tag{5.2}$$

where ϕ is a continuous function, $a \in \mathbb{R}$, $r \geq 0$.

1) *Case* $r' < \frac{r}{2} - 1$, $\int^\infty \frac{|\phi|}{t} < \infty$. Setting $q(t) = t^r$, (1.4) is easily verified and we can apply Th. 1.1 and Th. 1.2. Therefore for every solution u of (5.2) there exists the finite limit

$$\lim_{t\to\infty} \left(t^{\frac{r}{2}}|u(t)|^2 + t^{-\frac{r}{2}}|u'(t)|^2\right) = \mathcal{E}_u\,, \quad \text{with} \quad \mathcal{E}_u > 0 \quad \text{if} \quad u \not\equiv 0\,; \tag{5.3}$$

if $r > 0$ then every solution u of (5.2) is p-integrable for $p > \frac{4}{r}$.

1′) *Case* $r' \leq \frac{r}{2} - 1$, ϕ *bounded.* Setting $q(t) = t^r$ as above, condition (1.7) holds if we suppose if $3\|\phi\|_{L^\infty} + 4|a| < \frac{r}{2}$ ($3\|\phi\|_{L^\infty} < \frac{r}{2}$, if $r' < \frac{r}{2} - 1$). In this case, by Th. 1.3, every solutions u tends to 0 as $t \to \infty$.

Observe that in cases 1) and 1′) there are no restrictions on s. Even if $\phi \equiv 0$, the criterium of [1] is applicable only for $r > 2$ and $r' \leq 0$.

2) *Case* $\frac{r}{2} - 1 < r' < r$, $0 \leq s < 1$. In this case we must set $q = t^r + a\, t^{r'} \sin(t^s)$. Then, we have the inequalities

$$|(q^{-\frac{1}{2}})^{(h)}| \leq C\left(t^{-\frac{r}{2}-h} + t^{r'-\frac{3r}{2}+h(s-1)}\right) \quad \text{in} \quad [t_0, \infty)\,, \tag{5.4}$$

for all integers $h \geq 0$, provided t_0 is sufficiently large. This means that (1.4) is verified if we take a positive integer $m > \frac{2s+2r'-3r}{2(1-s)}$. Then we have:

– if $\int^\infty \frac{|\phi|}{t} < \infty$, we can apply Th. 1.1 and 1.2 as in the previous cases;
– if ϕ is bounded, condition (1.7) holds if $3\|\phi\|_{L^\infty} < \frac{r}{2}$. In this case, by Th. 1.3, every solutions u tends to 0 as $t \to \infty$.

3) *Case* $r' = r$, $0 \leq s < 1$. Setting $q = t^r + a\, t^r \sin(t^s)$, we have exactly the previous situation, provided $|a| < 1$.

In cases 2) and 3) the criterium of [9] is not applicable if q is not monotone nondecreasing, i.e. respectively if $s > r - r'$ and $s > 0$.

Remark 5.3. Let us consider equation (5.2) in $[1, \infty)$ with $r' = r \geq 0$, $|a| < 1$ and $s \geq 1$. Setting $q = t^r + a\, t^r \sin(t^s)$ as above, we have $q(t) \geq t^r(1 - |a|)$ in $[1, \infty)$. The assumptions stated in (1.8)–(1.10) of Remark 1.4 are easily verified if we suppose

$$r > 2(s-1) \tag{5.5}$$

and m is a sufficiently large, positive integer. In fact, from (5.4) we obtain

$$(q^{-\frac{1}{2}})^{1-\frac{1}{h}}\, |(q^{-\frac{1}{2}})^{(h)}|^{\frac{1}{h}} \leq C\, t^{-\frac{r}{2}+(s-1)} \quad \forall\, h \geq 1\,. \tag{5.6}$$

Thus (5.5) implies (1.8). In addition, (5.4) also gives

$$\left| q^{-\eta_0/2} \left(\frac{d}{dt} q^{-\frac{1}{2}}\right)^{\eta_1} \cdots \left(\frac{d^{m+1}}{dt^{m+1}} q^{-\frac{1}{2}}\right)^{\eta_{m+1}} \right| \leq C\, t^{-\frac{mr}{2}+(m+1)(s-1)}\,, \tag{5.7}$$

for all integers $\eta_0, \ldots, \eta_{m+1} \geq 0$ satisfying (1.10). Hence, (1.9) is verified if

$$r > \frac{2}{m} + 2(s-1)\frac{m+1}{m}\,. \tag{5.8}$$

From (5.5) again, we can see that (5.8) holds if m is large enough. As stated in Remark 1.4, we are therefore in a position to apply Th. 1.1, 1.2 and 1.3. More precisely, assuming $\int^\infty \frac{|\phi|}{t} < \infty$, we can apply Th. 1.1 and 1.2 as in the previous cases. Condition (1.7) holds if $3\|\phi\|_{L^\infty} < \frac{r}{2}$ and, in this case, every solution u tends to 0 as $t \to \infty$.

Example 5.4. Here we will show that the conclusion of Th. 1.3 ($u(t) \to 0$ as $t \to \infty$) may be false if we only require that (1.7) holds for an arbitrary $\mathcal{C} > 0$. In other words, we must suppose $\mathcal{C} \geq \mathcal{C}_0$, for a suitable $\mathcal{C}_0 > 0$. In fact, let us consider the equation

$$u'' + \gamma(t)u' + q(t)u = 0\,, \quad t \in [\tau, \infty)\,. \tag{5.9}$$

As it is known, if $\gamma \in C^1$, the substitution $u = v\, e^{-\frac{1}{2}\int_\tau^t \gamma\, dt}$ transforms (5.9) into

$$v'' + \big(q - \frac{\gamma'}{2} - \frac{\gamma^2}{4}\big)v = 0\,, \quad t \in [\tau, \infty). \tag{5.10}$$

Now, setting $q = t$, $\gamma = \frac{a}{t}$ ($a \in \mathbb{R}$) and $\tau = 1$ we obtain the equation

$$v'' + \left(t + \frac{a}{2t^2} - \frac{a^2}{4t^2}\right)v = 0 \quad t \in [1, \infty)\,. \tag{5.11}$$

Equation (5.11) satisfies the assumptions of Th. 1.2 in $[t_0, \infty) \subseteq [1, \infty)$, provided t_0 is large enough; for every nonzero solution v there exists finite and positive the limit

$$\lim_{t \to \infty} \left(t^{\frac{1}{2}}|v(t)|^2 + t^{-\frac{1}{2}}|v'(t)|^2\right) \stackrel{\text{def}}{=} \mathcal{E}_v\,. \tag{5.12}$$

Now, let $\tilde{v}$ be a fixed nonzero solution of (5.11), thus $\mathcal{E}_{\tilde{v}} > 0$. Since $\tilde{v}$ is oscillating, there exists a sequence $\{t_n\}_{n \geq 1}$, $t_n \to \infty$ as $n \to \infty$, such that

$$\big|\tilde{v}(t_n)\big| \geq \frac{\mathcal{E}_{\tilde{v}}}{2\sqrt[4]{t_n}}\,, \quad \forall\, n \geq 1\,. \tag{5.13}$$

Then $\tilde{u} = \tilde{v}\, e^{-\frac{1}{2}\int_1^t \gamma\, dt} = t^{-\frac{a}{2}}\,\tilde{v}$ is a solution of (5.9) in $[1, \infty)$ satisfying:

$$\big|\tilde{u}(t_n)\big| \geq \frac{\mathcal{E}_{\tilde{v}}}{2}\, t_n^{-\frac{a}{2} - \frac{1}{4}}\,, \quad \forall\, n \geq 1\,. \tag{5.14}$$

In particular, it follows that $\tilde{u}(t_n) \not\to 0$, as $n \to \infty$, if $a \leq -\frac{1}{2}$. Hence, for $a \leq -\frac{1}{2}$, the conclusion of Th. 1.3 cannot hold.

On the other hand, taking $q = t$, $\gamma = \frac{a}{t}$ it is easy to see that equation (5.9) satisfies condition (1.7) only if $|a| < \frac{1}{6}$.

References

[1] Bellman, R., *A stability property of solutions of linear differential equations*, Duke Math. J. **11** (1944), 513–516.

[2] Bellman, R., *Stability Theory of Differential Equations*, McGraw-Hill Book Company, New York, 1953.

[3] Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, 2nd ed., Springer-Verlag, Berlin, 1963.

[4] Galbraith, A., McShane, E. J., Parrish, G., *On the solutions of the linear second-order differential equations*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 247–249.

[5] Knowles, I., *On stability conditions for second order linear differential equations*, J. Differential Equations **34** (1979), 179–203.

[6] Manfrin, R., *Quadratic forms for the Liouville equation* $w_{tt} + \lambda^2 a(t)w = 0$ *with applications to Kirchhoff equation*, Portugal. Math. **65** (2008), 447–484.

[7] Manfrin, R., L^p *solutions of second order differential equations*, Funkcial. Ekvac. **52** (2009), 255–279.

[8] Manfrin, R., *On the boundedness of solutions of the equation* $u'' + (1 + f(t))u = 0$, Discrete Contin. Dynam. Systems **23** (2009), 991–1008.

[9] Opial, Z., *Nouvelles remarques sur l'équation différentielle* $u'' + a(t)u = 0$, Ann. Polon. Math. **6** (1959), 75–81.

[10] Trench, W. F., *On the asymptotic behavior of solutions of second order linear differential equations*, Proc. Amer. Math. Soc. **14** (1963), 12–14.

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